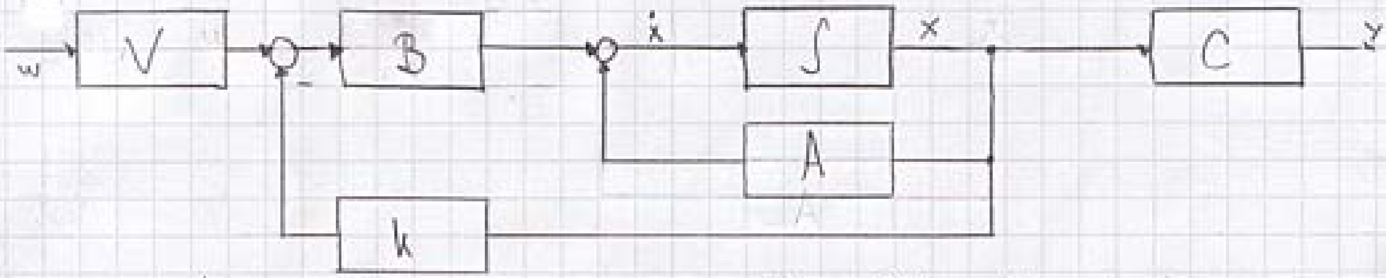
Control Theory (A)

A1

a)



A : system matrix
B : input matrix
C : output matrix
k : feedbackmatrix / feedback gain matrix

V : amplification matrix
x : state vector
y : output / controlled variable
w : reference / command variable

b) characteristic equation: $\det[\lambda I - A] = 0$

$$\begin{vmatrix} \lambda+\delta & -\omega \\ \omega & \lambda+\delta \end{vmatrix} = (\lambda+\delta)^2 + \omega^2 = \lambda^2 + 2\delta\lambda + \delta^2 + \omega^2 = 0$$

$$\lambda_{1/2} = -\delta \pm \sqrt{\delta^2 - \delta^2 - \omega^2} = -\delta \pm j\omega$$

the system is asymptotic stable if $Re\{\lambda_{1/2}\} < 0 \Rightarrow$ asymptotic stable if $\delta > 0$

c) $\dot{x}_1 = x_2$ linear

$$\dot{x}_2 = -k x_1 - c x_1^3 - d x_2 (1 + x_2^2) + u$$

$$\dot{\tilde{x}}_2 = \frac{\partial \dot{x}_2}{\partial x_1}\bigg|_{x_{1e}, x_{2e}} \tilde{x}_1 + \frac{\partial \dot{x}_2}{\partial x_2}\bigg|_{x_{1e}, x_{2e}} \tilde{x}_2 + \frac{\partial \dot{x}_2}{\partial u}\bigg|_{x_{1e}, x_{2e}} u$$

$$= (-k - 3c x_{1e}^2)\tilde{x}_1 + (-d - 3d x_{2e}^2)\tilde{x}_2 + u$$

$$= -k\tilde{x}_1 - d\tilde{x}_2 + u$$

$$\dot{\underline{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix}}_{A} \underline{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

d) Ljapunov matrix equation: $A^T P + P A = -Q$ has for a given symmetric, positive definite matrix Q a symmetric, positive definite solution P, if A is asymptotic stable

Set $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} -\delta & -\omega \\ \omega & -\delta \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} + \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} -\delta & \omega \\ -\omega & -\delta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -\delta p & -\omega p \\ \omega p & -\delta p \end{bmatrix} + \begin{bmatrix} -\delta p & \omega p \\ -\omega p & -\delta p \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$-2\delta p = -1 \Leftrightarrow p = \frac{1}{2\delta} > 0 \Rightarrow \delta > 0$

$p > 0 \Rightarrow$ P positive definit $\Rightarrow \delta > 0$ for A asymptotic stable

