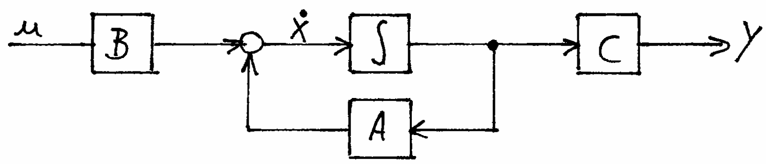
Problem 1

(a)



A system matrix
B input matrix
C output matrix

(b)

$$x(t) = e^{At} x_0 \implies y(t) = C e^{At} x_0$$

(c)

$$\begin{bmatrix} x_m \\ \dot{x}_m \end{bmatrix}^{\bullet} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} x_m \\ \dot{x}_m \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_m \\ \dot{x}_m \end{bmatrix}$$

(d)

$$\det(\lambda I - (A - BK)) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$$

desired eigenvalues: $\lambda_i \quad , i = 1, \dots, n$

$$\prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^n + b_{n-1}\lambda^{n-1} + \dots + b_1\lambda + b_0$$

$$a_i \overset{!}{=} b_i \implies K$$

(e) eigenvalues $\lambda_{1,2} = -3 \pm 4j$

$\implies$ stable

Problem 2

ⓐ $A-LC=\begin{bmatrix}0&1\\-2&-3\end{bmatrix}-\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}c_1&0\end{bmatrix}=\begin{bmatrix}-c_1&1\\-2-c_1&-3\end{bmatrix}$

$\det(\lambda I-(A-LC))=\begin{vmatrix}\lambda+c_1&-1\\2+c_1&\lambda+3\end{vmatrix}=\lambda^2+(3+c_1)\lambda+4c_1+2$

$\Rightarrow \lambda_{1,2}=\pm\sqrt{\left(\frac{3+c_1}{2}\right)^2-2-4c_1}-\frac{3+c_1}{2}=\frac{\pm\sqrt{(c_1-5)^2-24}-(3+c_1)}{2}$

Hurwitz: $a_2=1>0$, $a_1=3+c_1\overset{!}{>}0$, $a_0=4c_1+2\overset{!}{>}0$

$\Rightarrow c_1>-3$ and $c_1>-0{,}5$ $\Rightarrow H_1, H_2>0$

$\Rightarrow$ the observer is stable for $c_1>-0{,}5$

ⓑ $P(s)=\begin{bmatrix}sI-A&-B\\C&D\end{bmatrix}=\begin{bmatrix}s&-1&-b_1\\2&s+3&0\\c_1&0&0\end{bmatrix}$

$m=1$: $\det P(s_0)=b_1c_1(s_0+3)\overset{!}{=}0$

$\Rightarrow s_0=-3$ invariant zero

blocking zeros:

ⓘ $\operatorname{rank}[s_0I-A\ \ -B]<n \Rightarrow s_0$ input decoupling zero

ⓘⓘ $\operatorname{rank}\begin{bmatrix}s_0I-A\\C\end{bmatrix}<n \Rightarrow s_0$ output decoupling zero

ⓘ $[s_0I-A\ \ -B]=\begin{bmatrix}s_0&1&-b_1\\2&s_0+3&0\end{bmatrix}$

$\rightarrow b_1\neq0$: no input decoupling zero

$\rightarrow b_1=0$: $s_{01}=-1$, $s_{02}=-2$ are input decoupling zeros

ⓘⓘ $\begin{bmatrix}s_0I-A\\C\end{bmatrix}=\begin{bmatrix}s_0&-1\\2&s_0+3\\c_1&0\end{bmatrix}$

$\rightarrow c_1\neq0$: no output decoupling zero

$\rightarrow c_1=0$: $s_{01}=-1$, $s_{02}=-2$ are output decoupling zeros

ⓒ $Q_B=\begin{bmatrix}C\\CA\end{bmatrix}=\begin{bmatrix}c_1&c_2\\-2c_2&c_1-3c_2\end{bmatrix}$

