Non-Newtonian fluids with nonstandard rheology

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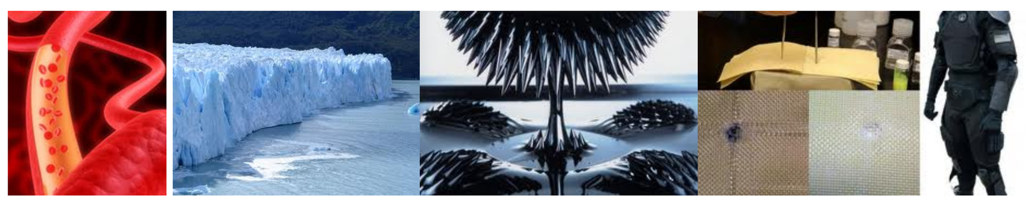
1. Introduction

- **The goal** We want to show existence of weak solutions to unsteady flow of incompressible fluids with nonstandard growth conditions of a stress tensor prescribed by quite general convex function. We wish to understand mathematical properties of equations describing fluid flow of type more general then power-low, i.e.:

$$\mathbf{S}(\mathbf{D}u) \approx |\mathbf{D}u|^{p-2}\mathbf{D}u.$$

- **Motivations** *Non-Newtonian fluids, whose properties are not described by a single constant value of viscosity. The viscosity may change in response to an applied force, shear stress, electric or magnetic field.*
  - shear thickening fluids - fluids which viscosity increases rapidly and significantly
  - shear thinning fluids fluids which viscosity decreases - blood flow, glacier-ice, paint, nail polish,
  - fluids which rheology is close to linear
  - anisotropic fluids - magnetorheological fluids



2. Considered Problem , Generalized Stokes system

If we assume that u is small or we consider the case of simple flow between two fixed parallel plates) then the convective term $\operatorname{div}(u\otimes u)$ can be neglected in the momentum equation.

Let $\Omega\subset\mathbb{R}^n$ be open bounded domain with a sufficiently smooth boundary $\partial\Omega$. We consider generalized Stokes system

$$\begin{aligned} \partial_t u - \operatorname{div}\mathbf{S}(t,x,\mathbf{D}u) + \nabla p &= f \quad \text{in } (0,T)\times\Omega,\\ \operatorname{div} u &= 0\\ u(0,x) &= u_0 \quad \text{in } \Omega,\\ u(t,x) &= 0 \quad \text{on } (0,T)\times\partial\Omega \end{aligned}$$

$\mathbf{D}u = \frac{1}{2}(\nabla u + \nabla^T u)$, $\mathbf{S} + \mathbf{I}p$ Cauchy tensor.

3. Anisotropic Orlicz Spaces

Definition 1 (Anisotropic $\mathcal{N}$-function) *Function $M : \mathbb{R}^{n\times n}_{\rm sym} \to \mathbb{R}_+$ is said to be an $\mathcal{N}$–function if it satisfies the following conditions*

1. *It is continuos and convex, $M(\xi) = M(-\xi)$ and $M(\xi) = 0$ iff $\xi = 0$*
2. $$\lim_{|\xi|\to 0}\frac{M(\xi)}{|\xi|} = 0, \qquad \lim_{|\xi|\to\infty}\frac{M(\xi)}{|\xi|} = \infty.$$

Definition 2 *The complementary function M^* to a function M is defined by*

$$M^*(\eta) = \sup_{\xi\in\mathbb{R}^{n\times n}_{\rm sym}} (\xi\cdot\eta - M(\xi)) \qquad \text{for } \eta\in\mathbb{R}^{n\times n}_{\rm sym}.$$

Examples:

- $M(\xi) = \frac{1}{p}|\xi|^p$, $M^*(\eta) = \frac{1}{p'}|\eta|^{p'}$, where $\frac{1}{p}+\frac{1}{p'} = 1$
- $M(\xi) = \exp(|\xi|^2) - 1$, $M^*(\eta) = |\eta|\sqrt{\log|\eta|}$ asymptotically
- $M(\xi) = \sum_i \frac{1}{p_i}|\xi|^{p_i}$, $M^*(\eta) = \frac{1}{p_i'}|\eta|^{p_i'}$, where $\frac{1}{p_i}+\frac{1}{p_i'} = 1$