Control Theory

②

A1

e) Check controllability

KALMAN $Q_s = [\ B \quad AB \quad A^2B\ ]$

$$= \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank $Q_S = 2 < n = 3$

⇒ not fully controllable

⇒ not possible to define complete dynamics by state control

A2 a) Let λ_i be the eigenvalues of the system $\dot{x} = Ax$

The system is stable if $Re\{\lambda_i\} \leq 0$ and A diagonalizable $\Leftrightarrow$ boundary stable

asymptotic stable if $Re\{\lambda_i\} < 0$

Characteristic equation: $\det[\lambda I - A] = 0$

$$\begin{vmatrix} \lambda+1 & 0 & 1 \\ 0 & \lambda-1 & 0 \\ 1 & 0 & \lambda+1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda+1 & 1 \\ 1 & \lambda+1 \end{vmatrix}$$

$$= (\lambda-1)(\lambda^2+2\lambda+1-1)$$

$$= (\lambda-1)\lambda(\lambda+2)$$

$$\Rightarrow \lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -2$$

$\Rightarrow$ unstable

b) Characteristic equation: $\det[\lambda I - A] = 0$

$$\begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ \frac{c_1}{m_1} & -\frac{c_1}{m_1} & \lambda & 0 \\ -\frac{c_1}{m_2} & \frac{c_1+c_2}{m_2} & 0 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & 0 & -1 \\ -\frac{c_1}{m_1} & \lambda & 0 \\ \frac{c_1+c_2}{m_2} & 0 & \lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & \lambda & -1 \\ \frac{c_1}{m_1} & -\frac{c_1}{m_1} & 0 \\ -\frac{c_1}{m_2} & \frac{c_1+c_2}{m_2} & \lambda \end{vmatrix}$$

$$= \lambda \quad \lambda \begin{vmatrix} \lambda & -1 \\ \frac{c_1+c_2}{m_2} & \lambda \end{vmatrix} - \left[-\lambda \begin{vmatrix} \frac{c_1}{m_1} & 0 \\ -\frac{c_1}{m_2} & \lambda \end{vmatrix} - \begin{vmatrix} \frac{c_1}{m_1} & -\frac{c_1}{m_1} \\ -\frac{c_1}{m_2} & \frac{c_1+c_2}{m_2} \end{vmatrix} \right]$$

$$= \lambda^2\left(\lambda^2 + \frac{c_1+c_2}{m_2}\right) + \lambda\left(\frac{c_1}{m_1}\lambda - 0\right) + \left(\frac{c_1}{m_1}\frac{c_1+c_2}{m_2} - \frac{c_1}{m_2}\frac{c_1}{m_1}\right)$$

$$= \lambda^4 + \left(\frac{c_1+c_2}{m_2} + \frac{c_1}{m_1}\right)\lambda^2 + \frac{c_1 c_2}{m_1 m_2}$$

$\Rightarrow$ unstable according to HURWITZ (coefficients for λ and λ^3 do not exist)

c) Observability matrix (KALMAN)

$$Q_B = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{c_1}{m_1} & \frac{c_1}{m_1} & 0 & 0 \\ \frac{c_1}{m_2} & -\frac{c_1+c_2}{m_2} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ stop here

$$Q_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$\Rightarrow$ observable for all c_1, c_2, m_1, m_2

A2 d)

$$u = [0 \quad 0 \quad k_1 \quad k_2]\, x$$

$$\dot{x} = Ax + B\,[0 \quad 0 \quad k_1 \quad k_2]\, x$$

$$= \underbrace{\left(A + B\,[0 \quad 0 \quad k_1 \quad k_2]\right)}_{A^*} x$$

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{1}{m_1} \\ 0 \end{bmatrix} [0 \quad 0 \quad k_1 \quad k_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{k_1}{m_1} & -\frac{k_2}{m_1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Characteristic equation $\det[\lambda I - A^*] = 0$

$$\begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ \frac{c_1}{m_1} & -\frac{c_1}{m_1} & \lambda + \frac{k_1}{m_1} & \frac{k_2}{m_1} \\ -\frac{c_1}{m_2} & \frac{c_1+c_2}{m_2} & 0 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & 0 & -1 \\ -\frac{c_1}{m_1} & \lambda + \frac{k_1}{m_1} & \frac{k_2}{m_1} \\ \frac{c_1+c_2}{m_2} & 0 & \lambda \end{vmatrix} - \begin{vmatrix} 0 & \lambda & -1 \\ \frac{c_1}{m_1} & -\frac{c_1}{m_1} & \frac{k_2}{m_1} \\ -\frac{c_1}{m_2} & \frac{c_1+c_2}{m_2} & \lambda \end{vmatrix}$$

