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Control Theory - Master examination March 1st, 2004

Problem 1

a) The eigenvalues characterize the dynamic properties of the system in a decisive manner.

$\det(\lambda I - A) = 0$ where λ_i are the eigenvalues, I is the unity matrix and A is the system matrix.

b) The poles of a system characterize the eigen behavior which the system performs as it's excited with the corresponding input.
All poles are also eigenvalues of A in state-space form but all eigenvalues are not poles. Poles are the zeros of the transfer function G(s) denominator.

$G(s) = C(sI-A)^{-1}B + D$

c) $Rank \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = n$ for all i eigenvalues λ_i of the system A

if λ_i do not fullfill this → this eigenvalue is not observable

d) All eigenvalues are observable but not controllable. Stabilisable: ~~Only~~ those on the right side of the s-plane are controllable, hence can be moved left to the stable side.
⇒ at least One eigenvalue on the left side is not controllable.

e) The eigenvalues of $(A-BK)$ should all have negative real values.
Should be investigated by the characteristic equation.

Problem 2

a) not fully controllable $\rightarrow \lambda_{1,2}$ not controllable (since $\lambda_{1,2}$ fixed)

stabilizable since $\lambda_{1,2}$ are stable and $\lambda_{3,..n}$ are controllable.

b) $$Q_s = [B \quad AB] = \begin{bmatrix} b & 2b \\ 2b & -ab+2ab-20b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & a-20 \end{bmatrix} b$$

$$\det Q_s = a-20-4 = a-24 \quad \neq 0$$

$a \neq 24$ in order to have a fully controllable system

Rosenbrock matrix: $$P(s) = \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s & -1 & -b \\ a & s+10-a & -2b \\ 10c & c & 0 \end{bmatrix}$$

c) $$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 10c & c \\ -ac & ac \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ -a & a \end{bmatrix} c$$

$$\text{rank} \begin{bmatrix} 10 & 1 \\ -a & a \end{bmatrix} = 2$$ full rank $\Rightarrow$ fully observable independent of a and c

$\Rightarrow$ detectable

Problem 2

(d) Ricatti: $A^T P + PA - PBR^{-1}B^T P + Q = 0$

$$A = \begin{bmatrix} 0 & 1 \\ -12 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad Q = \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix} \quad R = 1$$

$$\begin{bmatrix} 0 & -12 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -12 & 2 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 1 \cdot \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix} = 0$$

$P_{12} = P_{21}$

$$\begin{bmatrix} -12P_{12} & -12P_{22} \\ P_{11}+2P_{12} & P_{12}+2P_{22} \end{bmatrix} + \begin{bmatrix} -12P_{12} & P_{11}+2P_{12} \\ -12P_{22} & P_{12}+2P_{22} \end{bmatrix} - \begin{bmatrix} P_{11}^2 + 4P_{11}P_{12} + 4P_{12}^2 & P_{11}P_{12} + 2P_{12}^2 + 2P_{11}P_{22} + 4P_{12}P_{22} \\ P_{11}P_{12} + 2P_{11}P_{22} + 2P_{12}^2 + 4P_{22}P_{12} & P_{12}^2 + 2P_{22}P_{12} + 2P_{12}^2 + 4P_{22}^2 \end{bmatrix} +$$

$$+ \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix} = 0$$

$$-12P_{12} - 12P_{12} - P_{11}^2 - 4P_{11}P_{12} - 4P_{12}^2 + 9 = 0$$

$$-12P_{22} + P_{11} + 2P_{12} - P_{11}P_{12} - 2P_{12}^2 = 0$$
$$- 2P_{11}P_{22} - 4P_{12}P_{22} + 0 = 0$$

$$P_{12} + 2P_{22} + P_{12} + 2P_{22} - P_{12}^2 - 2P_{12}P_{22} - 2P_{12}^2 - 4P_{22}^2 + 3 = 0$$

$\Rightarrow$ 3 equations

Problem 3

(a) $\tilde{A} = V^{-1} A V$ $V = [v_1 \dots v_n]$

v_i of the eigenvectors of A

(b) $\tilde{C} = VC$ observable $\rightarrow$ all columns $\neq 0$ and no dependencies

$\tilde{B} = V^{-1} B$ Controllable $\rightarrow$ all rows $\neq 0$ and no dependencies

(c) Eigenvalues:

$$\det(\lambda I - A) = 0 \Rightarrow \det\begin{pmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 0 & \lambda-1 \end{pmatrix} = (\lambda-1)^3 \Rightarrow \lambda_{1,2,3} = 1$$