Definition 3 *An anisotropic Orlicz class $\mathcal{L}_M$*

$$\mathcal{L}_M := \left\{\xi : Q\to\mathbb{R}^{n\times n}_{\rm sym} \text{ measurable} \,\Big|\, \int_Q M(\xi(t,x))dxdt < \infty\right\}$$

An anisotorpic Orlicz space $L_M(Q)$,

$$L_M := \left\{\xi : Q\to\mathbb{R}^{n\times n}_{\rm sym} \text{ measurable} \,\Big|\, \int_Q M(\lambda\xi(x,t))dxdt \to 0 \quad \text{if } \lambda\to 0\right\}$$

The space L_M is a Banach space with Orlicz norm

$$\|\xi\|_M = \sup\left\{\int_Q \eta\cdot\xi\, dxdt : \eta\in L_{M^*}(Q), \int_Q M^*(\eta)dxdt \le 1\right\}.$$

By E_M we denote the closure of bounded functions in Orlicz space L_M. Note that: $(E_M)^* = L_{M^*}$.

Definition 4 *We say that $\mathcal{N}$-function M satisfies Δ_2 condition, if for some constant $C > 0$*

$$M(2\xi) \le CM(\xi) \quad \text{for every } \xi\in\mathbb{R}^{n\times n}_{\rm sym}.$$

Remark: If $M(\cdot)$ grows faster then polynomial then we are in troubles

- L_M is not separable, C^∞ is not dense in L_M
- $E_M \subsetneq \mathcal{L}_M \subsetneq L_M$
- $L_M(Q) \ne L_M(0,T;L_M(\Omega))$, Orlicz space is not reflexive

4. Stress tensor S

We assume that the stress tensor $\mathbf{S} : (0,T)\times\Omega\times\mathbb{R}^{3\times 3}_{\rm sym} \to \mathbb{R}^{3\times 3}_{\rm sym}$ satisfies the following conditions:

S1. $\mathbf{S}(t,x,\xi)$ is a Carathéodory function and $\mathbf{S}(t,x,0) = 0$.

S2. There exist a positive constant c_s, an $\mathcal{N}$–function M and M^* stands for the complementary function to M such that for all $\xi\in\mathbb{R}^{d\times d}_{\rm sym}$ and a.a. $t,x\in Q$ it holds

$$\mathbf{S}(t,x,\xi) : \xi \ge c_s[M(\xi) + M^*(\mathbf{S}(t,x,\xi))]$$

S3. $\mathbf{S}$ is monotone i.e. for all $\xi_1,\xi_2\in\mathbb{R}^{d\times d}_{\rm sym}$, a.a. $t,\ x\in Q$

$$[\mathbf{S}(t,x,\xi_1) - \mathbf{S}(t,x,\xi_2)] : [\xi_1 - \xi_2] \ge 0.$$

5. Existence Result

Below we define two function $\underline{m}, \overline{m} : \mathbb{R}_+\to\mathbb{R}_+$, which control the growth of N-function M:

$$\underline{m}(r) := \min_{\xi\in\mathbb{R}^{n\times n}_{\rm sym},|\xi|=r} M(\xi), \qquad \overline{m}(r) := \max_{\xi\in\mathbb{R}^{n\times n}_{\rm sym},|\xi|=r} M(\xi).$$

Theorem 1 (A.W., P. Gwiazda, A. Świerczewska-Gwiazda [2]) *Let condition D1. or D2. be satisfied*

(D1) Ω is a star-shaped domain,

(D2) Ω is a non-star-shaped domain, $\underline{m}$ satisfies Δ_2-cond. and for $r\in\mathbb{R}_+$

$$\overline{m}(r) \le c_m((\underline{m}(r))^{\frac{n}{n-1}} + |r|^2 + 1).$$

Let M be an N-function and $\mathbf{S}$ satisfy conditions (S1)-(S3). Then for given $u_0\in L^2_{\rm div}(\Omega)$ and $f\in E_{(\underline{m})^*}(Q)$ there exists $u\in Z^M_0$ such that

$$\int_Q -u\cdot\partial_t\varphi + \mathbf{S}(t,x,\mathbf{D}u)\cdot\mathbf{D}\varphi\, dxdt = \int_Q f\cdot\varphi\, dxdt - \int_\Omega u_0\varphi\, dx$$

for all $\varphi\in C^\infty_c(-\infty,T;\mathcal{V})$.