$\det Q_B=c_1(c_1-3c_2)+2c_2^2=(c_1-c_2)(c_1-2c_2)$

$\operatorname{rank}Q_B=2$ if $\det Q_B\neq0$

$\Rightarrow$ observable for $c_1\neq2c_2$ and $c_1\neq c_2$

(d) $A-BK=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}-\begin{bmatrix}-2\\ 3\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}=\begin{bmatrix}2k_1 & 1+2k_2\\ -2-3k_1 & -3-3k_2\end{bmatrix}$

$\det(\lambda I-(A-BK))=\begin{vmatrix}\lambda-2k_1 & -1-2k_2\\ 2+3k_1 & \lambda+3+3k_2\end{vmatrix}$

$=\lambda^2+(3+3k_2-2k_1)\lambda+2-3k_1+4k_2$

desired: $(\lambda+\frac{4}{3})(\lambda+\frac{3}{2})=\lambda^2+\frac{17}{6}\lambda+2$

comparing:

$-3k_1+4k_2+2\overset{!}{=}2 \Rightarrow k_1=\frac{4}{3}k_2$

$3+3k_2-2\cdot(\frac{4}{3}k_2)\overset{!}{=}\frac{17}{6} \Rightarrow k_2=-\frac{1}{2} \Rightarrow k_1=-\frac{2}{3}$

(e) $A-BK=\begin{bmatrix}-\frac{4}{3} & 0\\ 0 & -\frac{3}{2}\end{bmatrix}=A^*$

$Q_B=\begin{bmatrix}C\\ CA^*\end{bmatrix}=\begin{bmatrix}c_1 & c_2\\ -c_1\cdot\frac{4}{3} & -c_2\cdot\frac{3}{2}\end{bmatrix}$

$\det Q_B=-c_1c_2\cdot\frac{3}{2}+c_1c_2\cdot\frac{4}{3}$

$=-\frac{1}{6}c_1c_2$

rank $Q_B=2$ if $\det Q_B\neq 0$

$\Rightarrow$ the controlled system is observable for $c_1\neq 0$ and $c_2\neq 0$

Problem 3

ⓐ

$$\det(\lambda I-A)=\begin{vmatrix}\lambda+1 & 0 & 0\\ 0 & \lambda & -1\\ 0 & a & \lambda\end{vmatrix}$$

$$=(\lambda+1)(\lambda^2+a)=(\lambda+1)(\lambda-i\sqrt{a})(\lambda+i\sqrt{a})$$

$$\Rightarrow \lambda_1=-1 \;,\; \lambda_{2,3}=\pm i\sqrt{a}$$

$$\Rightarrow \begin{cases}\text{boundary stable for } a\geq 0\\ \text{unstable for } a<0\end{cases}$$

eigenvectors: $Av_i=\lambda_i v_i \Rightarrow (A-\lambda_i I)v_i=0$

$\lambda_1=-1$: $\begin{bmatrix}0&0&0\\0&1&1\\0&-a&1\end{bmatrix}v_1=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow v_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$

$\lambda_2=i\sqrt{a}$: $\begin{bmatrix}-1-i\sqrt{a}&0&0\\0&-i\sqrt{a}&1\\0&-a&-i\sqrt{a}\end{bmatrix}v_2=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow v_2=\begin{bmatrix}0\\1\\i\sqrt{a}\end{bmatrix}$

$\lambda_3=-i\sqrt{a}$: $\begin{bmatrix}-1+i\sqrt{a}&0&0\\0&i\sqrt{a}&1\\0&-a&i\sqrt{a}\end{bmatrix}v_3=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow v_3=\begin{bmatrix}0\\1\\-i\sqrt{a}\end{bmatrix}$

ⓑ $Q_S=\begin{bmatrix}B & AB & A^2B\end{bmatrix}=\begin{bmatrix}0&0&0\\0&b&0\\b&0&-ab\end{bmatrix}$

