Asymptotics behaviour in one dimensional model of interacting particles

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Introduction

We study the asymptotic behaviour of solutions to the one-dimensional initial value problem

$$u_t = \varepsilon u_{xx} + (u\,K' * u)_x \quad \text{for } x \in \mathbb{R},\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \tag{2}$$

where the initial datum $u_0 \in L^1(\mathbb{R})$ is nonnegative and $\varepsilon \geq 0$.

Motivations

Equation (1) arises in study of an animal aggregation as well as in some problems in mechanics of continous media. The unknown function $u = u(x,t)$ represents either the population density of a species or, in the case of materials applications, a particle density. Under our assumptions on interaction kernel $K' = K_x$, equation (1) describe a model in which particles are under some repulsive force.
Notice also that the one-dimensional parabolic-eliptic system of chemotaxis

$$u_t = \varepsilon u_{xx} + (uv_x)_x, \qquad -v_{xx} + v = u, \qquad x \in \mathbb{R},\ t > 0 \tag{3}$$

can be written as equation (1). Indeed, if we put $K(x) = -\frac{1}{2}e^{-|x|}$ into the (1), which is the fundamental solution of the operator $\partial_x^2 + \mathrm{Id}$, one can rewrite the second equation of (3) as $v = K * v$. Here, however, we should recall that we consider repulsive phenomena.

Main assumptions

First of all we assume that the interaction kernel has the form

$$K'(x) = -\frac{A}{2}H(x) + V(x), \tag{4}$$

where, H is the classical Heaviside function given by the formula:

$$H(x) := \begin{cases} -1 & \text{for} \quad x < 0, \\ 1 & \text{for} \quad x > 0 \end{cases} \qquad (x \in \mathbb{R})$$

Moreover, we assume that $A \in (0,\infty)$ is a constant and the function V satisfy

$$V \in W^{1,1}(\mathbb{R}) \tag{5}$$

$$\|V_x\|_{L^1} < A. \tag{6}$$

Remark

Notice that, under assumptions on function $V(x)$, we have the representation $V(x) = \int_{-\infty}^{x} V_y(y)\,dy$. Hence, we get immediately that $V \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, $\lim_{|x|\to\infty} V(x) = 0$ and the following estimate $\|V\|_\infty \leq \|V_x\|_1 < A$ hold true.

Recent works (existence)

Karch and Suzuki in their publication[2] showed that the initial value problem (1)-(2) have a unique and gobal-in-time solution for a large calss of initial conditions and interaction kernels. In particular, our assumption imply that $K' \in L^\infty(\mathbb{R})$, hence the kernel K' is mildly singular in the sense stated in [2, Thm 2.5]. In this case, results from [2] can be summarized as follows: for every $u_0 \in L^1(\mathbb{R})$ such that $u_0 \geq 0$, there exists the unique global-in-time solution u of problem (1)-(2) satisfying

$$u \in C\left([0,+\infty),\ L^1(\mathbb{R})\right) \cap C\left((0,+\infty),\ W^{1,1}(\mathbb{R})\right) \cap C^1\left((0,+\infty),\ L^1(\mathbb{R})\right).$$

In addition, the condition $u_0(x) \geq 0$ implies $u(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$.
Moreover we obtain the conservation of the L^1-norm of nonegative solutions:

$$\|u(t)\|_{L^1} = \int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_0(x)\,dx = \|u_0\|_{L^1}.$$

Recent works (asymptotic)

Karch and Suzuki in [1] studied the large time asymptotics of solutions to (1)-(2) under the assumption that $K' \in L^1(\mathbb{R})$. They showed that either the fundamental solution of heat equation or a nonlinear diffusion wave appear in the asymptotic expansion of solutions as $t \to \infty$. We would like to emphasise that, in all those results, a diffusion phenomena play a crucial role in the large time behaviour of solutions to problem (1)-(2).

Theorem (Decays of L^p norm)

Assume that $u = u(x,t)$ is a nonnegative solution to problem (1)-(2) where the interaction kernel satisfy assumptions (4)-(6). Suppose also that $u_0 \in L^1(\mathbb{R})$ is nonegative and $\varepsilon > 0$. Then for every $p \in [1,\infty]$ the following inequality hold true

$$||u(t)||_p \leq (A - ||V_x||_1)^{\frac{1-p}{p}}\,||u_0||_1^{1/p}\,t^{\frac{1-p}{p}} \tag{7}$$

for all $t > 0$.