$\mathcal{V} := \{\varphi\in C^\infty_c(\Omega;\mathbb{R}^n);\ \operatorname{div}\varphi = 0\}$, $L^2_{\rm div}$:= closure of $\mathcal{V}$ in L^2- norm.
Z^M_0 - the closure of smooth, compactly supported functions with respect to the weak* topology for gradients in L_M

$$Z^M_0 = \{\varphi\in L^\infty(0,T;L^2_{\rm div}(\Omega)), \mathbf{D}\varphi\in L_M(Q)\ |\ \exists\ \{\varphi^j\}^\infty_{j=1}\subset C^\infty_c((-\infty,T);\mathcal{V}) : \varphi^j \overset{*}{\rightharpoonup} \varphi \text{ in } L^\infty(0,T;L^2_{\rm div}(\Omega)), \mathbf{D}\varphi^j \overset{*}{\rightharpoonup} \mathbf{D}\varphi \text{ weakly* in } L_M(Q)\}.$$

6. Main steps in the proof

- Galerkin approximation with help of eigenvectors of Stokes operator - approximate solution u^k

$$-\int_Q u^k\cdot\partial_t\varphi\, dxdt + \int_Q \mathbf{S}(t,x,\mathbf{D}u^k) : \mathbf{D}\varphi\, dxdt = \int_Q f\cdot\varphi\, dxdt - \int_\Omega u_0\cdot\varphi\, dx$$

where $\varphi\in C^\infty_c((-\infty,T);\mathcal{V})$.

- A priori estimate

$$\begin{aligned}\frac{1}{2}\|u^k(t)\|^2_{L^2(\Omega)} + \frac{c}{2}\int_0^t\int_\Omega M(\mathbf{D}u^k)dxd\tau + c\int_0^t\int_\Omega M^*(\mathbf{S}(\tau,x,\mathbf{D}u^k))dxd\tau \\ \le \int_0^t\int_\Omega (\underline{m})^*(C|f|)dxd\tau + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}\end{aligned}$$

- $$\begin{aligned}\mathbf{D}u^k &\overset{*}{\rightharpoonup} \mathbf{D}u \quad \text{in } L_M(Q)\\ \mathbf{S}(\cdot,\cdot,\mathbf{D}u^k) &\overset{*}{\rightharpoonup} \chi \quad \text{in } L_{M^*}(Q)\\ u^k &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;L^2_{\rm div}(\Omega))\end{aligned}$$

- We pass to the limit with k for all $\varphi\in C^\infty_c((-\infty,T);\mathcal{V})$

$$-\int_Q u\cdot\partial_t\varphi\, dxdt + \int_Q \chi\cdot\mathbf{D}\varphi\, dxdt = \int_Q f\cdot\varphi\, dxdt - \int_\Omega u_0\cdot\varphi\, dx$$

for all $\varphi\in C^\infty_c((-\infty,T);\mathcal{V})$.

- We want to characterise now the nonlinear term χ, in particular, if

$$\chi \overset{?}{=} \mathbf{S}(t,x,\mathbf{D}u)$$

To this end we will use a monotonicity argument for non-reflexive spaces [3], but first we need an integration by parts formula.

Lemma 1 *If $u\in Y^M_0$, $\chi\in\mathcal{L}_{M^*}(Q)$, $f\in\mathcal{L}_{\underline{m}^*}(Q)$ and*

$$\partial_t u - \operatorname{div}\chi = f \quad \text{in } \mathcal{D}'(Q),$$

then for a.a. $s_0,\ s:\ 0 < s_0 < s < T$

$$\frac{1}{2}\|u(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(s_0)\|^2_{L^2(\Omega)} + \int_{s_0}^s\int_\Omega \chi\cdot\mathbf{D}u\, dxdt = \int_{s_0}^s\int_\Omega f\cdot u\, dxdt.$$

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- To prove the Lemma 1 we take a sufficiently regular test function

$$\varphi^j = u^{\delta,\lambda,\varepsilon}(t,x) := \sigma_\delta * ((\sigma_\delta * \varrho_\varepsilon * u^\lambda(t,x))\ \mathbb{1}_{(s_0,s)})$$

  where

$$u^\lambda(t,x) := u(t,\lambda(x-x_0)+x_0)$$

  ϱ_ε regularisation kernel w.r.t. space and σ_δ w.r.t. time.