$\operatorname{rank} Q_S=2<3 \Rightarrow$ not fully controllable

$\operatorname{rank}[\lambda_i I-A \;\; B]=n \Rightarrow \lambda_i$ controllable

$\lambda_1=-1$: $\operatorname{rank}\begin{bmatrix}0&0&0&0\\0&-1&-1&0\\0&a&-1&b\end{bmatrix}<3$

$\Rightarrow \lambda_1$ is not controllable

$\lambda_2=i\sqrt{a}$: $\operatorname{rank}\begin{bmatrix}i\sqrt{a}+1&0&0&0\\0&i\sqrt{a}&-1&0\\0&a&i\sqrt{a}&b\end{bmatrix}=3$ if $b\neq 0$

$\Rightarrow \lambda_2$ is controllable if $b\neq 0$

$\lambda_3=-i\sqrt{a}$: $\operatorname{rank}\begin{bmatrix}-i\sqrt{a}+1&0&0&0\\0&-i\sqrt{a}&-1&0\\0&a&-i\sqrt{a}&b\end{bmatrix}=3$ if $b\neq 0$

$\Rightarrow \lambda_3$ is controllable if $b\neq 0$

(c) $Q_B = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} c & 0 & 0 \\ -c & 0 & 0 \\ c & 0 & 0 \end{bmatrix}$

$\operatorname{rank} Q_B = 1 \Rightarrow$ not fully observable

$\operatorname{rank} \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = n \Rightarrow \lambda_i$ observable

$\lambda_1 = -1$:
$$\operatorname{rank} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & a & -1 \\ c & 0 & 0 \end{bmatrix} = 3 \quad \text{if } a \neq -1 \text{ and } c \neq 0$$

$\Rightarrow \lambda_1$ is observable if $a \neq -1$ and $c \neq 0$

$\lambda_2 = i\sqrt{a}$:
$$\operatorname{rank} \begin{bmatrix} i\sqrt{a} & 0 & 0 \\ 0 & i\sqrt{a} & -1 \\ 0 & a & i\sqrt{a} \\ c & 0 & 0 \end{bmatrix} = 2$$

$\Rightarrow \lambda_2$ is not observable

$\lambda_3 = -i\sqrt{a}$:
$$\operatorname{rank} \begin{bmatrix} -i\sqrt{a} & 0 & 0 \\ 0 & -i\sqrt{a} & -1 \\ 0 & a & -i\sqrt{a} \\ c & 0 & 0 \end{bmatrix} = 2$$

$\Rightarrow \lambda_3$ is not observable

(d) $Q_S = [B \;\; AB \;\; A^2B] = \begin{bmatrix} b_1 & -b_1 & b_1 \\ 0 & b_2 & 0 \\ b_2 & 0 & -a \cdot b_2 \end{bmatrix}$

$\det Q_S = -a b_1 b_2^2 - b_2^2 b_1 = (-a-1) b_2^2 b_1$

$\Rightarrow$ controllable for $a \neq 1$ and $b_2 \neq 0$ and $b_1 \neq 0$

(e) $b_1 = 0$: $A - BK = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ -b_2 k_1 & -a - b_2 k_2 & -b_2 k_3 \end{bmatrix}$

$\det(\lambda I - (A - BK)) = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda & -1 \\ b_2 k_1 & a + b_2 k_2 & \lambda + b_2 k_3 \end{vmatrix}$

$= (\lambda+1)(\lambda^2 + \lambda b_2 k_3 + a + b_2 k_2)$

desired: $(\lambda+1)(\lambda + \frac{10}{3})^2 = (\lambda+1)(\lambda^2 + \frac{20}{3}\lambda + \frac{100}{9})$

comparing:

$b_2 k_3 \overset{!}{=} \frac{20}{3} \quad \Rightarrow k_3 = \frac{20}{3} \frac{1}{b_2}$

$a + b_2 k_2 \overset{!}{=} \frac{100}{9} \quad \Rightarrow k_2 = (\frac{100}{9} - a) \frac{1}{b_2}$

$$\Rightarrow K = [k_1 \quad \frac{1}{b_2}\left(\frac{100}{9} - a\right) \quad \frac{1}{b_2}\frac{20}{3}]$$

The uncontrollable eigenvalue $\lambda_1 = -1$ is included in the set of desired eigenvalues. The remaining eigenvalues are controllable so that they can be shifted to any desired value using K.

