ON SOME VECTOR DIFFERENTIAL OPERATORS OF INFINITE ORDER

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Let E be a complex Banach space and $\varphi(z)=\sum_{n=0}^{\infty} C_n z^n$ be a formal power series that the coefficients are bounded linear operators in E.

We consider a problem on applicability of the following differential operator of infinite order $\varphi\left(\frac{d}{dz}\right)g=\sum_{n=0}^{\infty} C_n g^{(n)}$ to the space $H(\mathbb{C},E)$ of all entire E-valued functions. We obtained some integral representations of this operator. The space of entire E-valued functions we consider with the topology of uniform convergence on compacts. in the scalar case differential operators of infinite order were studied in different points of view in works of Valiron, Polya, Muggli, Sikkema, Korobeinik, Leont'ev, Dickcon and other mathematicians.

Example. Let $T:E\to E$ be a bounded linear operator, $\varphi(z)=\sum_{n=0}^{\infty} T^n z^n$, then $\varphi\left(\frac{d}{dz}\right)g=\sum_{n=0}^{\infty} T^n g^{(n)}$.

We denote that some of our following results don't have nontrivial scalar analogue. Let us go on to our main results.

Theorem 1 *Let $\varphi(z)=\sum_{n-0}^{\infty} C_n z^n$ be an entire operator-function of exponential type, i.e. $\|\varphi(z)\|\le \gamma e^{\beta|z|}$ for some $\gamma>0, \beta>0$ and all $z\in\mathbb{C}$.*

If $g\in H(\mathbb{C},E)$, then the series $\sum_{n-0}^{\infty} C_n g^{(n)}(z)$ converges uniformly in every disk and thus $\varphi\left(\frac{d}{dz}\right)$ is continuous linear operator on the space $H(\mathbb{C},E)$. In addition, not more than exponential growth of $\varphi(z)$ is the necessary condition: if $\varphi(z)=\sum_{n-0}^{\infty} C_n z^n$ is such power series, that the series $\sum_{n-0}^{\infty} C_n g^{(n)}(0)$ converges for all $g\in H(\mathbb{C},E)$, then $\varphi(z)$ is an entire function of exponential type.

From Theorem 1 we obtain the corollary of entire solutions of the following linear differential equation in a Banach space.

Corollary 1 *Let $T:E\to E$ be a quasinilpotent operator, the Fredholm resolvent $(1-zT)^{-1}$ have exponential type, and $g(z)$ be an arbitrary E-valued entire function. Then the differential equation*

$$Tw'+g(z)=w \tag{1}$$

has unique entire solution $w(z)=\sum_{n-0}^{\infty} T^n g^{(n)}(z)$.

It is clear that this corollary is interested only in a vector case. Note that it is not the classical Laplace integral representation. For obtaining the integral representation of differential operator of infinite order and in particularly the solution of differential Equation (1) we need to consider the notion of formal integral in the space of formal vector Laurent series.

Now let us go to a question of integral representation for differential Equation (1).

Let V be an arbitrary complex vector space and $V\left[\left[\zeta,\frac{1}{\zeta}\right]\right]$ be the space of all formal Laurent series with coefficient from V. For $f(\zeta)=\sum_{n=-\infty}^{\infty} b_n\zeta^n\in V\left[\left[\zeta,\frac{1}{\zeta}\right]\right]$ we set

$$\oint f(\zeta)d\zeta=2\pi i\, b_{-1}. \tag{2}$$

In the scalar case this formal integral is used in the functional theory and combinatorics, and in the vector case one is widely used in conformal and quantum field theory in additional in some questions of operator theory.

Theorem 2 *Let φ and g satisfy conditions of Theorem 1, $g(z)=\sum_{n-0}^{\infty} a_n z^n$ and $G(\zeta)=\sum_{n-0}^{\infty}\frac{n!\,a_n}{\zeta^{n+1}}$ be the formal Laplace-Borel' transformation of power series $g(z)$. Then the product $e^{z\zeta}\varphi(\zeta)G(\zeta)$ is well defined as an element of $E[[z]]\left[\left[\zeta,\frac{1}{\zeta}\right]\right]$ (i.e. in this case the space of V) and $\left(\varphi\left(\frac{d}{dz}\right)g\right)(z)=\frac{1}{2\pi i}\oint e^{z\zeta}\varphi(\zeta)G(\zeta)d\zeta$, where the integral is considered in the sense of (2).*

Corollary 2 *Let be held the conditions of Corollary 1. Then the unique solution of Equation (1) can be presented in the following integral form $w(z)=\frac{1}{2\pi i}\oint e^{z\zeta}(1-\zeta T)^{-1}G(\zeta)d\zeta$, where the integral is considered in the sense of (2).*

Note, that if $g(z)$ is not a function of exponential type, then its Laplace-Borel' formal transformation $G(\zeta)$ has zero radius of convergence.

Example *Let $E=C[0,1]$, $A=\frac{d}{dx}$ and $D(A)=\{u\in C^1[0,1]:u(0)=0\}$, $T=A^{-1}$. Then $(A^{-1}h)(x)=\int_0^x h(y)dy$, $(A^{-(n+1)}h)(x)=\frac{1}{n!}\int_0^x (x-y)^n h(y)dy$ and $\rho(A^{-1})=0$. By transition to real axes Equation $w'=Aw+f(z)$ has the form*

$$\begin{cases}\dfrac{\partial w}{\partial t}=\dfrac{\partial w}{\partial x}+f(t,x), t\in\mathbb{R}, x\in(0,1)\\ w(t,0)=0\end{cases} \tag{3}$$

If in the second variable f can be extended to an entire function of exponential type, then in this class of functions Problem (3) has the unique solution

$$(A^{-(n+1)}h)(x)=-\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^x (x-y)^n\frac{\partial^n f}{\partial t^n}(t,y)dy=-\int_0^x f(t+x-y,y)dy.$$

It is important to note, that Problem (3) has only zero solution for the homogeneous equation even in class of continuously differentiablle functions. In particular, A is not a Hille-Yosida operator (see W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace Transforms and Cauchy Problems, Monographs in Mathematics, 96:XI(2001), Section 3.5).

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