Primitive of solution

From now on, without loss of generality, we assume that $\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_0(x)\,dx = 1$. Indeed, it suffices to replace u in equation (1) by $\frac{u}{\int_{\mathbb{R}} u_0\,dx}$ and K' by $K' \int_{\mathbb{R}} u_0\,dx$.
Now, let us put

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\,dy - \frac{1}{2}, \tag{8}$$

where $u(x,t)$ is the solution of (1)-(2). Then, we show that the large time behaviour of U is described by a self-similar profile, given by a rarefaction wave, namely, the unique entropy solution of the Riemann problem for the scalar conservation law

$$W^R_t + AW^R W^R_x = 0 \tag{9}$$

$$W^R(x,0) = \frac{1}{2}H(x). \tag{10}$$

It is well-known that this rarefaction wave is given by explicit formula

$$W^R(x,t) := \begin{cases} -\frac{1}{2} & \text{for} \quad x < -\frac{At}{2}, \\ \frac{x}{At} & \text{for} \quad -\frac{At}{2} < x < \frac{At}{2}, \\ \frac{1}{2} & \text{for} \quad x > \frac{At}{2}. \end{cases} \tag{11}$$

Theorem (Convergence towards rarefaction waves)

Let the assumptions of above Theorem hold true and $\|u_0\|_1 = 1$. Assume, moreover, that

$$\int_{-\infty}^{x} u_0(y)\,dy \in L^1(-\infty,0), \quad \textit{and} \quad \int_{-\infty}^{x} u_0(y)\,dy - 1 \in L^1(0,\infty).$$

Then, there exist a constant $C > 0$ such that for every $t > 0$ and each $p \in (1,\infty]$ the following estimate hold true

$$\|U(\cdot,t) - W^R(\cdot,t)\|_p \leq Ct^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\left(\log(2+t)\right)^{\frac{1}{2}\left(1+\frac{1}{p}\right)},$$

where $U = U(x,t)$ is the primitive of solution of problem (1)-(2) given by (8) and $W^R = W^R(x,t)$ is the rarefaction wave qiven by (11).

Corolary

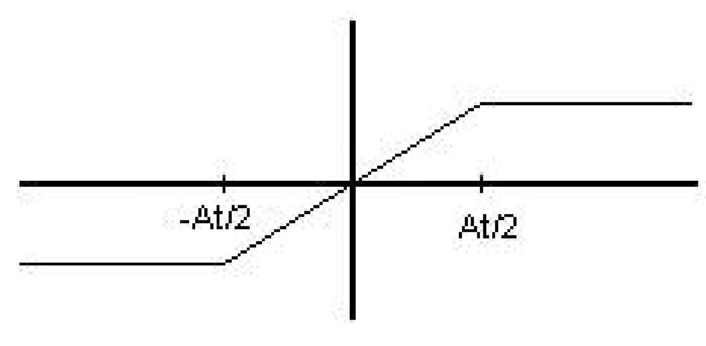
Let the assumptions of the second Theorem hold true. For the solution $u = u(x,t)$ of problem (1)-(2) we define its rescaled version $u^\lambda(x,t) = \lambda u(\lambda x, \lambda t)$ for $\lambda > 0$, $x \in \mathbb{R}$ and $t > 0$. Then, for every test function $\varphi \in C_c^\infty(\mathbb{R})$ and each $t_0 > 0$

$$\int_{\mathbb{R}} u^\lambda(x,t_0)\varphi(x)\,dx \xrightarrow{\lambda\to\infty} -\int_{\mathbb{R}} W^R(x,t_0)\varphi_x(x)\,dx.$$

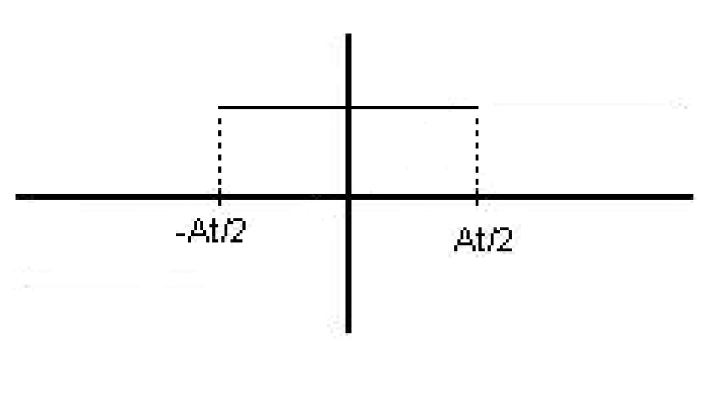
Final result

In other words, for each $t_0 > 0$, the family of functions $u^\lambda(\cdot,t_0)$ converges weakly as $\lambda \to \infty$ to $\left(W^R\right)_x(\cdot,t_0)$.

$W^R(x,t)$



$\left(W^R\right)_x(x,t)$



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