The gain k_1 has no influence on the dynamics of the controlled system.

Problem 4

(a)

$$m l \ddot{s} + m(l^2+r^2)\ddot{\theta} + m g l \theta = 2 l F_B \qquad (1)$$

$$m \ddot{s} + m l \ddot{\theta} + d \dot{s} = F_B \qquad (2)$$

$$(2)\cdot l - (1) \Rightarrow -m r^2 \ddot{\theta} + d l \dot{s} - m g l \theta = -F_B l$$

$$\Rightarrow \ddot{\theta} = \frac{1}{m r^2}(l F_B + d l \dot{s} - m g l \theta)$$

$$\text{in}(2) \Rightarrow \ddot{s} = \frac{r^2 - l^2}{m r^2} F_B - \left(\frac{d}{m} + \frac{d l^2}{m r^2}\right)\dot{s} + \frac{g l^2}{r^2}\theta$$

$$\begin{bmatrix} s \\ \theta \\ \dot{s} \\ \dot{\theta} \end{bmatrix}^{\bullet} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{g l^2}{r^2} & -\frac{d}{m} - \frac{d l^2}{m r^2} & 0 \\ 0 & -\frac{g l}{r^2} & \frac{d l}{m r^2} & 0 \end{bmatrix} \begin{bmatrix} s \\ \theta \\ \dot{s} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m} - \frac{l^2}{m r^2} \\ \frac{l}{m r^2} \end{bmatrix} \cdot F_B$$

(b)

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1000 & -101 & 0 \\ 0 & -500 & 50 & 0 \end{bmatrix} \Rightarrow \det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ 0 & -1000 & \lambda+101 & 0 \\ 0 & 500 & -50 & \lambda \end{vmatrix}$$

$$\det(\lambda I - A) = \lambda^2 \begin{vmatrix} \lambda+101 & 0 \\ -50 & \lambda \end{vmatrix} - \lambda \begin{vmatrix} -1000 & \lambda+101 \\ 500 & -50 \end{vmatrix}$$

$$= \lambda^4 + 101\lambda^3 + 500\lambda^2 + 500\lambda$$

Hurwitz: $a_0 = 0 \Rightarrow$ unstable

(c)

$$A - BK = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1000 & -101 & 0 \\ 0 & -500 & 50 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -9{,}9 \\ 5 \end{bmatrix} [k_1 \;\; 120 \;\; -12 \;\; 0{,}8]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9{,}9k_1 & 2188 & -219{,}8 & 7{,}92 \\ -5k_1 & -1100 & 110 & -4 \end{bmatrix} \Rightarrow \det(\lambda I - (A-BK)) = \begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ -9{,}9k_1 & -2188 & \lambda+219{,}8 & -7{,}92 \\ 5k_1 & 1100 & -110 & \lambda+4 \end{vmatrix}$$

$$\det(\lambda I - (A-BK)) = \lambda^2[\lambda^2 + 223{,}8\lambda + 879{,}2 + 39{,}6k_1] - \lambda[240680 - 1100\lambda - 241780] + \lambda[-9{,}9k_1\lambda - 39{,}6k_1 + 39{,}6k_1] + 1[-10890k_1 + 10940k_1]$$

$$= \lambda^4 + 223{,}8\lambda^3 + (1979{,}2 + 29{,}7k_1)\lambda^2 + 1100\lambda + 50k_1$$

