The Kneser Property for reaction-diffusion equations in some unbounded domains

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Setting of the problem

Let $\Omega \subset \mathbb{R}^N$ be a nonempty open set, not necessarily bounded, and suppose that Ω satisfies the Poincaré inequality, i.e., there exists a constant $\lambda_1 > 0$ such that

$$\int_\Omega |u(x)|^2 \, dx \le \lambda_1^{-1} \int_\Omega |\nabla u(x)|^2 \, dx \quad \forall u \in H_0^1(\Omega). \tag{1}$$

Let us consider the following problem for a non-autonomous reaction-diffusion equation with zero Dirichlet boundary condition in Ω,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(x,u) + h(t) \ \text{ in } \ \Omega \times (\tau, +\infty), \\ u = 0 \ \text{ on } \ \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{2}$$

where $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Caratheodory, i.e., $f(\cdot, u)$ is a measurable function for any $u \in \mathbb{R}$ and $f(x, \cdot) \in C(\mathbb{R})$ for almost every $x \in \Omega$, and satisfies that there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$, and $p \ge 2$ and positive functions $C_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$, $C_2(x) \in L^1(\Omega)$ such that

$$|f(x,s)|^{\frac{p}{p-1}} \le \alpha_1 |s|^p + C_1(x) \quad \forall s \in \mathbb{R},\ x \in \Omega, \tag{3}$$

$$f(x,s)s \le -\alpha_2 |s|^p + C_2(x) \quad \forall s \in \mathbb{R},\ x \in \Omega. \tag{4}$$

By $|\cdot|$ we denote the norm in $L^2(\Omega)$, by $\|\cdot\| = |\nabla \cdot|$ the norm in $H_0^1(\Omega)$ and by $\|\cdot\|_*$ the norm in $H^{-1}(\Omega)$.

Existence of solution

The following results can be found in [1].

Definition 1 *A weak solution of (2) is any function $u : (\tau, +\infty) \to L^p(\Omega) \cap H_0^1(\Omega)$, such that $u \in L^p(\tau, T; L^p(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega))$ for all $T > \tau$, and*

$$(u(t), w) + \int_\tau^t (\nabla u(s), w) \, ds = (u_\tau, w) + \int_\tau^t \langle f(x, u(s)) + h(s), w \rangle \, ds \quad \forall t \ge \tau, \tag{5}$$

for all $w \in L^p(\Omega) \cap H_0^1(\Omega)$.

Theorem 2 *Assume that Ω satisfies (1), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and f is Caratheodory and satisfies (3) and (4). Then, for all $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, there exists at least a weak solution u of (2).*

Preliminaries on the theory of pullback attractors

We recall some basic definitions for set-valued non-autonomous dynamical systems (see [3] and [4] for more details). Let $X = (X, d_X)$ be a metric space, and let $\mathcal{P}(X)$ denote the family of all nonempty subsets of X, and let us denote $\mathbb{R}^2_d := \{(t,s) \in \mathbb{R}^2 : t \ge s\}$. Let $\mathcal{D}$ be a class of sets parameterized in time, $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 3 *A multi-valued map $U : \mathbb{R}^2_d \times X \longrightarrow \mathcal{P}(X)$ is called a multi-valued non-autonomous dynamical system (**MNDS**) on X (also named a multi-valued process on X) if*

$$U(s,s,\cdot) = id_X(\cdot) \text{ for all } s \in \mathbb{R},$$

$$U(t,\tau,x) \subset U(t,s,U(s,\tau,x)) \text{ for all } \tau \le s \le t,\ x \in X.$$

*An **MNDS** is said to be strict if the second property is an equality.*

Definition 4 *An MNDS U on X is said to be **upper-semicontinuous** if for all $t \ge \tau$, for any $x_0 \in X$ and for every neighborhood $\mathcal{N}$ in X of the set $U(t,\tau,x_0)$, there exists $\delta > 0$ such that $U(t,\tau,y) \subset \mathcal{N}$ whenever $d_X(x_0, y) < \delta$.*

Definition 5 *We say that a family $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is **pullback $\mathcal{D}$-absorbing** if for every $\widehat{D} \in \mathcal{D}$ and every $t \in \mathbb{R}$, there exists $\tau(t, \widehat{D}) \le t$ such that $U(t,\tau,D(\tau)) \subset B(t)$ for all $\tau \le \tau(t, \widehat{D})$.*

