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"Reaction-Diffusion systems with positivity and mass control: global existence and singular limits"

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Abstract.

These notes come with a series of lectures given in Essen at the 2011 Spring School on Evolution Equations. They concern so-called "reaction-diffusion systems", which are mathematical models for evolution phenomena undergoing at the same time spatial diffusion and (bio-)chemical type of reactions. Interest has increased recently for these models, in particular for applications in biology, ecology, environment and population dynamics.

Two natural properties appear in most models: the nonnegativity of the solutions is preserved for all time; the total mass of the components is controlled for all time (sometimes even exactly preserved). We will mainly be interested in this family of systems.

The fact that the total mass of the components does not blow up in finite time suggests that solutions should exist for all time (mathematically speaking, solutions are actually bounded in L^1 uniformly in time). But, it turns out that the answer is not so simple. Some "good" situations do lead to global existence of classical solutions, and we will recall them. On the other hand, we will explain how "incomplete blow up" may occur. This implies that it is necessary to *give up looking for bounded classical solutions* and rather consider *weak solutions*. New kind of estimates are then necessary to provide global existence.

Although the systems under consideration offer an obvious L^1-structure, they surprisingly satisfy an a priori L^2-estimate which turns out to be useful for the global existence problems, and, at the same time, for several different questions. We will use them to describe some limit systems when reactions are so fast that they may be considered as being instantaneous, or when they are quite faster than the diffusion phenomena. This will lead us to the very active and open area of *"cross-diffusion systems"* which happen to model more and more concrete situations and which raise new and challenging mathematical questions.

Prerequisite announced for these lectures : only a good knowledge of systems of ordinary differential equations together with the main properties of the heat equation.

Contents

1 Introduction

As announced in the abstract above, we will describe some recent results and some new tools developed for the mathematical analysis of "reaction-diffusion" systems satisfying two main properties which hold in many applications, namely:

(p) *Positivity of the solutions is preserved for all time*
(m) *The total mass of the components is controlled for all time.*

The fact that the total mass of the components does not blow up in finite time suggests for instance that solutions should exist for all time. But, it turns out that the answer is not so simple. We will progressively describe the situation all along the lectures and in the present associated notes..

The analysis will require sophisticated tools from the theory of linear and nonlinear partial differential equations, which are interesting for themselves, and which turn out to be also very useful for questions of singular limits of reaction-diffusion systems. For instance, what happens when the reaction can be considered as being instantaneous or when it is quite faster than the diffusion?

We will be led to cross-diffusion systems for which many questions are still open. We will end by giving a list of open problems.

A good part of these notes is taken from the survey ([59]). The rest comes from recent and current work. We will mainly state and comments results without reproducing the proofs. Most of them will be given during the lectures and references are given here.

By "reaction-diffusion" systems, we mean systems of partial differential equations of the following type, set on a cylinder $(0,T)\times\Omega$, where Ω is a bounded open subset of $\mathbb{R}^N$ that *we will throughout assume to have an at least C^2-boundary*, and which writes:

$$\begin{cases} \forall i=1,...m, \\ \partial_t u_i - d_i\Delta u_i = f_i(u_1,...,u_m) \; on \; Q_T=(0,T)\times\Omega, \\ + \, boundary \; conditions \; on \; \partial\Omega \; for \; u_i, \\ u_i(0,\cdot)=u_i^0, \end{cases} \tag{1}$$

where, for all $i=1,...,m$, $d_i\in(0,\infty)$, $f_i:\mathbb{R}^m\to\mathbb{R}$ is a C^1-function, $u_i:(0,T)\times\Omega\to\mathbb{R}$ are the unknowns and u_i^0 is the initial data.

The positivity condition **(p)** is satisfied if and only if $f=(f_1,...,f_m)$ is **quasi-positive** which means that, for $i=1,...,m$

$$(\mathbf{P}) \;\; [r\in[0,\infty)^m] \;\Rightarrow\; [[f_i(r_1,...,r_{i-1},0,r_{i+1},...,r_m\geq 0]. \tag{2}$$

Condition **(m)** on the a priori bound of the total mass is satisfied for instance when, for some $a_i\in(0,+\infty)$:

$$(\mathbf{M}) \;\; \forall r=(r_1,...,r_m)\in[0,\infty)^m, \; \sum_{1\leq i\leq m} a_i f_i(r)\leq 0, \tag{3}$$

assuming moreover that boundary conditions on the u_i's are "correct". To see this, add up the m equations after multiplying each i-th line by a_i and integrate over $(0,t)\times\Omega$. For "correct" boundary conditions, we will have $-\int_\Omega \Delta u_i(t,x)\,dx\geq 0$, so that one obtains the a priori estimate

$$\forall t\in(0,T), \; \sum_{1\leq i\leq m}\int_\Omega a_i u_i(t,x)\,dx \leq \sum_{1\leq i\leq m}\int_\Omega a_i u_i(0,x)\,dx.$$

When the u_i are initially nonnegative, they remain nonnegative so that this implies

$$\forall i=1,...,m, \; \sup_{t\in(0,T)}\|u_i(t)\|_{L^1(\Omega)}<+\infty, \;\; \sup_{t\in[0,T]}\int_\Omega\sum_i u_i(t)<+\infty. \tag{4}$$

In other words, the total mass of the components is uniformly bounded in time as well as the $L^1(\Omega)$-norm of each component.

The same conclusion holds if we replace the property **(M)** by the following more general one which is more likely to appear in applications: for some $C\in[0,\infty)$

$$(\mathbf{M'}) \;\; \forall r=(r_1,...,r_m)\in[0,\infty)^m, \; \sum_{1\leq i\leq m} a_i f_i(r)\leq C\,[1+\sum_i r_i]. \tag{5}$$

For simplicity, throughout the notes (and the lectures), *we will consider homogeneous Neumann boundary conditions*, namely

$$\partial_\nu u_i = 0 \; on \; \Sigma_T=(0,T)\times\partial\Omega, \tag{6}$$

where $\partial_\nu = \nabla(\cdot) \cdot \nu$ denotes the exterior normal derivative to $\partial\Omega$. Most of the results we will mention extend to other "good" boundary conditions.

Remark. Note that some difficulties may however appear when boundary conditions are of different kind from one line to the other (see [49]).

2 Some examples of reaction-diffusion systems with properties (P)+(M')

We reproduce here the examples of reaction-diffusion systems gathered in [59] and that may be found here and there in the literature as models for very different applications and for which the two properties **(P)+(M)** or **(M')** hold.

- **The Brussellator.** Let us start with the classical so-called "Brussellator" appearing in the modeling of *chemical morphogenetic processes* ([63, 64, 66]):

$$\begin{cases} \partial_t u - d_1 \Delta u = -uv^2 + b\, v \\ \partial_t v - d_2 \Delta v = uv^2 - (b+1)\, v + a \\ u_{|\partial\Omega} = b/a, \ v_{|\partial\Omega} = a, \\ a, b, d_1, d_2 > 0. \end{cases} \tag{7}$$

  If we denote

$$f(u,v) = -uv^2 + bv, \ \ g(u,v) = uv^2 - (b+1)v + a,$$

  then for all $u, v \geq 0$,

$$f(0,v) = b\, v \geq 0, \ g(u,0) = a \geq 0, \ f(u,v) + g(u,v) \leq a,$$

  so that **(P)+(M')** holds. Global existence of classical solutions may be proved in small dimension using bootstrap arguments (see [65]). More sophisticated techniques are required in general, like those explained in the next section.

  Very similar systems are also used in *models of Glycolysis or in the so-called Gray-Scott models* (see [51] together with more general systems with "telescoping" nonlinearities arising in chemical kinetics).