- A little problem appears when we want to pass to the limit in nonlinear term

$$\int_{s_0}^s\int_\Omega u\cdot\partial_t(\varphi^j)dxdt = \int_{s_0}^s\int_\Omega \chi\cdot\mathbf{D}\varphi^j dxdt - \int_{s_0}^s\int_\Omega f\cdot\varphi^j dxdt.$$

  - If M^* satisfies Δ_2-condition, then $\chi\in L_{M^*} = E_{M^*}$ and we pass to the limit with weak* convergency.
  - In another case we need to pass to the limit using modular convergence.

Definition 5 *The sequence z^j converges in modular to z in L_M if there exists $\lambda > 0$ such, that $\int_Q M((z^j - z)/\lambda)dxdt \to 0$.*

Remark: We need the limit of $\mathbf{D}\varphi^j$ in weak* topology and in modular topology in L_M to coincide. It means that we need to show

$$Y^M_0 = Z^M_0.$$

$$Y^M_0 = \{\varphi\in L^\infty(0,T;L^2_{\rm div}(\Omega)), \mathbf{D}\varphi\in L_M(Q)\ |\ \exists\ \{\varphi^j\}^\infty_{j=1}\subset C^\infty_c((-\infty,T);\mathcal{V}) : \varphi^j \overset{*}{\rightharpoonup} \varphi \text{ in } L^\infty(0,T;L^2_{\rm div}(\Omega)),\ \mathbf{D}\varphi^j \overset{M}{\longrightarrow} \mathbf{D}\varphi \text{ modularly in } L_M(Q)\}$$

- In case of star-shaped domains it is not so difficult. We construct sequence by squeezing the function to the vintage point and regularising it w.r.t. space and time variable.
- In case of more general domain (at least Lipschitz) need to put much more effort.

  Since we consider the domain with at least Lipschitz boundary, then there exists a countable family of star-shaped Lipschitz domains $\{\Omega_i\}_{i\in J}$ such that $\Omega = \bigcup_{i\in J}\Omega_i$.
  We introduce the partition of unity θ_i with

$$0\le\theta_i\le 1 \quad \theta_i\in C^\infty_c(\Omega_i) \quad \operatorname{supp}\theta_i = \Omega_i \qquad \sum_{i\in J}\theta_i(x) = 1 \quad \text{for } x\in\Omega.$$

  We have to show

$$\mathbf{D}u^j\theta_i + \frac{1}{2}u^j\nabla^T\theta_i + \frac{1}{2}\nabla\theta(u^j)^T = \mathbf{D}(u^j\theta_i) \in L_M((0,T)\times\Omega_i)^{n\times n}.$$

  Unfortunately, the only information we have is $\mathbf{D}u\in\mathcal{L}_M$ and we need to control u in $L_{\overline{m}}$, not only in $L_{\underline{m}}$.
  Therefore we need to prove and apply here the Korn-Sobolev inequality for anisotropic Orlicz spaces

$$\|M(|u|)\|_{L^{\frac{n}{n-1}}(\Omega)} \le C_n\|M(|\mathbf{D}u|)\|_{L^1(\Omega)}. \qquad (1)$$

  and control the spread between $\overline{m}$ and $\underline{m}$.

References

[1] P. Gwiazda and A. Świerczewska–Gwiazda, *Parabolic equations in anisotropic Orlicz spaces with general $\mathcal{N}$–function*, Nonlinear Parabolic Problems: Herbert Amann Festschrift, 2010.

[2] P. Gwiazda, A. Świerczewska–Gwiazda and A. Wróblewska, *Generalized Stokes systems in Orlicz spaces.*

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