Hurwitz: $a_4 = 1 > 0$, $a_3 = 223{,}8 > 0$, $a_1 = 1100 > 0$

$a_2 = 1979{,}2 + 29{,}7k_1 \overset{!}{>} 0 \Rightarrow k_1 > -\frac{1979{,}2}{29{,}7} \approx -66{,}64$

$a_0 = 50k_1 \overset{!}{>} 0 \Rightarrow k_1 > 0$

$$H=\begin{bmatrix} 223,8 & 1100 & 0 & 0 \\ 1 & 1979,2+29,7k_1 & 50k_1 & 0 \\ 0 & 223,8 & 1100 & 0 \\ 0 & 1 & 1979,2+29,7k_1 & 50k_1 \end{bmatrix}$$

$H_1 = a_3 > 0$

$H_2 = 223,8(1979,2+29,7k_1) - 1100 = 441844,96 + 6646,86k_1 \overset{!}{>} 0$

$\Rightarrow k_1 > -\frac{441844,96}{6646,86} \approx -66,47$

$H_3 = 1100 \cdot H_2 - 223,8 \begin{vmatrix} 223,8 & 0 \\ 1 & 50k_1 \end{vmatrix}$

$= 486029456 + 4807224k_1 \overset{!}{>} 0 \Rightarrow k_1 > -\frac{486029456}{4807224} \approx -101,10$

$\Rightarrow k_1 > 0$

(d) desired:

$(\lambda+11,4731-11,3320i)(\lambda+11,4731+11,3320i)(\lambda+0,4269-6,1709i)\cdot$

$(\lambda+0,4269+6,1709i)$

$= (\lambda^2+22,9462\lambda+260,04625)\cdot(\lambda^2+0,8538\lambda+38,26225)$

$= \lambda^4+23,8\lambda^3+317,89997\lambda^2+1100,0007\lambda+9949,9546$

comparing with $\det(\lambda I-(A-BK))$ from part (c):

$23,8 \neq a_3 = 223,8$ for any k_1

$\Rightarrow$ there exists no $K=[k_1\ 120\ -12\ 0,8]$ so that the closed-loop system has the given eigenvalues

(e) Ljapunnov stability: → stable: 1., 4.

→ asymptotic stable: 4.

→ unstable: 2., 3.

state stability: → stable: 4.

→ boundary stable: 1.

→ unstable: 2., 3.

Problem 5

ⓐ $\begin{bmatrix} x \\ \dot{x} \end{bmatrix}^{\bullet} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f \;,\; y = [1 \;\; 0] \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

ⓑ $A = \begin{bmatrix} 0 & 1 \\ -4 & -20 \end{bmatrix} \Rightarrow \det(\lambda I - A) = \lambda(\lambda+20)+4 = \lambda^2 + 20\lambda + 4$

$\Rightarrow \lambda_{1,2} = -10 \pm \sqrt{100-4} = -10 \pm \sqrt{96}$

$\Rightarrow \lambda_1 \approx -0{,}2020 \;,\; \lambda_2 \approx -19{,}7980$

ⓒ eigenvectors: $A v_i = \lambda_i v_i \Rightarrow (A - \lambda_i I) v_i = 0$

$\lambda_1 = -0{,}202: \begin{bmatrix} -0{,}202 & 1 \\ -4 & -19{,}798 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ -0{,}202 \end{bmatrix}$

$\lambda_2 = -19{,}798: \begin{bmatrix} -19{,}798 & 1 \\ -4 & -0{,}202 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ -19{,}798 \end{bmatrix}$

ⓓ ⓘ $B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \Rightarrow Q_s = [B \;\; AB] = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & -\frac{20}{3} \end{bmatrix}$

$\Rightarrow \det Q_s = \frac{1}{9} \Rightarrow \operatorname{rank} Q_s = 2$

$\Rightarrow$ fully controllable

ⓘⓘ $B = \begin{bmatrix} b \\ 0 \end{bmatrix} \Rightarrow Q_s = [B \;\; AB] = \begin{bmatrix} b & 0 \\ 0 & -4b \end{bmatrix}$