Definition 6 *The MNDS U is **pullback asymptotically compact** with respect to a family $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ if for all $t \in \mathbb{R}$ and every sequence $\tau_n \le t$ tending to $-\infty$, any sequence $y_n \in U(t, \tau_n, B(\tau_n))$ is pre-compact.*

Definition 7 *A family $\widehat{A} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a **global pullback $\mathcal{D}$- attractor** for the MNDS U if it satisfies*

1) *$A(t)$ is compact for any $t \in \mathbb{R}$,*

2) *$\widehat{A}$ is pullback $\mathcal{D}$-attracting, i.e.*

$$\lim_{\tau \to -\infty} dist_X(U(t,\tau,D(\tau)), A(t)) = 0 \ \ \forall t \in \mathbb{R},$$

for all $\widehat{D} \in \mathcal{D}$, where $dist_X(\cdot,\cdot)$ denote the Hausdorff semidistance,

3) *$\widehat{A}$ is negatively invariant, i.e., $A(t) \subset U(t,\tau,A(\tau))$, for any $(t,\tau) \in \mathbb{R}^2_d$.*

*$\widehat{A}$ is said to be a **strict global pullback $\mathcal{D}$-attractor** if the invariance property in the third item is an equality.*

Theorem 8 *Assume that the MNDS U is upper-semicontinuous with closed values, and let $\widehat{B}$ be pullback $\mathcal{D}$- absorbing and such that U is pullback asymptotically compact with respect to $\widehat{B}$. Then, the following statements hold:*

1) *The set $\widehat{A}$ given by*

$$A(t) := \Lambda\left(\widehat{B}, t\right) := \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau,B(\tau))} \quad t \in \mathbb{R}, \tag{6}$$

is a pullback $\mathcal{D}$-attractor for the MNDS U. Moreover, suppose that $\mathcal{D}$ is inclusion closed (i.e., if $\widehat{D} \in \mathcal{D}$ and $\emptyset \ne D'(t) \subset D(t)$ for all $t \in \mathbb{R}$ then $\widehat{D}' \in \mathcal{D}$), $\hat{B} \in \mathcal{D}$ and $B(t)$ is closed in X for any $t \in \mathbb{R}$. Then $\widehat{A} \in \mathcal{D}$ and is the unique pullback $\mathcal{D}$-attractor with this property. In addition, in this case, if U is a strict MNDS, then $\widehat{A}$ is strictly invariant.

2) *If, in addition to the main assumptions, $U(t,\tau,\cdot)$ has connected values and $A(t) \subset C(t)$, where $\widehat{C} \in \mathcal{D}$ is connected, then $\widehat{A}$ is connected, which means that any $A(t)$ is connected for any $t \ge \tau$.*

The pullback attractor for system (2)

The following results can be found in [1].

For each $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega)$, let us denote $S(\tau, u_\tau)$ the set of all weak solutions of (2) defined for all $t \ge \tau$. We define a multi-valued map $U : \mathbb{R}^2_d \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ by

$$U(t,\tau,u_\tau) = \{u(t) : \ u \in S(\tau, u_\tau)\}, \quad \tau \le t, \quad u_\tau \in L^2(\Omega). \tag{7}$$

Lemma 9 *Under the assumptions of Theorem 2, the multi-valued mapping U defined by (7) is a strict MNDS on $L^2(\Omega)$.*

Let $\mathcal{R}_{\lambda_1}$ be the set of all functions $r : \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{t \to -\infty} e^{\lambda_1 t} r^2(t) = 0,$$

and denote by $\mathcal{D}_{\lambda_1}$ the class of all families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ such that $D(t) \subset \overline{B}_{L^2(\Omega)}(0, r_{\widehat{D}}(t))$ for some $r_{\widehat{D}} \in \mathcal{R}_{\lambda_1}$.

Theorem 10 *Suppose that Ω satisfies (1) and suppose that f is Caratheodory and satisfies (3) and (4). Let $h = \sum_{i=1}^N \frac{\partial h_i}{\partial x_i}$, with $h_i \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ for all $1 \le i \le N$, such that*

$$\sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 \, ds < +\infty \ \ \forall t \in \mathbb{R}. \tag{8}$$

Then, the MNDS U defined by (7) has a unique pullback $\mathcal{D}_{\lambda_1}$-attractor $\widehat{A}$ belonging to $\mathcal{D}_{\lambda_1}$, which is given by $A(t) := \Lambda\left(\widehat{B}_{\lambda_1}, t\right)$, where $B_{\lambda_1}(t) = \overline{B}_{L^2(\Omega)}(0, R_{\lambda_1}(t))$, where $R_{\lambda_1}(t)$ is the nonnegative number given for each $t \in \mathbb{R}$ by

$$R^2_{\lambda_1}(t) = 2e^{-\lambda_1 t} \sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 \, ds + 2\lambda_1^{-1} \|C_2\|_{L^1(\Omega)} + 1. \tag{9}$$

Moreover, $\widehat{A}$ is strictly invariant.