- **Combustion models.** Exothermic combustion in a gas may be modeled by a system of the following type (see e.g. [34])

$$\begin{cases} \partial_t Y - \mu \Delta Y = -H(Y,T) \\ \partial_t T - \lambda \Delta T = q\, H(Y,T), \end{cases} \tag{8}$$

  where Y is the concentration of a single reactant, T is the temperature and $H(0,T) = 0, H(Y,0) \geq 0$. Moreover, if $f(Y,T) = -H(Y,T), g(Y,T) = qH(Y,T)$, we see that $q\, f(Y,T) + g(H,T) = 0$ so that **(P)+(M)** is satisfied. A typical function H is given by $H(Y,T) = Y^m e^T$. Similar equations appear for different applications in [51, 55].

- **Lotka-Volterra systems.** A general class of Lotka-Volterra Systems may be written (see for instance in [45], [28])

$$\forall i = 1...m, \ \ \partial_t u_i - d_i \Delta u_i = e_i u_i + u_i \sum_{1 \le j \le m} p_{ij} u_j, \tag{9}$$

  with $e_i, p_{ij} \in \mathbb{R}$ and various boundary conditions. Condition **(P)** is always satisfied, and so is **(M)** -see (5)- if for instance for some $a_i > 0$ (see e.g. [45])

$$\forall w \in R^m, \ \sum_{i,j=1}^{m} a_i p_{ij} w_i w_j \le 0,$$

- **Quadratic chemical reactions.** Many chemical reactions, when modeled through the mass action law, lead to reaction-diffusion systems with the above **(P)+(M)** structure. Let us first take a typical example that we will discuss later in this paper. We consider the reversible reaction

$$U_1 + U_2 \underset{k^-}{\overset{k^+}{\rightleftharpoons}} U_3 + U_3.$$

  Then according to the mass action law, and with a Fickian diffusion, the evolution of the concentrations u_i of U_i is governed by the following reaction-diffusion system:

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k^+ u_1\, u_2 + k^- u_3\, u_4 \\ \partial_t u_2 - d_2 \Delta u_2 = -k^+ u_1\, u_2 + k^- u_3\, u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = k^+ u_1\, u_2 - k^- u_3\, u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = k^+ u_1\, u_2 - k^- u_3\, u_4, \end{cases} \tag{10}$$

  with $k^+, k^- > 0$. Our two conditions are obviously satisfied here. We may also exploit that the entropy is decreasing : see (43) (this is actually the case in all reversible reactions).

- **Superquadratic reaction-diffusion systems.** We consider a general reversible chemical reaction of the form

$$p_1 U_1 + p_2 U_2 + ... + p_m U_m \underset{k^-}{\overset{k^+}{\rightleftharpoons}} q_1 U_1 + q_2 U_2 + ... + q_m U_m, \tag{11}$$

  where p_i, q_i are nonnegative integers. According to the usual mass action kinetics and with classical diffusion operators, we model the evolution of the concentrations u_i of U_i by the following system of reaction-diffusion

$$\partial_t u_i - d_i \Delta u_i = (q_i - p_i) \left(k^+ \Pi_{j=1}^m u_j^{q_j} - k^- \Pi_{j=1}^m u_j^{p_j} \right), \forall i = 1...m,$$

  where d_i are positive diffusion coefficients. A classical conservation property states that $\sum_i m_i p_i = \sum_i m_i q_i$ for some $m_i \in (0, \infty), i = 1...m$. Denoting by f_i the nonlinearity in the i-th equation, this implies $\sum_{i=1}^m m_i\, f_i = 0$, whence the condition (**M**). The quasipositivity **(P)** is satisfied as well.

- **Another quadratic model.** Another model for diffusive calcium dynamics with quadratic terms may also be found in H.G. Othmer [55] (see also more comments on it in [51]):

$$\left\{\begin{array}{l} \partial_t u_1 = d_1 \Delta u_1 + \lambda(\gamma_0 + \gamma_1 u_4)(1 - u_1) - \frac{p_1 u_1^4}{p_2^4 + u_1^4} \\ \partial_t u_2 - d_2 \Delta u_2 = -k_1 u_2 + k_1' u_3 \\ \partial_t u_3 - d_3 \Delta u_3 = -k_1' u_3 - k_2 u_1 u_3 + k_1 u_2 + k_2' u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = k_2 u_1 u_3 + k_3' u_5 - k_2' u_4 - k_3 u_1 u_4 \\ \partial_t u_5 - d_5 \Delta u_5 = k_3 u_1 u_4 - k_3' u_5. \end{array}\right. \tag{12}$$

- **Electrodynamics.** The following 6×6 system for the electro-deposition of nickel-iron alloy is studied for instance in [1]: it offers a similar semilinear structure, but now coupled with extra terms: $\forall i = 1...5$

$$\left\{\begin{array}{l} \partial_t w_i - d_i (w_i)_{xx} + b(x)(w_i)_x - [w_i \Phi_x]_x = S_i(w) \\ S_1 = S_2 = 0,\ S_3(w) = S_4(w) = -S_5(w) \\ -[\Phi]_{xx} = \sum_{i=1}^{5} z_i w_i, \quad z_i \in \mathbb{R}, \ +\text{bdy cond.} \end{array}\right. \tag{13}$$

  Here the functions S_i are nonlinearities which preserve nonnegativity and their structure implies **(M)**. Two extra terms are present: a convection term $b(x)(w_i)_x$: if b is a regular enough function, then, for the question of global existence, this perturbation may essentially be treated as if $b \equiv 0$ and may be 'included' in the linear p.d.e. part. The second convection terms $[w_i \Phi_x]_x$ is different since the regularity of the transport coefficient Φ_x depends itself on estimates on the $w_i's$. Therefore, it is important to obtain a priori estimates on w_i from the only **(P)+(M)** structure.

  Let us also refer to [18] for the study of a model with similar features which is used in cardiac electrophysiology.

- **Diffusion of pollutants in atmosphere.** Another interesting example comes from the modeling of pollutants transfer in atmosphere (here $N = 3$): this system of 20 equations is studied in [29] and, more recently in [62] (we refer to these two papers for more references):

$$\left\{\begin{array}{l} \partial_t \phi_i = d_i \, \partial^2_{zz} \phi_i + \omega \cdot \nabla \phi + f_i(\phi) + g_i, \ \forall i = 1...20, \\ + \text{Bdy and initial conditions .} \end{array}\right. \tag{14}$$