$\Rightarrow \det Q_s = -4b^2 \Rightarrow \operatorname{rank} Q_s = 2$ for $b \neq 0$

$\Rightarrow$ fully controllable for $b \neq 0$

ⓔ $\dot{x} = Ax + Bu + Nn \quad , \quad N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$n \approx H \cdot v$

$\dot{v} = S \cdot v$

ⓕ $\begin{bmatrix} x \\ v \end{bmatrix}^{\bullet} = \begin{bmatrix} A & NH \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \;,\; y = [C \;\; 0] \begin{bmatrix} x \\ v \end{bmatrix}$

$$\begin{bmatrix} x \\ \dot{x} \\ v \end{bmatrix}^{\bullet} = \left[\begin{array}{cc|c} 0 & 1 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot H \\ -4 & -20 & \\ \hline 0 & 0 & S \end{array}\right] \cdot \begin{bmatrix} x \\ \dot{x} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \end{bmatrix} f \;,\; y = [1 \; 0 \; 0] \cdot \begin{bmatrix} x \\ \dot{x} \\ v \end{bmatrix}$$

ⓖ $A_e = \begin{bmatrix} 0 & 1 & 0 \\ -4 & -20 & 1 \\ 0 & 0 & 0{,}1 \end{bmatrix} \;,\; C_e = [1 \;\; 0 \;\; 0]$

$\left(A_e \text{ results from the approximation } x \approx 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot v \right.$
$\left. \text{and from the model } \dot{v} = 0{,}1 \cdot v \right)$

$$Q_B = \begin{bmatrix} C_e \\ C_e A_e \\ C_e A_e^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -20 & 1 \end{bmatrix}$$

$\operatorname{rank} Q_B = 3 \Rightarrow$ the extended system is fully observable

$$A_e - L \cdot C_e = \begin{bmatrix} 0 & 1 & 0 \\ -4 & -20 & 1 \\ 0 & 0 & 0{,}1 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \cdot [1 \ 0 \ 0] = \begin{bmatrix} -l_1 & 1 & 0 \\ -4-l_2 & -20 & 1 \\ -l_3 & 0 & 0{,}1 \end{bmatrix}$$

$$\det(\lambda I - (A_e - L C_e)) = \begin{vmatrix} \lambda + l_1 & -1 & 0 \\ 4 + l_2 & \lambda + 20 & -1 \\ l_3 & 0 & \lambda - 0{,}1 \end{vmatrix}$$

$$= (\lambda + l_1)[(\lambda+20)(\lambda-0{,}1)] + [(4+l_2)(\lambda - 0{,}1) + l_3]$$

$$= \lambda^3 + (19{,}9 + l_1)\lambda^2 + (2 + l_2 + 19{,}9 l_1)\lambda + (-0{,}4 - 2l_1 - 0{,}1 l_2 + l_3)$$

desired: $(\lambda+1)(\lambda+2)(\lambda+3) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$

comparing: $19{,}9 + l_1 \stackrel{!}{=} 6 \Rightarrow l_1 = -13{,}9$

$2 + l_2 + 19{,}9 \cdot (-13{,}9) \stackrel{!}{=} +11 \Rightarrow l_2 = 285{,}61$

$-0{,}4 - 2 \cdot (-13{,}9) - 0{,}1 \cdot 285{,}61 + l_3 \stackrel{!}{=} 6 \Rightarrow l_3 = 7{,}161$

Problem 6

(a) $$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & \frac{c}{m} & \lambda + \frac{c}{d} \end{vmatrix}$$

$$= \lambda \left[\lambda^2 + \frac{c}{d}\lambda + \frac{c}{m} \right]$$

$$\Rightarrow \lambda_1 = 0$$

$$\lambda^2 + \frac{c}{d}\lambda + \frac{c}{m} \overset{!}{=} 0 \Rightarrow \lambda_{2,3} = -\frac{c}{2d} \pm \sqrt{\frac{c^2}{4d^2} - \frac{c}{m}}$$