The Kneser property

When we consider a partial differential equation with non-uniqueness of the Cauchy problem it is interesting to consider **the Kneser property**, that is, the connectedness and compactness of the set of values attained by the solutions at any moment of time. In particular, this problem has been studied for reaction-diffusion equations by several authors so far. In [6] the results of [5] were extended to unbounded domains. However, due to technical difficulties it was necessary to assume an additional condition concerning the derivatives of the nonlinear terms. In this work we improve the method of the proof given in [6] in order to avoid such condition.

Remark 11 *We note that the compactness of $U(t,\tau,u_\tau)$ in $L^2(\Omega)$ is a consequence of Proposition 16 in [1].*

We shall obtain now that $U(t,\tau,u_\tau)$ is conected in $L^2(\Omega)$ and for this aim we need some preliminary lemmas. We take a sequence $0 < \epsilon_k < 1$ converging to 0 as $k \to \infty$ and define a sequence of smooth functions $\psi_k : \mathbb{R}^+ \longrightarrow [0,1]$ satisfying

$$\psi_k(s) := \begin{cases} 1, \text{ if } 0 \le s \le \sqrt{\epsilon_k}, \\ 0 \le \psi_k \le 1, \text{ if } \sqrt{\epsilon_k} \le s \le 2\sqrt{\epsilon_k}, \\ 0, \text{ if } 2\sqrt{\epsilon_k} \le s \le 1/\epsilon_k, \\ 0 \le \psi_k \le 1, \text{ if } 1/\epsilon_k \le s \le 1/\epsilon_k + 1, \\ 1, \text{ if } s \ge 1/\epsilon_k + 1. \end{cases}$$

Let $\rho_{\epsilon_k} : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a mollifier, that is, $\rho_{\epsilon_k} \in \mathbb{C}_0^\infty(\mathbb{R}; \mathbb{R})$, supp$\rho_{\epsilon_k} \subset B_{\epsilon_k}$, $\int_{\mathbb{R}} \rho_{\epsilon_k}(s) \, ds = 1$ and $\rho_{\epsilon_k}(s) \ge 0$ for all $s \in \mathbb{R}$, where $B_{\epsilon_k} = \{u \in \mathbb{R} : |u| \le \epsilon_k\}$. We define the following approximating function

$$f^k(x,u) := \psi_k(|u|) \left(C_0^1 |u|^{p-2} u + f(x,0) \right) + (1 - \psi_k(|u|)) \int_{\mathbb{R}} \rho_{\epsilon_k}(s) f(x, u - s) \, ds,$$

where $k \ge 1$, $p \ge 2$, and C_0^1 is a negative constant. Then for a.a. $x \in \Omega$ it is easy to check that

$$\sup_{|u| \le A} \left| f^k(x,u) - f(x,u) \right| \longrightarrow 0, \text{ as } k \longrightarrow \infty, \text{ for any } A > 0.$$

Lemma 12 *Assume (3)-(4). Then the function f^k satisfy conditions (3)-(4), i.e., there exist constants $\widehat{\alpha}_1$, $\widehat{\alpha}_2 > 0$, and positive functions $\widehat{C}_1(x) \in L^1(\Omega) \cap L^\infty(\Omega)$ and $\widehat{C}_2(x) \in L^1(\Omega)$, not depending on k, such that*

$$\left| f^k(x,u) \right|^{\frac{p}{p-1}} \le \widehat{\alpha}_1 |u|^p + \widehat{C}_1(x) \quad \forall u \in \mathbb{R},\ x \in \Omega,$$

$$f^k(x,u)u \le -\widehat{\alpha}_2 |u|^p + \widehat{C}_2(x) \ \ \forall u \in \mathbb{R},\ x \in \Omega,$$

for k great enough.

Lemma 13 *Assume (3)-(4). Then, there exist D_{ϵ_k} such that*

$$f_u^k(x,u) \le D_{\epsilon_k},\ \forall u \in \mathbb{R}, \text{ for a.a. } x \in \Omega, \tag{10}$$

where f_u^k is the derivative with respect to u.