  Here the nonlinearities f_i are given by

$$\left\{\begin{array}{rcl} f_1(\phi) & = & -k_1\phi_1 + k_{22}\phi_{19} + k_{25}\phi_{20} + k_{11}\phi_{13} + k_9\phi_{11}\phi_2 + k_3\phi_5\phi_2 \\ & + & k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 - k_{14}\phi_1\phi_6 + k_{12}\phi_{10}\phi_2 - k_{10}\phi_{11}\phi_1 - k_{24}\phi_{19}\phi_1, \\ f_2(\phi) & = & k_1\phi_1 + k_{21}\phi_{19} - k_9\phi_{11}\phi_2 - k_3\phi_5\phi_2 - k_2\phi_2\phi_4 - k_{12}\phi_{10}\phi_2 \\ f_3(\phi) & = & k_1\phi_1 + k_{17}\phi_4 + k_{19}\phi_{16} + k_{22}\phi_{19} - k_{15}\phi_3 \\ f_4(\phi) & = & -k_{17}\phi_4 + k_{15}\phi_3 - k_{16}\phi_4 - k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 \\ f_5(\phi) & = & 2k_4\phi_7 + k_7\phi_9 + k_{13}\phi_{14} + k_6\phi_7\phi_6 - k_3\phi_5\phi_2 + k_{20}\phi_{17}\phi_6 \\ f_6(\phi) & = & 2k_{18}\phi_{16} - k_8\phi_9\phi_6 - k_6\phi_7\phi_6 + k_3\phi_5\phi_2 - k_{20}\phi_{17}\phi_6 - k_{14}\phi_1\phi_6 \\ f_7(\phi) & = & -k_4\phi_7 - k_5\phi_7 + k_{13}\phi_{14} - k_6\phi_7\phi_6 \\ f_8(\phi) & = & k_4\phi_7 + k_5\phi_7 + k_7\phi_9 + k_6\phi_7\phi_6 \\ f_9(\phi) & = & -k_7\phi_9 - k_8\phi_9\phi_6 \\ f_{10}(\phi) & = & k_7\phi_9 + k_9\phi_{11}\phi_2 - k_{12}\phi_{10}\phi_2 \\ f_{11}(\phi) & = & k_{11}\phi_{13} - k_9\phi_{11}\phi_2 + k_8\phi_9\phi_6 - k_{10}\phi_{11}\phi_1 \\ f_{12}(\phi) & = & k_9\phi_{11}\phi_2 \\ f_{13}(\phi) & = & -k_{11}\phi_{13} + k_{10}\phi_{11}\phi_1 \\ f_{14}(\phi) & = & -k_{13}\phi_{14} + k_{12}\phi_{10}\phi_2 \\ f_{15}(\phi) & = & k_{14}\phi_1\phi_6 \\ f_{16}(\phi) & = & -k_{19}\phi_{16} - k_{18}\phi_{16} + k_{16}\phi_4 \\ f_{17}(\phi) & = & -k_{20}\phi_{17}\phi_6 \\ f_{18}(\phi) & = & k_{20}\phi_{17}\phi_6 \\ f_{19}(\phi) & = & -k_{21}\phi_{19} - k_{22}\phi_{19} + k_{25}\phi_{20} + k_{23}\phi_1\phi_4 - k_{24}\phi_{19}\phi_1 \\ f_{20}(\phi) & = & -k_{25}\phi_{20} + k_{24}\phi_{19}\phi_1. \end{array}\right. \tag{15}$$

where the k_i's are positive real numbers. These nonlinearities may seem complicated, but they are quadratic and, obviously satisfy **(P)+(M')**. The main new point in this system is that diffusion occurs only in the vertical direction. As a consequence, many of the tools, which are based on the regularizing effects of the diffusion, need to be revisited. Even the transport term may cause new difficulties due to the lack of diffusion in two directions. However, the general methods described in the next sections may be used to obtain some global existence results for this degenerate system.

We could go on with more and more examples arising in applications with **(P)+(M)**. We refer for instance to the books [33, 65, 45, 26, 27, 54, 56].

3 A starting point: local existence of classical solutions for System (1),(6).

Lemma 1 *Assume $u_0 \in L^\infty(\Omega)^m$. Then, there exist $T > 0$ and a unique* classical solution *to System (1),(6) on $[0,T]$. If T^* denotes the supremum of these T's, then*

$$\left[\sup_{t\in[0,T^*)}\left(\sum_{1\le i\le m}\|u_i(t)\|_{L^\infty(\Omega)}\right)<+\infty\right]\Rightarrow[T^*=+\infty]. \tag{16}$$

If, moreover, the nonlinearity $(f_i)_{1\le i\le m}$ is **quasi-positive** *(see condition* **(P)** *in (2)), then*

$$[\forall i=1,...,m,\ u_{i0}\ge 0]\Rightarrow[\forall i=1,...,m,\ \forall t\in[0,T^*),\ u_i(t)\ge 0]\,.$$

By "classical solutions", we mean that the $\partial_t u_i, \Delta u_i$ are continuous and the equations (1),(6) are satisfied in the classical sense, including boundary conditions.

As for systems of ODE's, this local existence result is proved via a fixed-point argument; the difference here is that we need to work in infinite dimensional functional spaces. More precisely, one chooses an adequate ball $\mathcal{B}$ of $C(Q_T, \mathbb{R}^m)$ equipped with the $L^\infty(Q_T)$-norm and we consider the mapping $\hat{u} \in \mathcal{B} \to u = (u_1, ..., u_m) \in \mathcal{B}$, solution of

$$\partial_t u_i - d_i\Delta u_i = f_i(\hat{u})\ on\ Q_T,\ \partial_\nu u_i = 0\ on\ \Sigma_T,\ u(0) = u^0.$$

By the locally Lipschitz property of f, this is a strict contraction if T is small enough. Whence the existence of a solution on $[0,T]$ and on a maximal interval $[0,T^*)$. The characterization (16) follows from the fact that T^* depends only on the L^∞-norm of the initial data.

Finally, the solution obtained in this way is regular thanks to the strong regularizing properties of the heat operator. For the rest of the present notes, we will remember that *any "weak" solution with values in $L^\infty(Q_T)$ is actually regular enough to be a "classical solution"*, and it is even C^∞ in the interior of Q_T if f is itself C^∞.

According to (16), in order to prove global existence of classical solutions for system (1),(6), it is sufficient to prove that, if $T^* < +\infty$, then the solutions u_i are uniformly bounded on $[0, T^*)$. *Thus, a priori L^∞-bounds imply global existence.*

It is actually the case in the system (1),(6), when **(P)+(M)** holds, if all the diffusion coefficients are equal : $\forall i = 1, ..., m, d_i = d$. Indeed, in this case

$$\partial_t \left(\sum_i a_i u_i \right) - d\Delta \left(\sum_i a_i u_i \right) \le 0.$$

By maximum principle

$$\forall t \in [0, T^*),\ \| \sum_i a_i u_i(t) \|_{L^\infty(\Omega)} \le \| \sum_i a_i u_i^0 \|_{L^\infty(\Omega)}.$$

Together with positivity, this implies a uniform $L^\infty(\Omega)$ bound on each $u_i(t)$, whence $T^* + \infty$.

The situation is quite more complicated if the diffusion coefficients are different from each other and this will be analyzed all along these notes.

4 A first global existence result.

Let us first state a typical result for 2×2 systems.

Proposition 1 *Let $m = 2$ in System (1),(6) where we assume* **(P)+(M)**. *We assume also that the growth of f at ∞ is at most polynomial and $f_1 \le 0$. Then, the system (1),(6) has a global classical solution.*

The idea is as follows: since $f_1 \le 0$, by maximum principle

$$\|u_1(t)\|_{L^\infty(\Omega)} \le \|u_1^0\|_{L^\infty(\Omega)}.$$

Next, it follows from

$$a_2 \left[\partial_t u_2 - d_2 \Delta u_2\right] \le -a_1 \left[\partial_t u_1 - d_1 \Delta u_1\right],$$

that, for all $p \in (1, \infty)$ and all $T > 0$, there exists $C \in (0, \infty)$ such that

$$\|u_2\|_{L^p(Q_{T^*})} \le C \, \|u_1\|_{L^p(Q_{T^*})}.$$

This follows by duality from the L^p-regularity theory for the heat operator. Then, u_2 is bounded in $L^p(Q_{T^*})$ for all $p < +\infty$. So is $f_2(u)$ thanks to the assumption of at most polynomial growth. But, as soon as $p > (N + 1)/2$, this implies that u_2 is in $L^\infty(Q_{T^*})$, whence global existence.

This approach may be generalized to $m \times m$ systems as follows.

Theorem 1 *Assume the growth of f at ∞ is at most polynomial and that f satisfies the quasipositivity* **(P)**. *Assume moreover that there exist $\mathbf{b} \in \mathbb{R}^m$ and a lower triangular invertible $m \times m$ matrix P with nonnegative entries such that*

$$\forall r \in [0, \infty)^m,\ Pf(r) \le [1 + \sum_{1 \le i \le m} r_i] \, \mathbf{b} \tag{17}$$

where the usual order in $\mathbb{R}^m$ is used. Then, the system (1),(6) has a global classical solution.