$\lambda = 1 \rightarrow$ System is unstable

(d) $Q_B = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$ $C = [0 \ 1 \ 0]$ $CA = [0 \ 1 \ 1]$ $CA^2 = [0 \ 1 \ 2]$

$\det Q_B \neq 0$ $\det Q_B = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 0 \rightarrow$ not observable

(e) $Q_S = [B \ AB \ AB^2]$ $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $AB = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $AB^2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$$\det Q_S = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = (1-2) + 2 - 2 + 0 + 0 = -1 \neq 0$$

$\Rightarrow$ fully controllable

Problem 4

ⓐ $$\begin{vmatrix} \lambda-3 & -4 & -5 \\ -1 & \lambda-2 & -3 \\ -1 & 0 & \lambda-1 \end{vmatrix} = -1\cdot\begin{vmatrix} -4 & -5 \\ \lambda-2 & -3 \end{vmatrix} + (\lambda-1)\begin{vmatrix} \lambda-3 & -4 \\ -1 & \lambda-2 \end{vmatrix} =$$

$$= -\left(12+5(\lambda-2)\right) + (\lambda-1)\left((\lambda-3)(\lambda-2)-4\right) = \lambda^3 - 6\lambda^2 + 2\lambda - 4 = 0$$

Hurwitz: negative coefficients $\Longrightarrow$ not stable

ⓑ $Q_B = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$ $\quad C = [0 \ 0 \ 1] \quad CA = [1 \ 0 \ 1] \quad CA^2 = [4 \ 4 \ 6]$

$$\det Q_B = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 4 & 4 & 6 \end{vmatrix} = 0\cdot\begin{vmatrix} 0 & 1 \\ 4 & 6 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & 1 \\ 4 & 6 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = 4 \neq 0$$

$\Rightarrow$ fully observable

ⓒ $Q_S = [B \ \ AB \ \ A^2B]$ $\quad B = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad AB = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad AB = \begin{bmatrix} 14 \\ 6 \\ 2 \end{bmatrix}$

$$\det Q_S = \begin{vmatrix} -1 & 2 & 14 \\ 0 & 2 & 6 \\ 1 & 0 & 2 \end{vmatrix} = -1\cdot\begin{vmatrix} 2 & 6 \\ 0 & 2 \end{vmatrix} - 0\cdot\begin{vmatrix} 2 & 14 \\ 0 & 2 \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & 14 \\ 2 & 6 \end{vmatrix} =$$

$$= -4 + 12 - 28 = -20 \neq 0$$ $\Rightarrow$ fully controllable

Problem 4

(d) Yes. The system is fully controllable, hence stabilizable.

(e) charcteristic equation of an observer : $\det(\lambda I - A + LC) = 0$

$$A-LC = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} [0 \ 0 \ 1] = \begin{bmatrix} 3 & 4 & 5-\ell_1 \\ 1 & 2 & 3-\ell_2 \\ 1 & 0 & 1-\ell_3 \end{bmatrix}$$

$$\det(\lambda I - A + LC) = \begin{vmatrix} \lambda-3 & -4 & \ell_1-5 \\ -1 & \lambda-2 & \ell_2-3 \\ -1 & 0 & \lambda-1+\ell_3 \end{vmatrix} = (\lambda-3)\begin{vmatrix} \lambda-2 & \ell_2-3 \\ 0 & \lambda-1+\ell_3 \end{vmatrix} - (-1)\cdot\begin{vmatrix} -4 & \ell_1-5 \\ 0 & \lambda-1+\ell_3 \end{vmatrix}$$

$$+(-1)\cdot\begin{vmatrix} -4 & \ell_1-5 \\ \lambda-2 & \ell_2-3 \end{vmatrix} = \lambda^3 + \lambda^2(\ell_3-6) + \lambda(2-5\ell_3+\ell_1) + 4\ell_2 + 2\ell_3 - 2\ell_1 - 4$$

Desired charc. equation: $(\lambda+1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1$