Let $T > \tau$ be arbitrary. In order to prove the Kneser property let us consider the following auxiliary problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f^k(x,u) + h(t) \ \text{ in } \ \Omega \times (\gamma, +\infty), \\ u = 0 \ \text{ on } \ \partial\Omega \times (\gamma, +\infty), \\ u(x,\gamma) = u^\gamma(x), \quad x \in \Omega, \end{cases} \tag{11}$$

where $\gamma \in [\tau, T]$. In view of Lemma 12 for all $k \ge 1$ the function f^k satisfies (3) and (4), so that by Theorem 2 for any $u^\gamma \in L^2(\Omega)$ problem (11) has at least one weak solution $u_\gamma^k(\cdot)$ defined on $[\gamma, T]$. Using Lemma 13 it is standard to check that the solution of (11) is unique.

Lemma 14 *Suppose that Ω satisfies (1) and suppose that f is Caratheodory and satisfies (3) and (4). Let $h = \sum_{i=1}^N \frac{\partial h_i}{\partial x_i}$, with $h_i \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ for all $1 \le i \le N$, such that*

$$\sum_{i=1}^N \int_{-\infty}^t e^{\lambda_1 s} |h_i(s)|^2 \, ds < +\infty \ \ \forall t \in \mathbb{R}. \tag{12}$$

Then there exists $R = R(B,T)$ (not depending neither on γ nor k), where B is a bounded set of $L^2(\Omega)$, such that

$$\left| u_\gamma^k(t) \right| \le R, \ \ \forall t \in [\gamma, T], \tag{13}$$

and

$$\left\| u_\gamma^k(\cdot) \right\|_{L^p(\gamma,T;L^p(\Omega))} \le R, \tag{14}$$

for any $u^\gamma \in B$, where $u_\gamma^k(\cdot)$ is the unique solution to (11) with $u_\gamma^k(\gamma) = u^\gamma$.

Lemma 15 *Under the assumptions in Lemma 14, let K be a relatively compact set in $L^2(\Omega)$. Then, for all $\tau \le T$ and $\varepsilon > 0$ there exists $M = M(\gamma, T, \varepsilon, K)$ such that*

$$\int_{\Omega \cap \{|x|_{\mathbb{R}^N} \ge 2m\}} u_\gamma^k(x,t)^2 dx \le \varepsilon,\ \forall t \in [\gamma, T], \quad \forall \gamma \in [\tau, T],\ \ \forall m \ge M,$$

for any $u^\gamma \in K$, where $u_\gamma^k(\cdot)$ is the unique solution to (11) with $u_\gamma^k(\gamma) = u^\gamma$.

Now, we can deduce the following result.

Theorem 16 *Under the assumptions in Lemma 14, the set $U(t,\tau,u_\tau)$ is connected in $L^2(\Omega)$ for any $t \in [\tau, T]$.*

Connectedness of the pullback attractor

Our aim now is to obtain that the attractor is connected in $L^2(\Omega)$.

Theorem 17 *Under the assumptions in Lemma 14, the MNDS U defined by (7) has a unique pullback $\mathcal{D}_{\lambda_1}$-attractor $\widehat{A}$ belonging to $\mathcal{D}_{\lambda_1}$, which is strictly invariant and connected.*

Proof. In Theorem 10 it is shown the existence of a unique pullback $\mathcal{D}_{\lambda_1}$-attractor $\widehat{A}$ for U which is strictly invariant and belongs to $\mathcal{D}_{\lambda_1}$. On the other hand, we shall study the connectedness of the pullback $\mathcal{D}_{\lambda_1}$-attractor $\widehat{A}$. By Theorem 16, $U(t,\tau,u_\tau)$ has connected values in $L^2(\Omega)$. By Lemma 12 in [1] we have that $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing, then in particular we can deduce that there exists $\tau(t, \widehat{A}) \le t$ such that

$$U(t,\tau,A(\tau)) \subset B_{\lambda_1}(t) \ \text{ for all } \tau \le \tau(t, \widehat{A}).$$

Since $\widehat{A}$ is negatively invariant, we have

$$A(t) \subset B_{\lambda_1}(t) = \overline{B}_{L^2(\Omega)}(0, R_{\lambda_1}(t)),$$

where $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$ is connected. Hence, all conditions of the second statement of Theorem 8 are satisfied. Then, we have that $\widehat{A}$ is connected. ■

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