(b) stability check for $\lambda_{2,3}$ with Hurwitz:

$$\lambda^2 + \frac{c}{d}\lambda + \frac{c}{m} \quad \Rightarrow a_2 = 1 > 0,\ a_1 = \frac{c}{d} \overset{!}{>} 0,\ a_0 = \frac{c}{m}$$

$$\Rightarrow a_2, a_1, a_0 > 0 \text{ for } c, d, m > 0 \Rightarrow H_1, H_2 > 0$$

$\Longrightarrow$ with $\lambda_1 = 0$: the system is boundary stable

(c) $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0{,}25 & -2 \end{bmatrix}$$

$b_1 = 1$: $$Q_S = [B \ \ AB \ \ A^2B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & b_2 \\ 1 & 0 & 0 & b_2 & -0{,}25 & -2b_2 \\ 0 & b_2 & -0{,}25 & -2b_2 & 0{,}5 & 3{,}75 b_2 \end{bmatrix}$$

rank $Q_S = 3 \Rightarrow$ fully controllable for $b_1 = 1$

$b_1 = 1, b_2 = 0$: $$Q_S = [B \ \ AB \ \ A^2B] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -0{,}25 \\ 0 & -0{,}25 & 0{,}5 \end{bmatrix}$$

rank $Q_S = 3 \Rightarrow$ fully controllable for $b_1 = 1, b_2 = 0$

$c_2 = 1$:

$$Q_B = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & c_1 \\ 0 & -0{,}25 & -2 \end{bmatrix}$$

rank $Q_B = 3$ for $c_1 \neq 0$

$\Rightarrow$ fully observable for $c_2 = 1$ if $c_1 \neq 0$

(d) $$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0{,}25 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -k_1 & -k_2 & 1-k_3 \\ -1 & -1{,}25 & -3 \end{bmatrix}$$

$$\det(\lambda I - (A - BK)) = \begin{vmatrix} \lambda & -1 & 0 \\ k_1 & \lambda + k_2 & -1 + k_3 \\ 1 & 1{,}25 & \lambda + 3 \end{vmatrix}$$

$$= \lambda\left[(\lambda+k_2)(\lambda+3) - 1{,}25(-1+k_3)\right] + \left[k_1(\lambda+3) - 1\cdot(-1+k_3)\right]$$

$$= \lambda^3 + (3+k_2)\lambda^2 + (3k_2 - 1{,}25k_3 + 1{,}25 + k_1)\lambda + 3k_1 - k_3 + 1$$

desired: $(\lambda+1)(\lambda+2)(\lambda+3) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$

comparing: $3 + k_2 \stackrel{!}{=} 6 \Rightarrow k_2 = 3$

$3\cdot k_1 - k_3 + 1 \stackrel{!}{=} 6 \Rightarrow k_3 = -5 + 3k_1$

$3k_2 - 1{,}25k_3 + 1{,}25 + k_1 \stackrel{!}{=} 11 \Rightarrow 16{,}5 - 2{,}75k_1 = -5{,}5$

$\Rightarrow k_1 = 2,\ k_3 = 1 \quad \Rightarrow K = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(e) $A - LC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0{,}25 & -2 \end{bmatrix} - \begin{bmatrix} 1 & l_1 \\ 1 & l_2 \\ 1 & l_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1-l_1 & 0 \\ -1 & -l_2 & 1 \\ -1 & -0{,}25-l_3 & -2 \end{bmatrix}$

$$\det(\lambda I - (A-LC)) = \begin{vmatrix} \lambda+1 & -1+l_1 & 0 \\ 1 & \lambda+l_2 & -1 \\ 1 & 0{,}25+l_3 & \lambda+2 \end{vmatrix}$$