This L^p approach has been first introduced in [36] for 2×2 systems and then extended in several places to $m \times m$ *triangular* systems in the sense of (17), (see [52, 53, 28] and the references herein; a proof can be found in the survey [59]). Note the polynomial growth in the assumptions: this can be weakened a little bit, but not much: see Section 11 of open problems.

For some more specific nonlinearities f, there are also "direct" approaches using clever Lyapunov functions to get estimates of the $L^p(Q_T)$-norms; they may even be extended to reach some exponential growth (see [50, 32, 7, 11, 39, 41]).

5 The structure (P)+(M) does not keep from blowing up

Theorem 1 may seem unsatisfactory, since, even for a 2×2 polynomial system satisfying **(P)+(M)**, it requires a strong extra assumption to provide global classical solutions, namely $f_1 \leq 0$ for 2×2 systems or, more generally (17). This leaves for instance open the question of global existence in apparently "simple" systems like (18), (20) below:

$$\left\{ \begin{array}{l} \partial_t u - d_1 \Delta u = \lambda u^p v^q - uv^\beta (= f(u,v)\ ) \\ \partial_t v - d_2 \Delta v = -u^p v^q + uv^\beta (= g(u,v)\ ). \end{array} \right. \tag{18}$$

Note that here

$$f(u,v) + g(u,v) = (\lambda - 1) u^p v^q \leq 0 \text{ if } \lambda \in [0,1]. \tag{19}$$

But, except for good values of the exponents, one cannot conclude in general global existence of classical solutions: none of the equations is "good" and none of the u or v is a priori known to be bounded. The situation is the same for systems like

$$\left\{ \begin{array}{l} \partial_t u - d_1 \Delta u = -c(t,x) u^\alpha v^\beta \\ \partial_t v - d_2 \Delta v = c(t,x) u^\alpha v^\beta. \end{array} \right. \tag{20}$$

Here, if $\forall (t,x), c(t,x) \geq 0$ or $\forall (t,x), c(t,x) \leq 0$, then we may apply Proposition 1 (the dependence of f, g in t, x does not matter). But, the problem is open otherwise: blow up could occur at places where c changes sign.

It turns out that $L^\infty(\Omega)$*-blow up may occur in finite time* for polynomial 2×2 systems satisfying **(P)+(M)** as proved in [60, 61] where the two following theorems may be found (B denotes the open unit ball in $\mathbb{R}^N$ and $Q_T = (0,T) \times B$):

Theorem 2 *One can find C^∞ functions f, g, with polynomial growth and satisfying* **(P)+(M)***, together with $d_1, d_2 > 0, u_0, v_0 \in C^\infty(\overline{B}), \beta_1, \beta_2 \in C^\infty([0,T])$ and u, v nonnegative classical solutions on $(0,T)$ of*

$$\left\{ \begin{array}{l} \partial_t u - d_1 \Delta u = f(u,v) \\ \partial_t v - d_2 \Delta v = g(u,v) \\ u(0,\cdot) = u_0(\cdot) \geq 0,\ v(0,\cdot) = v_0(\cdot) \geq 0, \\ u = \beta_1, \quad v = \beta_2 \text{ on } (0,T) \times \partial\Omega \end{array} \right. \tag{21}$$

with $T < +\infty$ and

$$\lim_{t \to T} \|u(t)\|_{L^\infty(\Omega)} = \lim_{t \to T} \|v(t)\|_{L^\infty(\Omega)} = +\infty.$$

Theorem 3 *One can find $p, q > 1, d_1, d_2 \in (0, \infty), u_0, v_0 \in C^\infty(\overline{B}), \beta_1, \beta_2 \in C^\infty([0,T]), c_1, c_2 \in C^k(\overline{Q}_T)$ with $k \geq 0$, $c_1(t,x) + c_2(t,x) \leq 0$ and u, v nonnegative classical solutions on $(0,T)$ of*

$$\begin{cases} \partial_t u - d_1 \Delta u = c_1(t,x) u^p v^q \\ \partial_t v - d_2 \Delta v = c_2(t,x) u^p v^q \\ u(0,\cdot) = u_0(\cdot) \geq 0, \ v(0,\cdot) = v_0(\cdot) \geq 0, \\ u = \beta_1, \ \ v = \beta_2 \ on \ (0,T) \times \partial\Omega \end{cases} \tag{22}$$

with $T < +\infty$ and

$$\lim_{t \to T} \|u(t)\|_{L^\infty(\Omega)} = \lim_{t \to T} \|v(t)\|_{L^\infty(\Omega)} = +\infty.$$

Remark 1 Examples of Theorems 2 and 3 are actually more surprising than expected since, not only we have $f + g \leq 0, c_1 + c_2 \leq 0$, but we even have a second independent relation

$$\exists \lambda \in (0,1); \ f + \lambda g \leq 0, \ c_1 + \lambda c_2 \leq 0.$$

We will see in next Section that this richer structure allows global existence of weak solutions. In other words, the solutions constructed in the two above theorems blow up at T^*, but continue to live "in a weak sense".

6 Existence of global weak supersolutions

Approximate problem: We consider an approximation of System (1),(6), namely a classical solution $u^n = (u_1^n, ..., u_m^n)$ of

$$\begin{cases} \forall i = 1, ..., m, \\ \partial_t u_i^n - d_i \Delta u_i^n = f_i^n(u_1^n, ..., u_m^n) \ on \ Q_\infty, \\ \partial_\nu u_i^n = 0 \ on \ \Sigma_\infty, \\ u_i^n(0,\cdot) = u_{i0}^n, \end{cases} \tag{23}$$

where the f_i^n are essentially "truncations" of the f_i's. More precisely, we assume that the f_i^n are regular, satisfy **(P)** (not necessarily **(M)** though), that $|f_i^n|$ is uniformly bounded for each n, and that, for all $M > 0$, ϵ_M^n tends to zero in $L^1(Q_T)$ and *a.e.* where

$$\epsilon_M^n(t,x) = \sup_{0 \leq |r| \leq M, 1 \leq i \leq m} |f_i^n(t,x,r) - f_i(t,x,r)|. \tag{24}$$

Since f^n is uniformly bounded for fixed n, there exists a global classical solution u^n to (23) by Theorem 1.

We denote $\mathcal{D} = \{\psi \in \mathcal{C}^\infty(\overline{Q}_T); \psi \geq 0, \ \psi(\cdot, T) = 0\}$.

Theorem 4 *[58] Let $u^n = (u_1^n, ..., u_m^n)$ be a nonnegative solution to the approximate system (23) satisfying*

$$\forall\, i = 1, ..., m, \ \sup_{n \geq 1} \int_{Q_T} |f_i^n(u^n)| < +\infty. \tag{25}$$

Assume that, for $i = 1, ..., m$, u_{i0}^n tends to u_{i0} in $L^1(\Omega)$. Then, up to a subsequence, u^n converges in $L^1(Q_T)$ and a.e. to a super-solution *of System (1),(6) which means*

$$\begin{cases} \forall i = 1, ..., m, \ f_i(u) \in L^1(Q_T), \nabla u_i \in L^1(Q_T), \\ for all\, \psi \in \mathcal{D}, - \int_\Omega \psi(0) u_{i0} + \int_{Q_T} [-\psi_t u_i + d_i \nabla\psi \nabla u_i] \geq \int_{Q_T} \psi f_i(u). \end{cases} \tag{26}$$

If one now reintroduces also the property **(M)**, we get global existence of "true" solutions.

