On the Navier boundary condition for viscous fluids in a thin domain covered by very small asperities

F.J. Suárez-Grau

Departamento de Matemáticas, Universidad de Huelva, Spain

fjsgrau@us.es

Universidad de Huelva

1. Introduction

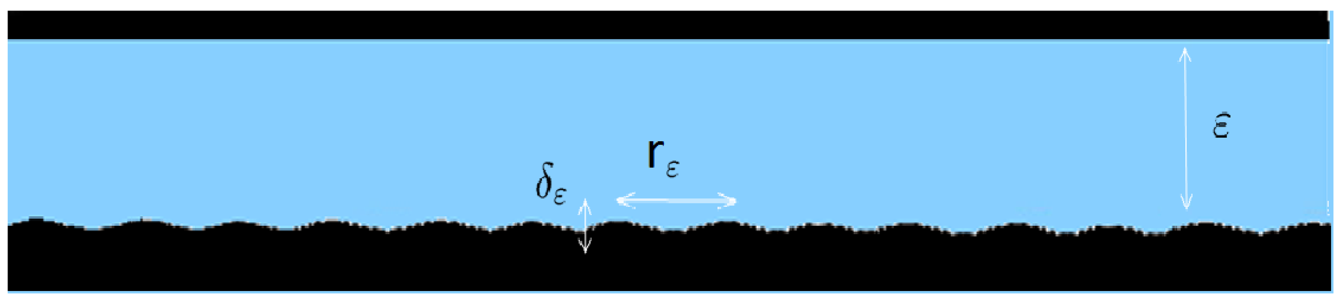
FOR a viscous fluid in an open set of $\mathbb{R}^3$ with a rugous boundary, it is known that if the normal velocity vanishes on the boundary (Navier condition), then the fluid behaves as if the whole velocity vector vanishes on the boundary (adherence condition). This gives a mathematical explanation of why it is usual for a viscous fluid to impose the adherence condition. The equivalence between the Navier and adherence conditions was proved in [2] for a periodic rough boundary of small period ε and amplitude ε. In [3] it was considered the case of a weak roughness, namely the boundary was described by a periodic function of small period ε and amplitude δ_ε, with $\delta_\varepsilon/\varepsilon$ converging to zero.

Our aim in the present work is to study the relation between the Navier and adherence conditions in the case of a domain of small height ε. Namely, for a Lipschitz bounded open set $\omega\subset\mathbb{R}^2$ and a function Ψ in $W^{2,\infty}_{loc}(\mathbb{R}^2)$, periodic of period $Z'=(-1/2,1/2)^2$, we define Ω_ε by

$$\Omega_\varepsilon=\left\{x=(x',x_3)\in\omega\times\mathbb{R}:\ -\delta_\varepsilon\Psi\left(\frac{x'}{r_\varepsilon}\right)<x_3<\varepsilon\right\},\tag{1}$$

where the parameters r_ε, δ_ε are chosen non-negative and satisfying

$$\lim_{\varepsilon\to0}\frac{r_\varepsilon}{\varepsilon}=0,\quad \lim_{\varepsilon\to0}\frac{\delta_\varepsilon}{r_\varepsilon}=0,\quad i.e.\quad \delta_\varepsilon\ll r_\varepsilon\ll\varepsilon.$$



We consider a fluid satisfying the Stokes system in Ω_ε, the Navier condition on the rough boundary

$$\Gamma_\varepsilon=\left\{x=(x',x_3)\in\omega\times\mathbb{R}:\ x_3=-\delta_\varepsilon\Psi\left(\frac{x'}{r_\varepsilon}\right)\right\}\tag{2}$$

and (to simplify) the adherence condition on the rest of the boundary $\partial\Omega_\varepsilon\setminus\Gamma_\varepsilon$,

$$\begin{cases}-\mu\Delta u_\varepsilon+\nabla p_\varepsilon=f \text{ in } \Omega_\varepsilon, & \text{div}\, u_\varepsilon=0 \text{ in } \Omega_\varepsilon,\\ u_\varepsilon=0 \text{ on } \partial\Omega_\varepsilon\setminus\Gamma_\varepsilon, & \\ u_\varepsilon\cdot\nu=0 \text{ on } \Gamma_\varepsilon, & \mu\dfrac{\partial u_\varepsilon}{\partial\nu} \text{ parallel to } \nu \text{ on } \Gamma_\varepsilon.\end{cases}\tag{3}$$

