Dispersive Properties for Discrete Schrödinger Equations

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Linear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = 0, \; x \in \mathbb{R}, \; t \neq 0, \\ u(0,x) = \varphi(x), \; x \in \mathbb{R}, \end{cases}$$

Conservation of the L^2-norm:

$$\|S(t)\varphi\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$$

Dispersive estimate:

$$|S(t)\varphi(x)| \leq \frac{1}{(4\pi|t|)^{1/2}} \|\varphi\|_{L^1(\mathbb{R})}$$

Coupled DLSE

System of two discrete linear Schrodinger equations:

$$\begin{cases} iu_t(j) + (\Delta u)(j) = 0 & j \leq -1, \\ iv_t(j) + (\Delta v)(j) = 0 & j \geq 1, \\ u(t,0) = v(t,0), & t > 0, \\ u(t,-1) - u(t,0) = v(t,0) - v(t,1), & t > 0 \\ u(0,j) = \varphi(j), & j \leq -1, \\ v(0,j) = \varphi(j), & j \geq 1. \end{cases} \quad (1)$$

Theorem. For any $\varphi \in l^2(\mathbb{Z}\setminus\{0\})$ there exist a unique solution $(u,v) \in C([0,\infty, l^2(\mathbb{Z}\setminus\{0\}))$ of equation (1) which satisfies the dispersive estimate

$$\|(u,v)(t)\|_{l^\infty(\mathbb{Z}\setminus\{0\})} \leq c(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}\setminus\{0\})}. \quad (2)$$

Sketch of the proof: analyzing two problems: with Dirichlet(resp. Neumann)boundary condition satisfied by

$$S(j) = \frac{v(j)+u(-j)}{2}, \text{ and } D(j) = \frac{v(j)-u(-j)}{2}, j \geq 0.$$

Remark. This is a particular case of model (3) (take $b_1 = b_2 = 1$).

DLSE with discontinuous coefficients

Our paper proves dispersive estimates for the following model (3):

$$\begin{cases} iu_t(j) + b_1^{-2}(\Delta u)(j) = 0 & j \leq -1, \\ iv_t(j) + b_2^{-2}(\Delta v)(j) = 0 & j \geq 1, \\ u(t,0) = v(t,0), & t > 0, \\ b_1^{-2}(u(t,-1) - u(t,0)) = b_2^{-2}(v(t,0) - v(t,1)), & t > 0 \\ u(0,j) = \varphi(j), & j \leq -1, \\ v(0,j) = \varphi(j), & j \geq 1. \end{cases}$$

Matrix formulation

$U = (u(j))_{j\neq 0}$ satisfies $iU_t + AU = 0$ where A is given by

$$\begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1^{-2} & -2b_1^{-2} & b_1^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1^{-2} & -b_1^{-2} - \frac{1}{b_1^2+b_2^2} & \frac{1}{b_1^2+b_2^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{b_1^2+b_2^2} & -\frac{1}{b_1^2+b_2^2} - b_2^{-2} & b_2^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_2^{-2} & -2b_2^{-2} & b_2^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}$$

Main result

Theorem. For any $\varphi \in l^2(\mathbb{Z}\setminus\{0\})$ there exists a unique solution $(u,v) \in C(\mathbb{R}, l^2(\mathbb{Z}\setminus\{0\}))$ of system (3). Moreover, there exists a positive constant $C(b_1,b_2)$ such that

$$\|(u,v)(t)\|_{l^\infty(\mathbb{Z}\setminus\{0\})} \leq C(b_1,b_2)(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}\setminus\{0\})}, \; \forall t \in \mathbb{R},$$

holds for all $\varphi \in l^1(\mathbb{Z}\setminus\{0\})$.

Sketch of the proof

Use of the resolvent. Idea from reference (2).

For any b_1 and b_2 positive the spectrum of the operator A satisfies

$$\sigma(A) \subset I = [-4\max\{b_1^{-2}, b_2^{-2}\}, 0].$$

For any $\omega \in I$ define $R^{\pm}(\omega) = \lim_{\epsilon\downarrow 0} R(\omega \pm i\epsilon)$. Then

$$R^-(\omega) = \overline{R}^+(\omega), \quad \forall \omega \in I,$$

A formula for our solutions

$$e^{itA}\varphi = \frac{1}{\pi}\int_I e^{it\omega}\mathrm{Im}R^+(\omega)\varphi d\omega,$$

where $R^+(\omega)\varphi(j)$ can be computed explicitly.

Oscillatory integrals

Lemma(Van der Corput)

Suppose ψ is real-valued and smooth in I, and that $|\psi^{(k)}(x)| \geq 1$ for all $x \in I$. Then

$$\left|\int_I e^{i\lambda\psi(x)}\phi(x)dx\right| \leq c_k\lambda^{-1/k}(\|\phi\|_{L^\infty(I)} + \int_I |\phi'|).$$

Lemma(Kenig, Ponce, Vega 91)

Under certain hypothesis over ϕ, the following

$$\left|\int_a^b e^{i(t\phi(\xi)-x\xi)}|\phi''(\xi)|^{1/2}\psi(\xi)d\xi\right|$$

$$\leq c_\phi|t|^{-1/2}\{\|\psi\|_{L^\infty(a,b)} + \int_a^b |\psi'(\xi)|d\xi\}.$$

holds for all real numbers x and t.

Lemma(L. Ignat, DS 2010)

Assuming that at the critical points we have

$$\phi'(\xi) \sim \xi^\alpha, \alpha \geq 2$$

then

$$I(x,t) = \int_\Omega e^{i(t\phi(\xi)-x\xi)}|\phi'''(\xi)|^{\frac{1}{3}}d\xi \leq ct^{-\frac{1}{3}}.$$

Open problems

I. Give sufficient conditions for a symmetric matrix A with a finite number of diagonals not identically vanishing such that for the equation $iU_t + AU = 0$ we can prove similar decay properties, even with other type of decay: $t^{-1/4}$, etc..
II. Coupling more than two equations.
III. Discrete Schrödinger equations on trees, graphs.

References

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