Theorem 5 *Let us consider the System (1),(6) with* $u_0 \in L^1(\Omega)^m, u_0 \geq 0$. *Assume that the structure* **(P)+(M)** *holds together with the following a priori estimate on the solution* u^n *just defined*

$$\forall\, i = 1, ..., m,\ \sup_{n \geq 1} \int_{Q_T} |f_i^n(u^n)| < +\infty. \tag{27}$$

Then, system (1),(6) has a weak solution on $(0,T)$ *(i.e. equality holds in (26)).*

Corollary 1 *Assume* $m = 2$ *in System (1),(6),* **(P)** *and*

$$\forall r, s \in [0, \infty), (f_1 + f_2)(r, s) \leq C[1 + r + s], (f_1 + \lambda f_2)(r, s) \leq C[1 + r + s] \tag{28}$$

for some $\lambda \in [0, +\infty) \setminus \{1\}$. *Then, there exists a global weak solution to System (1),(6).*

The two conditions in (28) imply an a priori $L^1(Q_T)$-bound on the nonlinear terms so that we can apply Theorem 5. This corollary applies to System (18) with $\lambda \in [0, 1)$ and to systems with non polynomial growth like

$$\begin{cases} \partial_t u - d_1 \Delta u = -u^\alpha\, e^{v^2} \\ \partial_t v - d_2 \Delta v = u^\alpha\, e^{v^2}. \end{cases}$$

Corollary 1 applies as well to the two counterexamples mentioned in Theorems 2 and 3. Thus, they are examples where L^∞-blow up does occur while global existence of weak solutions holds.

We can extend Corollary 1 to the $m \times m$ system (1),(6):

Corollary 2 *Assume there exists an invertible* $m \times m$ *matrix* P *with nonnegative entries and* $\mathbf{b} \in \mathbb{R}^m$ *such that*

$$\forall r \in [0, \infty)^m, \quad P\, f(r) \leq \mathbf{b}\, [1 + \sum_i r_i].$$

Then, System (1),(6) has a global weak solution.

As for $m = 2$ in Corollary 1, we check that the existence of P implies an a priori $L^1(Q_T)$-estimate on the nonlinear terms. Note that the main difference with Theorem 1 is that the matrix P is not any more assumed to be triangular.

The statement can be sometimes weakened by not requiring that all entries of P are nonnegative. But, in any case, they need to be positive for one line at least to ensure a uniform $L^1(\Omega)$-bound on the solution.

A somehow stronger result is stated below in Theorem 6.

7 Global existence of weak solutions for quadratic systems with (P)+(M)

An interesting and very general consequence of Theorem 5 and of next Section is that global existence of weak solutions holds for all systems with the structure **(P)+(M)** and *whose nonlinearities are at most quadratic.* Note that this includes all the following systems mentioned in Section 2: (7),(9),(10),(12) and (14),(15) with nondegenerate diffusion.

Proposition 2 *We assume* **(P)+(M)** *in System (1),(6) and*

$$\forall i = 1, ..., m,\ |f_i(u)| \leq C[1 + \sum_{1 \leq i \leq m} u_i^2], \tag{29}$$

for some $C \geq 0$. *Assume also* $u_0 \in [L^2(\Omega)]^m$. *Then, there exists a global weak solution to System (1),(6).*

The main new idea for the proof of this proposition is that, strangely enough, the structure **(P)+(M)** implies also **an a priori** $L^2(Q_T)$**-estimate** on the solutions, as stated in the next section. If the nonlinearities are at most quadratic, according to (29), they are consequently bounded in $L^1(Q_T)$ and we are in the situation of Theorem 5. Actually, we can even prove the following more general result with only *unilateral quadratic growth* (and which may be compared to Corollary 2 above):

Theorem 6 *In System (1),(6), we assume that* **(P)+(M)** *holds and that there exist an invertible* $m \times m$ *matrix* P *with nonnegative entries and* $\mathbf{b} \in \mathbb{R}^m$ *such that*

$$\forall r \in [0, \infty)^m,\ P\,f(r) \leq \mathbf{b}[1 + \sum_{1 \leq i \leq m} r_i^2]. \tag{30}$$

Assume also $u_0 \in [L^2(\Omega)]^m, u_0 \geq 0$. *Then, there exists a global weak solution to System (1),(6).*

This applies, for instance, to the 3×3 system where

$$f_1 = f_2 = -f_3 = u_2^2 u_3^2 - u_1 u_2. \tag{31}$$

Indeed $f_1 + f_2 + 2f_3 = 0$ and (30) is satisfied as follows:

$$f_1 + f_3 = 0,\ f_2 + f_3 = 0,\ f_3 \leq u_1 u_2 \leq u_1^2 + u_2^2.$$

Similarly, we can treat the systems modeling the chemical reactions (11) when $\sum_i p_i \leq 2$ or $\sum_j q_j \leq 2$.

8 An L2-estimate...and even an L2-compactness

Proposition 3 *Assume* **(P)+(M)** *in System (1),(6). Let* u *be a nonnegative classical solution on* $(0, T)$ *of (1),(6) with* $u_0 \in [L^2(\Omega)]^m, u_0 \geq 0$. *Then, for some* C *depending on the data*

$$\int_{Q_T} \sum_{1 \leq i \leq m} u_i^2(t, x)\, dt\, dx \leq C.$$

Let us explain the idea of this L^2-estimate.

Let $u_i, i = 1...m$ be the solution of System (1),(6). We set $W = \sum_i u_i, Z = \sum_i d_i u_i$. Assume for simplicity that $\sum_i f_i \leq 0$. Then

$$W_t - \Delta Z \leq 0 \ \text{ or } \ W_t - \Delta(A\,W) \leq 0, \tag{32}$$

where we set $A = Z/W$. The point is that, thanks to the nonnegativity of the u_i, we have

$$0 < \underline{d} = \min_i d_i \leq A \leq \max_i d_i = \overline{d} < +\infty.$$

Therefore, the operator $W \to \partial_t W - \Delta(A\,W)$ is parabolic. It is not of divergence form and, moreover, no continuity may a priori be expected on A. But the parabolicity is enough to imply the following estimate

$$\int_{Q_T} W^2 \le C \int_\Omega W(0)^2, \;\; C = C(\underline{d}, \overline{d}, T).$$

By duality, this inequality is equivalent to the dual inequality $\|\phi(0)\|_{L^2(\Omega)} \le C\|\Theta\|_{L^2(Q_T)}$ for the solution ϕ of the dual problem

$$-\left[\phi_t + A\Delta\phi\right] = \Theta \ge 0, \;\; \phi(T) = 0 \;+\; \textit{boundary conditions}. \tag{33}$$

This dual inequality is easily obtained by multiplying the equation (33) by $-\Delta\phi$ which gives, for instance with $\phi = 0$ on Σ_T, and after integration by parts of the first term:

$$\int_{Q_T} -\frac{1}{2}\partial_t|\nabla\phi|^2 + A(\Delta\phi)^2 = \int_{Q_T} -\Theta\Delta\phi. \tag{34}$$

Recalling $0 < \underline{d} \le A \le \overline{d} < +\infty$ and using

$$\int_{Q_T} -\Theta\Delta\phi \le \frac{\underline{d}}{2}\int_{Q_T}(\Delta\phi)^2 + \frac{1}{2\underline{d}}\int_{Q_T}\Theta^2, \tag{35}$$

we obtain

$$\int_\Omega |\nabla\phi(0)|^2 + \underline{d}\int_{Q_T}(\Delta\phi)^2 \le \underline{d}^{-1}\int_{Q_T}\Theta^2. \tag{36}$$

Going back to (33), we also have

$$\|\phi(0)\|_{L^2(\Omega)} = \|\int_0^T \partial_t\phi\|_{L^2(\Omega)} \le T^{1/2}\|\left[\Theta + A\Delta\phi\right]\|_{L^2(Q_T)} \le C\|\Theta\|_{L^2(Q_T)}.$$