Here $f=(f',f_3)\in L^2(\omega)^3$, ν denotes the unitary outside normal vector to Ω_ε in Γ_ε and $\mu>0$ corresponds to the viscosity of the fluid.

It is well known that (3) has a unique solution $(u_\varepsilon,p_\varepsilon)\in H^1(\Omega_\varepsilon)^3\times L^2_0(\Omega_\varepsilon)$ ($L^2_0(\Omega_\varepsilon)$ denotes the space of functions in $L^2(\Omega_\varepsilon)$ whose integral in Ω_ε is zero). Moreover, we can show the following estimates

$$\int_{\Omega_\varepsilon}|u_\varepsilon|^2dx\le C\varepsilon^4,\qquad \int_{\Omega_\varepsilon}|Du_\varepsilon|^2dx\le C\varepsilon^2,\qquad \int_{\Omega_\varepsilon}|p_\varepsilon|^2dx\le C.\tag{4}$$

Our purpose is to study the asymptotic behavior of this system when ε tends to zero. We show that it depends on

$$\lambda=\lim_{\varepsilon\to0}\frac{\delta_\varepsilon}{r_\varepsilon^{\frac{3}{2}}}\sqrt{\varepsilon}\in[0,+\infty].\tag{5}$$

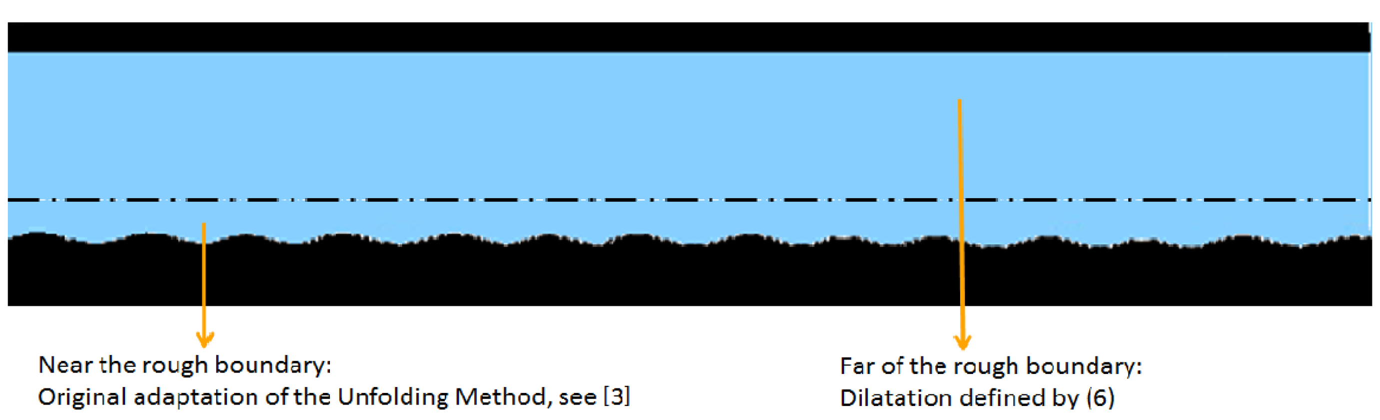
2. Changes of variables

OUR aim is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, we use a suitable combination of two changes of variables:

(1) Far of the rough boundary Γ_ε we use a dilatation in the variable x_3 in order to have the functions defined in an open set of fixed height. Namely, we take $\Omega=\omega\times(0,1)$ and we define $\tilde u_\varepsilon\in H^1(\Omega)^3$, $\tilde p_\varepsilon\in L^2_0(\Omega)$ by

$$\tilde u_\varepsilon(y)=u_\varepsilon(y',\varepsilon y_3),\quad \tilde p_\varepsilon(y)=p_\varepsilon(y',\varepsilon y_3),\quad \text{a.e. } y\in\Omega.\tag{6}$$

(2) Near Γ_ε we use an original adaptation ([3]) of the *Unfolding Method* ([1], [5]), which is very related to the two-scale convergence method.