$$= -(-1)\left[(\lambda+1)(0{,}25+l_3) - 1\cdot(-1+l_1)\right] + (\lambda+2)\left[(\lambda+1)(\lambda+l_2) + 1 - l_1\right]$$

$$= \lambda^3 + (3+l_2)\lambda^2 + (3{,}25 - l_1 + 3l_2 + l_3)\lambda$$
$$+\ 3{,}25 - 3l_1 + 2l_2 + l_3$$

desired: $(\lambda+4)(\lambda+5)(\lambda+6) = \lambda^3 + 15\lambda^2 + 74\lambda + 120$

comparing: $3 + l_2 \stackrel{!}{=} 15 \Rightarrow l_2 = 12$

$3{,}25 - l_1 + 3\cdot(12) + l_3 \stackrel{!}{=} 74 \Rightarrow l_3 = 34{,}75 + l_1$

$3{,}25 - 3l_1 + 2\cdot(12) + (34{,}75 + l_1) \stackrel{!}{=} 120 \Rightarrow l_1 = -29 \quad \Rightarrow l_3 = 5{,}75$

$\Rightarrow L = \begin{bmatrix} 1 & -29 \\ 1 & 12 \\ 1 & 5{,}75 \end{bmatrix}$

(f) $\dot{e} = \dot{x} - \dot{\hat{x}}$

$= Ax + Bu - (A\hat{x} + Bu + L(y - \hat{y}))$

$= (A - LC)e$

$\dot{e} = \begin{bmatrix} -1 & 30 & 0 \\ -1 & -12 & 1 \\ -1 & -6 & -2 \end{bmatrix} e$

(g) $\dot{x} = Ax + Bu \quad , u = -K\hat{x}$

$\Rightarrow \dot{x} = Ax - BK\hat{x} = (A - BK)x + BK(x - \hat{x})$

$$\Rightarrow \begin{bmatrix} x \\ e \end{bmatrix}^{\bullet} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

$$\begin{bmatrix} x \\ e \end{bmatrix}^{\bullet} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & -3 & 0 & 2 & 3 & 1 \\ -1 & -1{,}25 & -3 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 30 & 0 \\ 0 & 0 & 0 & -1 & -12 & 1 \\ 0 & 0 & 0 & -1 & -6 & -2 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

eigenvalues:

$\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$, (resulting from d)

$\lambda_4 = -4$, $\lambda_5 = -5$, $\lambda_6 = -6$ (resulting from e)

seperation principle

h) 2–6 × faster than the dynamics of the controlled system

i)

$$G(s) = C(sI-A)^{-1}B + D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0{,}25 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{1}{s(s^2+2s+0{,}25)} \begin{bmatrix} s^2+2s+0{,}25 & s+2 & 1 \\ 0 & s^2+2s & s \\ 0 & -0{,}25s & s^2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{s(s^2+2s+0{,}25)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ s^2+2s & s \\ -0{,}25s & s^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0{,}75s+2}{s(s^2+2s+0{,}25)} & \frac{s^2+1}{s(s^2+2s+0{,}25)} \\ \frac{s+2}{s^2+2s+0{,}25} & \frac{1}{s^2+2s+0{,}25} \end{bmatrix}$$

$\det G(s) \stackrel{!}{=} 0 \Rightarrow$ (numerator $\stackrel{!}{=} 0$)

$(0{,}75s+2)\cdot 1 - (s^2+1)(s+2) = -s(s^2+2s+0{,}25) \stackrel{!}{=} 0$

$\Rightarrow s_1 = 0$, $s_{2,3} = -1 \pm \sqrt{\frac{3}{4}}$ are transmission zeros

$s_1 = 0$, $s_{2,3} = -1 \pm \sqrt{\frac{3}{4}}$ are poles

j) $Y_2(s) = G_{21}(s)\, U_1(s)$

$$= \frac{s+2}{s^2+2s+0{,}25}\, U_1(s)$$