Whence the expected estimate on $\phi(0)$. But this computation shows that $\phi(0)$ is not only bounded in $L^2(\Omega)$, but it is even bounded in $H^1(\Omega)$ so that *the mapping* $\Theta \in L^2(Q_T) \to \phi(0) \in L^2(\Omega)$ *is not only continuous but compact* ! By duality, the original operator is also compact. More generally, we have the following *compactness result* where we denote $\mathcal{L} = L^2(\Omega) \times L^2(Q_T)$ (see [59]):

Theorem 7 *Let* $0 < \underline{d} \le \overline{d} < +\infty$*. For* $(W_0, H) \in \mathcal{L}$*, let* $\mathcal{F}_{\underline{d},\overline{d},W_0,H}$ *denote the family of functions* $W \in H^1(Q_T)$ *such that for some* $Z \in H^1(Q_T)$*, for all* $\psi \in \mathcal{C}^\infty(\overline{Q}_T), \psi \ge 0, \psi(T) = 0$

$$\begin{cases} -\int_\Omega \psi(0)W_0 + \int_{Q_T} -\psi_t W + \nabla\psi\nabla Z = \int_{Q_T} H\psi, \\ 0 \le W, Z;\;\; \underline{d} \le Z/W \le \overline{d}, \end{cases} \tag{37}$$

Then, for all bounded $\mathcal{K} \subset \mathcal{L}$*, the family* $\{\mathcal{F}_{\underline{d},\overline{d},W_0,H}; (W_0, H) \in \mathcal{K}\}$ *is relatively compact in* $L^2(Q_T)$*.*

The L^2-estimate of Proposition 3 is very robust and may be generalized to quite more general diffusion operators (see e.g. [25, 62, 12]), and may even allow some degeneracy in the diffusion coefficients (see e.g. [25]).

A further improvement (see [35, 23]): it may be proved that there exists $\eta > 0$ such that, under the assumptions of Proposition 3 and with $u_0 \in L^{2+\eta}(\Omega)$, the solution is bounded in $L^{2+\eta}(Q_T)$. This implies compactness of the solution in $L^{2+\eta'}(Q_T)$ for all $\eta' < \eta$.

9 Quasi-steady-state approximation for a system with fast intermediate

The L^2-estimates and compactness of the previous section are also the main ingredient to prove the convergence of the system modeling the reaction

$$A + B \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C \underset{k_4}{\overset{k_3}{\rightleftharpoons}} P + Q$$

to the limit system

$$A + B \underset{k_1'}{\overset{k_2'}{\rightleftharpoons}} P + Q$$

when the decay of C into the products P and Q, or back to the educts A and B, is extremely fast, which means that $k_2 + k_3$ is very large (see [13, 15, 16]).

Here the intermediate species C is viewed as a transition complex which is highly unstable, i.e. the decay of C into the products P and Q or back to the educts A and B is extremely fast. According to the mass action law and the Fick's law, and assuming that the whole system is isolated, we obtain the following set of reaction-diffusion equations:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \partial_t a - d_a\Delta a = -k_1 a\, b + k_2 c \\ \partial_t b - d_b\Delta b = -k_1 a\, b + k_2 c \\ \partial_t c - d_c\Delta c = k_1 a\, b - k_2 c - k_3 c + k_4 pq \\ \partial_t p - d_p\Delta p = -k_4 pq + k_3 c \\ \partial_t q - d_q\Delta q = -k_4 pq + k_3 c \end{array}\right\} \text{ on } (0,+\infty)\times\Omega, \\ \partial_n a = \partial_n b = \partial_n c = \partial_n p = \partial_n q = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ a(0,\cdot) = a_0, b(0,\cdot) = b_0, c(0,\cdot) = c_0, p(0,\cdot) = p_0, q(0,\cdot) = q_0. \end{array}\right. \tag{38}$$

Here a, b, c, p, q denote the molar concentrations of the reacting species, the d_j ($j = a, b, c, p, q$) are positive diffusion coefficients, k_i ($i = 1 \ldots 4$) are positive rate constants and a_0, b_0, c_0, p_0, q_0 are the initial concentrations which we assume to be nonnegative and bounded. As mentioned above, we are mainly interested in what happens to the solutions of this system when the life-time of the complex C is very short, or, in other words, when the rate constants k_2, k_3 tend to $+\infty$.

This question has recently been studied in [13, 15].

Global existence in time of classical nonnegative solutions to (38) is a nice application of Theorem 1.

We assume

$$d_i,\ k_i \in (0,\infty), \qquad a_0, b_0, c_0, p_0, q_0 \in L^\infty(\Omega;\mathbb{R}_+). \tag{39}$$

We study the limit of instantaneous decay of the intermediate when

$$k := k_2 + k_3 \to +\infty, \quad \frac{k_2}{k_2+k_3} \to \alpha \in [0,1] \quad \frac{k_3}{k_2+k_3} \to \alpha' := 1-\alpha. \tag{40}$$

The following result is proved in [15]:

Theorem 8 *Assume (39) and (40). Then, up to a subsequence, the solution $U_k = (a_k, b_k, c_k, p_k, q_k)$ of (38) converges in $L^2(\Omega)^5$ to $U = (a, b, 0, p, q)$, where*

(a, b, p, q) *is a* **weak solution** *on* $[0, +\infty)$ *of*

$$\begin{cases} \left.\begin{aligned} \partial_t a - d_a \Delta a &= -\alpha' k_1 a\, b + \alpha k_4 p\, q \\ \partial_t b - d_b \Delta b &= -\alpha' k_1 a\, b + \alpha k_4 p\, q \\ \partial_t p - d_p \Delta p &= \alpha' k_1 a\, b - \alpha k_4 p\, q \\ \partial_t q - d_q \Delta q &= \alpha' k_1 a\, b - \alpha k_4 p\, q, \end{aligned}\right\} \text{ on } [0, +\infty) \times \Omega, \\ \partial_n a = \partial_n b = \partial_n p = \partial_n q = 0 \quad \text{on} \quad (0, +\infty) \times \partial\Omega, \\ a(0, \cdot) = a_0 + \alpha c_0, b(0, \cdot) = b_0 + \alpha c_0, p(0, \cdot) = p_0 + \alpha' c_0, q(0, \cdot) = q_0 + \alpha' c_0. \end{cases}$$

Note that the original initial value for finite k_2, k_3 is in the limit projected onto the manifold $\{c = 0\}$ in such a manner that the quantities $a - b$, $p - q$ and $a + b + 2c + p + q$ are conserved. The constants in the limit system are given by $k_1' = \alpha' k_1, k_2' = \alpha k_4$.

10 Instantaneous limit for a fast reversible reaction

Another nice application of the L^2-estimates of Section 8 is for the identification of the limit as $k \to +\infty$ for the following reaction-diffusion system

$$(R^K) \begin{cases} \left.\begin{aligned} \partial_t c_1 - d_1 \Delta c_1 &= -k(c_1 c_2 - \kappa c_3) \\ \partial_t c_2 - d_2 \Delta c_2 &= -k(c_1 c_2 - \kappa c_3) \\ \partial_t c_3 - d_3 \Delta c_3 &= +k(c_1 c_2 - \kappa c_3) \end{aligned}\right\} \text{ on } (0, +\infty) \times \Omega, \\ \partial_\nu c_1 = \partial_\nu c_2 = \partial_\nu c_3 = 0 \text{ on } (0, +\infty) \times \partial\Omega, \\ c_1(0, \cdot) = c_1^0;\ c_2(0, \cdot) = c_2^0;\ c_3^0(0, \cdot) = c_3^0. \end{cases} \tag{41}$$

where $\kappa, d_i \in (0, \infty)$ and the initial data c_i^0 are nonnegative. We denote $K = (k, \kappa)$. This system is a classical model for the chemical reaction

$$C_1 + C_2 \underset{k\kappa}{\overset{k}{\rightleftharpoons}} C_3\ ,$$

when the reaction takes place in an isolated domain represented by Ω where diffusion of the species C_i occurs : here $c_i(t, x)$ represents the concentration of the species C_i at time t and at $x \in \Omega$. We assume that the reaction follows the law of mass action and that a linear Fickian diffusion holds, with diffusion constants $d_1,\ d_2,\ d_3 > 0$. We impose no-flux conditions on the boundary. This modelization leads to the system (R^K).