3. Main Result

Let $(u_\varepsilon,p_\varepsilon)\in H^1(\Omega_\varepsilon)^3\times L^2_0(\Omega_\varepsilon)$ be the solution of the Stokes system (3) and let $\tilde u_\varepsilon$, $\tilde p_\varepsilon$ be defined by (6). Then, there exist $v'\in H^1(0,1;L^2(\omega))^2$, $w\in H^2(0,1;H^{-1}(\omega))$ and $p\in L^2_0(\omega)$, where p does not depend on y_3, such that, up to a subsequence,

$$\frac{\tilde u'_\varepsilon}{\varepsilon^2}\rightharpoonup v' \text{ in } H^1(0,1;L^2(\omega))^2,\qquad \frac{\tilde u_{\varepsilon,3}}{\varepsilon^3}\rightharpoonup w \text{ in } H^2(0,1;H^{-1}(\omega)),$$

$$\tilde p_\varepsilon\rightharpoonup p \text{ in } L^2(\Omega).$$

According to the value of λ defined by (5), we obtain the different expressions for v' and w depending on p which satisfies a *Reynolds equation*:

(i) If $\lambda=+\infty$, then denoting by $P_{W^\perp}$ the orthogonal projection from $\mathbb{R}^2$ to the orthogonal of the space $W=\{\nabla_{z'}\Psi(z')\in\mathbb{R}^2:\ z'\in Z'\}$, we have that v' and p are given by

$$v'(y)=\frac{(y_3-1)}{2\mu}\Big(y_3I+P_{W^\perp}\Big)\left(\nabla_{y'}p(y')-f'(y')\right),\quad \text{a.e. } y\in\Omega,$$

$$-\text{div}_{y'}\left(\left(\frac{1}{3}I+P_{W^\perp}\right)(\nabla_{y'}p-f')\right)=0 \text{ in } \omega,\quad \left(\frac{1}{3}I+P_{W^\perp}\right)(\nabla_{y'}p-f')\cdot\nu=0 \text{ on } \partial\omega.$$

Moreover, the distribution w is given by $\quad w(y)=-\displaystyle\int_0^{y_3}\text{div}_{y'}v(y',s)ds,\ $ in Ω. $\qquad$ (7)

(ii) If $\lambda\in(0,+\infty)$, then defining $(\widehat\phi^i,\widehat q^i)$, $i=1,2$, as solutions of the Stokes systems

$$\begin{cases}-\mu\Delta_z\widehat\phi^i+\nabla_z\widehat q^i=0 \text{ in } \mathbb{R}^2\times(0,+\infty), & \text{div}_z\,\widehat\phi^i=0 \text{ in } \mathbb{R}^2\times(0,+\infty),\\ \widehat\phi^i_3(z',0)+\partial_{z_i}\Psi(z')=0,\ \partial_{z_3}(\widehat\phi^i)'(z',0)=0, & \widehat\phi^i(.,z_3),\ \widehat q^i(.,z_3) \text{ periodic of period } Z',\\ D_z\widehat\phi^i\in L^2(Z'\times(0,+\infty))^{3\times3},\ \widehat q^i\in L^2(Z'\times(0,+\infty)), & \end{cases}$$

and $R\in\mathbb{R}^{2\times2}$ by $\quad R_{ij}=\mu\displaystyle\int_{Z'\times(0,+\infty)}D_z\widehat\phi^i:D_z\widehat\phi^j\,dz,\quad \forall i,j\in\{1,2\}$, we have