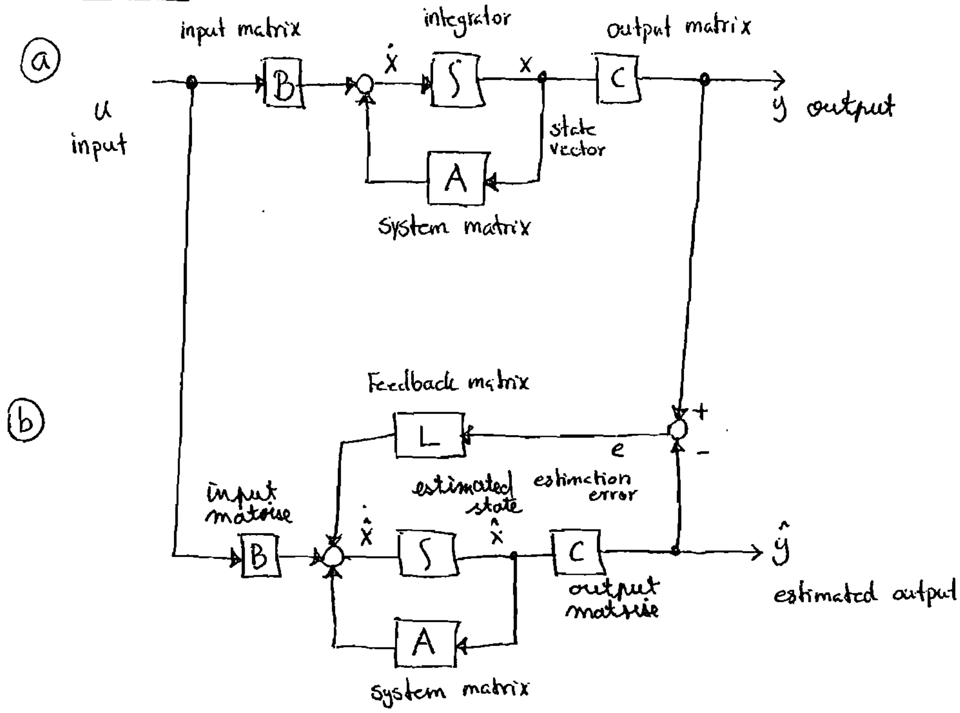
$\ell_3 - 6 = 3 \rightarrow \ell_3 = 9$

$2 - 5\ell_3 + \ell_1 = 3 \rightarrow \ell_1 = 46$

$4\ell_2 + 2\ell_3 - 2\ell_1 - 4 = 1 \rightarrow \ell_2 = 79/4$

$$\Rightarrow L = \begin{bmatrix} 46 \\ 79/4 \\ 9 \end{bmatrix}$$

Problem 5



(c) At least one state of the system has to be measured with a sensor if the system (with this output) is fully observable. A microcontroller is necessary to simulate the ~~system for~~ the estimated states. The input, system and output matrices must be known. Observer dynamics should be faster than the system to be observed:

$$Re\ \lambda_i(A-LC) < Re\ \lambda_i(A)$$

(d) State feedback $u(t) = \underbrace{-K\hat{x}}_{\text{feedback}} + \underbrace{Vw}_{\text{input}}$

System $\dot{x} = Ax + Bu = Ax + BVw - BK\hat{x}$

Observer $\dot{\hat{x}} = A\hat{x} + Bu + LCx - LC\hat{x} = A\hat{x} - BK\hat{x} + BVw + LCx - LC\hat{x}$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} BV \\ BV \end{bmatrix} w$$

Problem 5

(e) $v \approx H v_1 \quad , \dot{v}_1 = S \cdot v_1$

$$\underbrace{\begin{bmatrix} x \\ v_1 \end{bmatrix}^{\cdot}}_{\dot{x}_e} = \underbrace{\begin{bmatrix} A & H \\ 0 & S \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x \\ v_1 \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_e} u \quad , y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C_e} \begin{bmatrix} x \\ v_1 \end{bmatrix}$$

$$\dot{x}_e - \dot{\hat{x}}_e = \dot{e} = (A_e - L C_e) e$$

$$\dot{\hat{x}}_e = A_e \hat{x}_e + B_e u + L C_e (x_e - \hat{x}_e)$$

Problem 6

State vector: $x^T = [\alpha \ \beta \ \dot{\alpha} \ \dot{\beta} \ e]$

Equations of motions:

$$\ddot{\alpha} = -\frac{c}{J_2}(\alpha-\beta) = -\omega^2(\alpha-\beta) = \omega^2\beta - \omega^2\alpha$$

$$\ddot{\theta} = -\frac{c}{iJ_1}(\beta-\alpha) + \frac{e}{J_1} \qquad \theta = i\beta \quad , M = J_1$$

$$\ddot{\beta} = -\frac{c}{i^2J_1}(\beta-\alpha) + \frac{e}{J_1 i} = \frac{e}{J_1 i} - \Omega^2\beta + \Omega^2\alpha$$

$$\dot{e} = -\frac{1}{T}e + \frac{K}{T}u$$

$$\dot{x} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \ddot{\alpha} \\ \ddot{\beta} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\omega^2 & \omega^2 & 0 & 0 & 0 \\ \Omega^2 & -\Omega^2 & 0 & 0 & 1/J_1 i \\ 0 & 0 & 0 & 0 & -\frac{1}{T} \end{bmatrix}}_{A} \begin{bmatrix} \alpha \\ \beta \\ \dot{\alpha} \\ \dot{\beta} \\ e \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{K}{T} \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{C} x$$