Again, existence of a global classical solution to (R^K) follows from Theorem 1. The following is proved in [17]

Theorem 9 *Let* $K^n := (k_n, \kappa_n) \overset{n \to +\infty}{\longrightarrow} (+\infty, \kappa^\infty)$ *with* $\kappa^\infty > 0$ *and let* $c^n = (c_1^n, c_2^n, c_3^n)$ *be the solution of* (R^{K_n}) *on* $[0, \infty)$ *with initial data* $c^0 = (c_1^0, c_2^0, c_3^0) \in L^\infty(\Omega, \mathbb{R}_+^3)$. *Then, up to a subsequence,* $(c^n)_{n \in N}$ *converges for all* $T > 0$ *in* $L^2(Q_T)^3$ *to a limit* $c = (c_1, c_2, c_3)$, *solution of the following for all* $T > 0$:

$$\begin{cases} \forall i = 1, 2, 3,\ c_i \in L^2(Q_T),\ \nabla c_i \in L^{\frac{4}{3}}(Q_T)^N, \quad c \geq 0, \quad c_1 c_2 = \kappa^\infty c_3, \\ \forall \psi \in C^\infty(\overline{Q_T}) \text{ such that } \psi(T) = 0, \\ \qquad \begin{cases} \int_\Omega \psi(0)(c_1^0 + c_3^0) + \int_{Q_T} -\psi_t(c_1 + c_3) + \nabla\psi . \nabla(d_1 c_1 + d_3 c_3) = 0, \\ \int_\Omega \psi(0)(c_2^0 + c_3^0) + \int_{Q_T} -\psi_t(c_2 + c_3) + \nabla\psi . \nabla(d_2 c_2 + d_3 c_3) = 0. \end{cases} \end{cases} \tag{42}$$

Here, it follows from the L^2-compactness result of Theorem 7 that $c_1^n + c_3^n$ and $c_2^n + c_3^n$ are relatively compact in $L^2(Q_T)$. An extra argument is needed to prove that, separately, each c_i^n is relatively compact. The following entropy inequality, classically valid for reversible reactions, is exploited to obtain this extra information and also to prove that $c_1c_2 = \kappa^\infty c_3$ at the limit. We define the nonnegative functions

$$W_i^n = c_i^n \log\left(\frac{c_i^n}{c_i^{n*}}\right) - (c_i^n - c_i^{n*}), \quad W^n = \sum_{i=1}^3 W_i^n, \quad Z^n = \sum_{i=1}^3 d_i W_i^n,$$

where $c_1^{n*}, c_2^{n*}, c_3^{n*}$ are positive numbers such that $c_1^{n*}c_2^{n*} = \kappa_n c_3^{n*}$. Then

$$k_n \int_{Q_T} (c_1^n c_2^n - \kappa c_3^n)(\log(c_1^n c_2^n) - \log(\kappa_n c_3^n)) + \sum_{i=1}^3 d_i \int_{Q_T} \frac{|\nabla c_i^n|^2}{c_i^n} \tag{43}$$

$$= \int_\Omega W^n(0,\cdot) - \int_\Omega W^n(T,\cdot) \leq \int_\Omega W^n(0,\cdot). \tag{44}$$

The limit system (42) is a of a new kind. Note that we have

$$\begin{cases} \partial_t(c_1 + c_3) - \Delta(d_1c_1 + d_3c_3) \ \textit{in}\ Q_\infty, \\ \partial_t(c_2 + c_3) - \Delta(d_2c_2 + d_3c_3) \ \textit{in}\ Q_\infty, \end{cases}$$

together with the algebraic relation $c_1c_2 = \kappa^\infty c_3$. If all the d_i are equal, then $c_1 + c_3$ and $c_2 + c_3$ are the solution of a heat equation. Coupling with the algebraic relation $c_1c_2 = \kappa^\infty c_3$, they are uniquely defined. But, if the d_i are different, the system is a "true" *cross-diffusion system.*

We may eliminate c_3 in this system and rewrite it as a 2×2 cross-diffusion system by introducing the new unknown functions:

$$x(c_1, c_2) := c_1 + c_3 = c_1 + c_1c_2; \quad y(c_1, c_2) := c_2 + c_3 = c_2 + c_1c_2\ . \tag{45}$$

Then, the function ψ defined by

$$\begin{array}{rccc} \psi: & (0,+\infty)^2 & \longrightarrow & (0,+\infty)^2 \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} \psi_1(x,y) \\ \psi_2(x,y) \end{pmatrix} := \begin{pmatrix} d_1c_1 + c_1c_2 \\ d_2c_2 + c_1c_2 \end{pmatrix}(x,y), \end{array}$$

is a C^∞-diffeomorphism from $(0,\infty)^2$ onto itself. The limit problem (42) can then be (at least formally) rewritten

$$\begin{cases} \partial_t x - \Delta\psi_1(x,y) & = 0 & \text{in } Q_T \\ \partial_t y - \Delta\psi_2(x,y) & = 0 & \text{in } Q_T \\ \partial_\nu\left(\psi_1(x,y)\right) = \partial_\nu\left(\psi_2(x,y)\right) & = 0 & \text{on } \Sigma_T \\ x(0,\cdot) = x^0;\ y(0,\cdot) & = y^0 & \text{in } \Omega\ . \end{cases} \tag{46}$$

The new system is a nonlinear cross-diffusion system. The spectrum of $D\psi(x,y)$ is in $\{z \in \mathbb{C} : \text{Re } z > 0\}$ for all $(x,y) \in (0,+\infty)^2$. We may therefore apply Amann's local existence theory [5, 6] and state:

Proposition 4 *Let $p > N$ and $c^0 \in W^{1,p}(\Omega, \mathbb{R}_+^2)$. Then, there exists a unique classical and nonnegative solution in $C([0,T^*) \times \overline{\Omega}) \cap C^\infty((0,T^*) \times \Omega)$ for the problem (46) on a maximal time interval $[0,T^*)$.*

Global existence would follow from a uniform bound in $W^{1,p}(\Omega)$ on $[0, T^*)$, but this question is still open. However, the existence result of Theorem 9 does provide a *global* weak solution to the system (46). We do not know in general if it coincides with the regular one obtained in Proposition 4, even on the interval $[0, T^*)$. However, it does in some cases and the following is proved in [17].

Theorem 10 *Let $D = \{(d_1, d_2) \in \mathbb{R}^2_+ : (\frac{d_1}{d_3} - 1)^2(\frac{d_2}{d_3} - 1)^2 < 16\frac{d_1 d_2}{d_3^2}\}$. For $(d_1, d_2) \in D$, there exists a unique solution on $(0, \infty)$ to (42).*

11 Open problems

- **Problem 1.** A most challenging problem is to understand whether global solutions exist for a system, even 2×2, for which the structure **(P)+(M)** holds, but for which there is no obvious a priori $L^1(Q_T)$-bound on the nonlinearities. Let us indicate two simple examples of this situation:

$$\begin{cases} \partial_t u - d_1 \Delta u = u^3 v^2 - u^2 v^3 & \textit{on } Q_T, \\ \partial_t v - d_2 \Delta v = u^2 v^3 - u^3 v^2 & \textit{on } Q_T, \\ + \text{ initial and boundary conditions.} \end{cases}$$

$$\begin{cases} \partial_t u - d_1 \Delta u = -c(t,x) u^2 v^2 & \textit{on } Q_T, \\ \partial_t v - d_2 \Delta v = c(t,x) u^2 v^2 & \textit{on } Q_T, \\ + \text{ initial and boundary conditions,} \end{cases}$$

  when $c(t, x)$ is not of constant sign. Here, it could well be that the nonlinearities are not bounded in $L^1(Q_T)$ in general. Therefore, even the definition of weak solution is not obvious in this case. We feel that one should truncate the nonlinearities and introduce some sort of *renormalized solution*: but, this is not available yet for this kind of semilinear systems. A direction could be to better understand the proof of Theorem 4: it heavily involves truncation operators and the result looks like providing some sort of maximum principle for the system.