$$v'(y)=\frac{(y_3-1)}{2\mu}\left(y_3I+\left(I+\frac{\lambda^2}{\mu}R\right)^{-1}\right)\left(\nabla_{y'}p(y')-f'(y')\right),\quad \text{a.e. } y\in\Omega,$$

where p satisfies

$$\begin{cases}-\text{div}_{y'}\left(\left(\dfrac{1}{3}I+\left(I+\dfrac{\lambda^2}{\mu}R\right)^{-1}\right)(\nabla_{y'}p-f')\right)=0 \text{ in } \omega,\\ \left(\dfrac{1}{3}I+\left(I+\dfrac{\lambda^2}{\mu}R\right)^{-1}\right)(\nabla_{y'}p-f')\cdot\nu=0 \text{ on } \partial\omega.\end{cases}$$

Moreover, the distribution w is given by (7).

(iii) If $\lambda=0$, then $\quad v'(y)=\dfrac{(y_3^2-1)}{2\mu}(\nabla_{y'}p(y')-f'(y')),\quad$ a.e. $y\in\Omega$,

where p satisfies $\quad -\Delta_{y'}p=-\text{div}_{y'}f'$ in ω, $\quad \dfrac{\partial p}{\partial\nu}=f'\cdot\nu$ on $\partial\omega$.

Moreover, the distribution w is zero.

4. Conclusions

• For $\lambda=+\infty$, the main result shows that u_ε, p_ε behave as if in (3) we had assumed that Γ_ε was the plane boundary $\{x_3=0\}$ and that the boundary condition on Γ_ε was

$$u_\varepsilon\in W^\perp\times\{0\}\ \text{ on } \Gamma_\varepsilon,\quad \partial_3u'_\varepsilon\in W.\tag{8}$$

In particular, if W agrees with $\mathbb{R}^2$ we deduce that the Navier condition in (3) is equivalent to the adherence condition $u_\varepsilon=0$ on $\{x_3=0\}$.

• For $\lambda\in(0,+\infty)$, the result shows that the asymptotic behavior of u_ε and p_ε is the same that if Γ_ε was the plane boundary $\{x_3=0\}$ and the boundary condition on Γ_ε was

$$u_{\varepsilon,3}=0\ \text{ on } \Gamma_\varepsilon,\quad -\mu\partial_3u'_\varepsilon+\lambda^2Ru'_\varepsilon=0\ \text{ on } \Gamma_\varepsilon,\tag{9}$$

i.e. although the roughness is not strong enough to deduce that the Navier condition on Γ_ε is equivalent to (8), it is sufficient to provide the friction coefficient $\lambda^2Ru'_\varepsilon$ in (9).

• For $\lambda=0$, the roughness is so weak that u_ε and p_ε behave as if Γ_ε was plane.

The critical size $\lambda\in(0,+\infty)$ can be considered as the general one. In fact, the cases $\lambda=0$ and $\lambda=+\infty$ can be obtained from this one by taking the limit when λ tends to zero and infinity respectively.

References

[1] *T. Arbogast, J. Douglas, U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory,* SIAM J. Math. Anal. 21 (1990) 823-836.

[2] J. Casado-Díaz, E. Fernández-Cara, J. Simon, Why viscous fluids adhere to rugose walls: A mathematical explanation, J. Differential Equations 189 (2003) 526-537.

[3] J. Casado-Díaz, M. Luna-Laynez, F.J. Suárez-Grau, Asymptotic behavior of a viscous fluid with slip boundary conditions on a slightly rough wall, M3AS 20 (2010) 121-156.

[4] J. Casado-Díaz, M. Luna-Laynez, F.J. Suárez-Grau, A viscous fluid in a thin domain satisfying the slip condition on a slightly rough boundary, C. R. Acad. Sci. Paris, Ser. I (2010).

[5] D. Cioranescu, A. Damlamian, G. Griso, Periodic unfolding and homogenization, C.R. Acad. Sci. Paris, Ser. I 335 (2002) 99-104.