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Global Existence and Blow-up Results for Some Problems in Nonlinear Nonlocal Elasticity

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INTRODUCTION

We study the initial-value problem for a general class of nonlinear nonlocal wave equations arising in nonlocal elasticity. The model involves a convolution integral operator with a general kernel function whose Fourier transform is nonnegative. Some well-known examples of nonlinear wave equations, such as Boussinesq-type equations, follow from the present model for suitable choices of the kernel function. We establish global existence of solutions of the model assuming enough smoothness on the initial data together with some positivity conditions on the nonlinear term. Furthermore, conditions for finite time blow-up are provided.

The integro-differential equation modeling the propagation of longitudinal waves in a one-dimensional nonlocal nonlinear elastic medium

$$u_{tt} = (\beta * (u + g(u)))_{xx} \quad x \in \mathbb{R}, \quad t > 0 \tag{1}$$

and the system modeling the propagation of two transverse waves in a nonlocal nonlinear elastic medium

$$\begin{aligned} u_{1tt} &= (\beta_1 * (u_1 + g_1(u_1, u_2)))_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \\ u_{2tt} &= (\beta_2 * (u_2 + g_2(u_1, u_2)))_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \end{aligned}$$

are derived. Note that, $u = u(x,t)$ and $u_i = u_i(x,t)$ represent strains and $g(u)$ and $g_i(u_1,u_2)$ $i = 1,2$ represent the nonlinear parts of the strain energy functions, subscripts denote partial derivatives, the symbol $*$ denotes convolution in the spatial domain and β is an integrable function whose Fourier transform, $\hat{\beta}(\xi)$, satisfies the growth condition

$$0 \le \hat{\beta}(\xi) \le C(1+\xi^2)^{-r/2} \quad \text{for all } \xi$$

for a suitable constant C and $r \ge 2$. The number r is closely related to the smoothness of β. As the decay rate r gets larger the regularizing effect of the nonlocal behaviour increases.

Moreover the nonlinear functions g_i $(i = 1,2)$ satisfy the exactness condition

$$\frac{\partial g_1}{\partial u_2} = \frac{\partial g_2}{\partial u_1}$$

or equivalently there exists some $G(u_1,u_2)$ satisfying

$$g_i = \frac{\partial G}{\partial u_i} \qquad (i = 1,2).$$

Examples for the Kernel

Two kernel examples used in the literature are given below:

- *The Dirac measure*: $\beta = \delta$. In this case $r = 0$ and (1) becomes a nonlinear wave equation,
$$u_{tt} - u_{xx} = (g(u))_{xx}.$$
- *The exponential kernel*: $\beta(x) = \frac{1}{2}e^{-|x|}$. Since $\hat{\beta}(\xi) = (1+\xi^2)^{-1}$, we have $r = 2$ and (1) becomes the generalized improved Boussinesq equation,
$$u_{tt} - u_{xx} - u_{xxtt} = (g(u))_{xx}.$$

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GLOBAL EXISTENCE

Theorem 1. *Consider*

$$u_{tt} = [\beta * (u + g(u))]_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{2}$$
$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{3}$$

Let $s \ge 0$ and $r > 3$. If there is some $k > 0$ such that

$$G(u) \ge -ku^2 \quad \text{for all} \quad u \in R\,,$$

where $G(u) = \int_0^u g(p)dp$, then the Cauchy problem (2)-(3) has a global solution for initial data in Sobolev space H^s.

Theorem 2. *Consider*

$$u_{1tt} = [\beta_1 * (u_1 + g_1(u_1,u_2))]_{xx}, \quad x \in R, \quad t > 0, \tag{4}$$
$$u_{2tt} = [\beta_2 * (u_2 + g_2(u_1,u_2))]_{xx}, \quad x \in R, \quad t > 0, \tag{5}$$
$$u_1(x,0) = \varphi_1(x), \quad u_{1t}(x,0) = \psi_1(x) \tag{6}$$
$$u_2(x,0) = \varphi_2(x), \quad u_{2t}(x,0) = \psi_2(x). \tag{7}$$

Let $s \ge 0$, $r_1, r_2 > 3$. If there is some $k > 0$ so that

$$G(a,b) \ge -k(a^2+b^2),$$

for all $a, b \in R$, then the Cauchy problem (4)-(7) has a global solution for initial data in H^s.

BLOW-UP

Define an operator P as $Pv = \mathcal{F}^{-1}\left(|\xi|^{-1}\left(\widehat{\beta}(\xi)\right)^{-1/2}\widehat{v}(\xi)\right)$ with the inverse Fourier transform $\mathcal{F}^{-1}$. Observe that $P^2(\beta * v)_{xx} = -v$.

Lemma 3. *(Conservation of Energy)*
For a solution u of (2)-(3), the energy

$$E(t) = \|Pu_t\|^2 + \|u\|^2 + 2\int_R G(u(x,t))dx$$

is constant. (where the norms are $L^2(R)$ norms)

Theorem 4. *Let $s > 1/2$ and $r_1, r_2 \ge 2$. If $E(0) < 0$ and there exists some $\nu > 0$ satisfying*

$$uf(u) \le 2(1+2\nu)F(u)$$

where $F(u) = \int_0^u f(\rho)d\rho$ with $f(u) = u + g(u)$, then the solution u of (2)-(3) blows up in finite time.

Define an operator P_i as $P_i\omega = \mathcal{F}^{-1}(|\xi|^{-1}\hat{\beta}_i(\xi))^{-1/2}\hat{\omega}(\xi)$.

Lemma 5. *For solutions (u_1,u_2) of (4)-(7), the energy*

$$E(t) = \|P_1u_{1t}\|^2 + \|P_2u_{2t}\|^2 + \|u_1\|^2 + \|u_2\|^2 + 2\int_R G(u_1,u_2)dx$$

is constant.

Theorem 6. *Let $s > 1/2$ and $r_1, r_2 \ge 2$. If $E(0) < 0$ and there exists some $\nu > 0$ satisfying*

$$u_1f_1(u_1,u_2) + u_2f_2(u_1,u_2) \le 2(1+2\nu)F(u_1,u_2)$$

where $F(u_1,u_2) = \frac{1}{2}(u_1^2+u_2^2) + G(u_1,u_2)$ and $f_i = \frac{\partial F}{\partial u_i}$ $(i = 1,2)$ then the solution (u_1,u_2) of (4)-(7) blows up in finite time.

References

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[2] N. Duruk, A. Erkip, H.A. Erbay, *Blow-up and global existence for a general class of nonlocal nonlinear coupled wave equations*, Journal of Differential Equations, **250**, (2011) 1448-1459.