  The question is open also for the chemical systems (11) of Section 2 when they are not quadratic, or not bounded above by a quadratic polynomial. See a partial interesting result in [44]

- **Problem 2.** What about uniqueness of weak solutions? Working with uniformly bounded solutions is satisfactory, since they come with a uniqueness property and the problem is well posed in this class. Unfortunately, as shown in Section 5, even for regular initial data, the solution may leave $L^\infty(\Omega)$ so that we must give up with this confortable framework. But we do not know any more what happens with uniqueness. The question is certainly delicate since it is known that there is not uniqueness of weak solutions even for the simple *equation*:

$$u_t - \Delta u = u^3, \; u(0) = u_0 \geq 0, u = 0 \textit{ on } \partial\Omega,$$

  and even for $\mathcal{C}^\infty$ initial data (see [8, 31]). The right question to ask is probably: is there a way to select the "good" solution among the possible several ones? But, is there a "good" solution? In the previous example, the smallest one, which is uniformly bounded, seems to be the "good" one. In a system without maximum principle, it is not clear. This question of uniqueness could be generalized to some kind of adequate renormalized solution ('adequate' actually requires that uniqueness holds).

- **Problem 3.** Once we have proved global existence of weak solutions for a system, it remains interesting to decide whether it is uniformy bounded (and therefore classical) or not. Let us take for instance the quadratic system

$$\begin{cases} \partial_t a - d_1 \Delta a = -a\,b + c\,d \\ \partial_t b - d_2 \Delta b = -a\,b + c\,d \\ \partial_t c - d_3 \Delta c = a\,b - c\,d \\ \partial_t d - d_4 \Delta d = a\,b - c\,d, \end{cases} \tag{47}$$

for which global existence of weak solutions holds in any dimension. It is proved that in dimensions $N = 1, 2$ and for bounded initial data, the solutions are bounded (see [24, 30]). What happens in higher dimensions? A result in this direction may be found in [30] where it is proved in dimensions $N \geq 3$ that the Hausdorff dimension of the possible set of blow up in Q_T is at most $(N^2 - 4)/N$. But, is blow up indeed possible? Same question for

$$\begin{cases} \partial_t u - d_1 \Delta u = -u^\alpha\, e^{v^2} \\ \partial_t v - d_2 \Delta v = u^\alpha\, e^{v^2}. \end{cases}$$

Note that, as proved in [11], L^∞-bounds hold for the slightly better system

$$f(u, v) = -u^\alpha (1 + v^2) e^{v^2}, \;\; g(u, v) = u^\alpha\, e^{v^2}.$$

- **Problem 4.** What happens for initial data in $L^1(\Omega)$ only? The general existence result of Theorem 5 is stated for L^1-initial data. However, when applied for instance to system (47), it requires that the initial data be in $L^2(\Omega)$. What happens for this system if they are only in $L^1(\Omega)$, or even worse only bounded measures? The same question is of interest for several other systems. It is actually very connected with the global existence question, since we need to extend solutions after an L^∞-blow up at time T^* in situations where merely an L^1-estimate holds on $u(T^*)$. See [14] for some results in this direction.

- **Problem 5.** What happens for degenerate diffusions? We saw that most of the estimates were based on the regularizing effects of the diffusions. We loose them in general when degeneracies appear, and it is the case in many applications of interest: for instance, when nonlinear diffusions occur like in

$$\begin{cases} \partial_t u - \Delta u^m = f(u, v) & \text{on } Q_T \\ \partial_t v - \Delta v^p = g(u, v) & \text{on } Q_T. \end{cases}$$

See some results in this direction in [43]. Even the case of linear degenerate diffusions is of interest; see some examples in [25] or also system (14-15) (see [29, 62] for some contributions).

- **Problem 6.** How far is it possible to extend the results recalled in this survey to situations where the nonlinearities depend also on the gradient of the solutions, like they do in several models? A 2×2 model would be of the form

$$\begin{cases} \partial_t u - d_1 \Delta u = f(u, v, \nabla u, \nabla v) \\ \partial_t v - d_2 \Delta v = g(u, v, \nabla u, \nabla v), \end{cases}$$

together with conditions of the kind $f + g \leq 0$. All questions about global existence of classical, or of weak solutions, are of interest. See for instance [2, 19] for some results in this direction and for more references.

- **Problem 7.** How do the known techniques extend to cross-diffusions, namely

$$\begin{cases} \partial_t u - d_1 \Delta u - \nabla \cdot (a_1(u,v)\nabla u + a_2(u,v)\nabla v) = f(u,v) & \textit{on } Q_T \\ \partial_t v - d_2 \Delta v - \nabla \cdot (b_1(u,v)\nabla u + b_2(u,v)\nabla v) = g(u,v) & \textit{on } Q_T? \end{cases}$$

More and more pertinent models require these cross-diffusions. Conditions are required to preserve positivity. Next, global existence remains a natural question (see [41, 21, 22, 12]).

- **Problem 8.** Instead of having an L^1-structure of type **(M)**, $a\,f + b\,g \leq 0$ with $a, b > 0$, there are systems for which a more general Lyapunov structure holds like

$$h_1'(u) f(u,v) + h_2'(v) g(u,v) \leq 0,$$

where $h_1, h_2 : [0, \infty) \to [0, \infty)$ satisfy $\lim_{r\to\infty} h_i(r) = +\infty$. Global existence would still hold for the associated O.D.E. But, what about the P.D.E. system, even for convex h_i? Some results may be deduced directly from the remark that, due to the convexity of the h_i, $h(u) = (h_1(u_1), .., h_m(u_m))$ is a subsolution of a system satisfying **(P)+(M)**.

- **Problem 9.** We dealt in this paper with evolution problems governed by parabolic operators. But, the same type of questions may be asked for elliptic systems. To progress, it may actually be a good idea to address them in this context where the technicality is sometimes easier. For instance, we may look at existence results for

$$\begin{cases} u - \Delta u - \lambda u_{x_1 x_1} = f(u,v) + F \textit{ on } \Omega \\ v - \Delta v = g(u,v) + G \textit{ on } \Omega, \end{cases}$$

where F, G are regular nonnegative given functions on Ω and where f, g satisfy **(P)+(M)**. Here, when $\lambda > 0$ is large, the two diffusion operators are very different from each other, like the two parabolic operators $\partial_t - d_1\Delta$ and $\partial_t - d_2\Delta$ are when d_1/d_2 is away from 1. The difficulties are of the same kind. Some results may be found in [47].

- **Problem 10.** Section 10 leads to interesting uniqueness and regularity questions, for instance:
  - Does the uniqueness stated in Theorem 10 for weak solutions of (42)extend to all values of the d_i?
  - In the range of the d_i considered in Theorem 10, the unique weak solution of (42) coincides with the unique strong solution of (46). But, it may a priori happen that this unique solution loses its regularity at some time: does it happen or not?

We give next a wide (but certainly incomplete) list of references strongly related to these questions.

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