Problem 6

$$\begin{vmatrix} \lambda & 0 & -1 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 \\ \omega^2 & -\omega^2 & \lambda & 0 & 0 \\ -\Omega^2 & \Omega^2 & 0 & \lambda & -\frac{1}{J,i} \\ 0 & 0 & 0 & 0 & \lambda+\frac{1}{T} \end{vmatrix} = \left(\lambda+\frac{1}{T}\right) \begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ \omega^2 & -\omega^2 & \lambda & 0 \\ -\Omega^2 & \Omega^2 & 0 & \lambda \end{vmatrix} = \lambda\left(\lambda+\frac{1}{T}\right) \begin{vmatrix} \lambda & 0 & -1 \\ -\omega^2 & \lambda & 0 \\ \Omega^2 & 0 & \lambda \end{vmatrix} -$$

$$-\left(\lambda+\frac{1}{T}\right) \begin{vmatrix} 0 & \lambda & -1 \\ \omega^2 & -\omega^2 & 0 \\ -\Omega^2 & \Omega^2 & \lambda \end{vmatrix} = \lambda^2\left(\lambda+\frac{1}{T}\right) \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - \lambda\left(\lambda+\frac{1}{T}\right) \begin{vmatrix} -\omega^2 & \lambda \\ \Omega^2 & 0 \end{vmatrix} + \lambda\left(\lambda+\frac{1}{T}\right) \begin{vmatrix} \omega^2 & 0 \\ -\Omega^2 & \lambda \end{vmatrix} +$$

$$+\left(\lambda+\frac{1}{T}\right) \begin{vmatrix} \omega^2 & -\omega^2 \\ -\Omega^2 & \Omega^2 \end{vmatrix} = \lambda^2\left(\lambda+\frac{1}{T}\right)\left(\lambda^2+\Omega^2+\omega^2\right)$$

c) $P(\lambda)=0 \Rightarrow \lambda_{1,2}=0 \qquad \lambda_3=-\frac{1}{T} \qquad \lambda_{4,5}=\pm\sqrt{-\Omega^2-\omega^2}=\pm j\sqrt{\Omega^2+\omega^2}$

d) $\lambda_{1,2}=0 \rightarrow$ system is not asymptotic stable

e)

$$Q_S = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\omega^2 K}{J,iT} \\ 0 & 0 & \frac{K}{J,iT} & -\frac{K}{T^2J,i} & -\frac{\Omega^2K}{J,iT}+\frac{K}{T^3J,i} \\ 0 & 0 & 0 & \frac{\omega^2K}{J,iT} & \frac{-\omega^2K}{T^2J,i} \\ 0 & \frac{K}{J,iT} & -\frac{K}{T^2J,i} & -\frac{\Omega^2K}{J,iT}+\frac{K}{T^3J,i} & \frac{\Omega^2K}{T^2J,i}-\frac{K}{T^4J,i} \\ \frac{K}{T} & -\frac{K}{T^2} & \frac{K}{T^3} & \frac{-K}{T^4} & \frac{K}{T^5} \end{bmatrix}$$

Rank $Q_S=5=n$

$\rightarrow$ full rank

$\rightarrow$ fully controllable

f)

$$Q_B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ \Omega^2 & -\Omega^2 & 0 & 0 & \frac{1}{J,i} \\ 0 & 0 & \Omega^2 & -\Omega^2 & -\frac{1}{TJ,i} \\ \Omega^4 & \Omega^4 & 0 & 0 & -\frac{\Omega^2}{J,i}+\frac{1}{T^2J,i} \\ 0 & 0 & \Omega^4 & \Omega^4 & \frac{\Omega^2}{TJ,i}-\frac{1}{T^3J,i} \end{bmatrix}$$

Rank $Q_B=5=n$

$\rightarrow$ full rank $\rightarrow$ fully observable