$$= \lambda\left(\lambda + \frac{k_1}{m_1}\right) \begin{vmatrix} \lambda & -1 \\ \frac{c_1+c_2}{m_2} & \lambda \end{vmatrix} + \lambda \begin{vmatrix} \frac{c_1}{m_1} & \frac{k_2}{m_1} \\ -\frac{c_1}{m_2} & \lambda \end{vmatrix} + \begin{vmatrix} \frac{c_1}{m_1} & -\frac{c_1}{m_1} \\ -\frac{c_1}{m_2} & \frac{c_1+c_2}{m_2} \end{vmatrix}$$

$$= \left(\lambda^2 + \lambda\frac{k_1}{m_1}\right)\left(\lambda^2 + \frac{c_1+c_2}{m_2}\right) + \lambda\left(\lambda\frac{c_1}{m_1} + \frac{c_1 k_2}{m_1 m_2}\right) + \frac{c_1(c_1+c_2)}{m_1 m_2} - \frac{c_1^2}{m_1 m_2}$$

$$= \underbrace{\lambda^4}_{a_4} + \underbrace{\frac{k_1}{m_1}}_{a_3}\lambda^3 + \underbrace{\left(\frac{c_1+c_2}{m_2} + \frac{c_1}{m_1}\right)}_{a_2}\lambda^2 + \underbrace{\left(\frac{k_1}{m_1}\frac{c_1+c_2}{m_2} + \frac{c_1 k_2}{m_1 m_2}\right)}_{a_1}\lambda + \underbrace{\frac{c_1 c_2}{m_1 m_2}}_{a_0}$$

Stability check with HURWITZ:

i) all coefficients exist $a_4, a_2, a_0 > 0$ independent of k_1 and k_2

$$a_3 = \frac{k_1}{m_1} > 0 \Rightarrow \boxed{k_1 > 0}$$

$$a_1 = \frac{k_1}{m_1}\frac{c_1+c_2}{m_2} + \frac{c_1}{m_1 m_2}k_2 > 0 \Leftrightarrow k_1(c_1+c_2) > -c_1 k_2 \Rightarrow \boxed{k_2 > -\frac{c_1+c_2}{c_1}k_1}$$

ii) HURWITZ matrix

$$H = \begin{bmatrix} a_3 & a_1 & 0 & 0 \\ a_4 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \end{bmatrix}$$

$$\det H_1 = a_3 > 0 \Rightarrow k_1 > 0$$

$$\det H_2 = a_3 a_2 - a_1 a_4 = \frac{k_1}{m_1}\left(\cancel{\frac{c_1+c_2}{m_2}} + \frac{c_1}{m_1}\right) - \left(\frac{k_1}{m_1}\cancel{\frac{c_1+c_2}{m_2}} + \frac{c_1 k_2}{m_1 m_2}\right) = \frac{k_1 c_1}{m_1^2} - \frac{c_1 k_2}{m_1 m_2} > 0$$

$$\Leftrightarrow \frac{k_1 c_1 m_2 - k_2 c_1 m_1}{m_1 m_2} > 0 \Leftrightarrow k_1 m_2 - k_2 m_1 > 0 \Rightarrow \boxed{k_2 > \frac{m_2}{m_1}k_1}$$

$$\det H_3 = -a_3 \begin{vmatrix} a_3 & 0 \\ a_4 & a_0 \end{vmatrix} + a_1 \det H_2 = -a_3^2 a_0 + a_1 \det H_2 = -\frac{k_1^2}{m_1^2}\frac{c_1 c_2}{m_1 m_2} + \left(\frac{k_1}{m_1}\frac{c_1+c_2}{m_2} + \frac{c_1 k_2}{m_1 m_2}\right)\left(\frac{k_1 c_1}{m_1^2} - \frac{c_1 k_2}{m_1 m_2}\right)$$

$$= -\frac{k_1^2}{m_1^2}\frac{c_1 c_2}{m_1 m_2} + \frac{k_1^2 c_1 (\cancel{c_1} + c_2)}{m_1^3 m_2} - \frac{k_1 k_2 c_1 (c_1+c_2)}{m_1^2 m_2 \, m_2} + \frac{c_1^2 k_1 k_2}{m_1^3 m_2} - \frac{c_1^2 k_2^2}{m_1^2 m_2^2}$$

$$= \frac{k_1^2 c_1 c_2}{m_1^3 m_2} - \frac{k_1 k_2 c_1 (c_1+c_2)}{m_1^2 m_2^2} + \frac{c_1^2 k_1 k_2}{m_1^3 m_2} - \frac{c_1^2 k_2^2}{m_1^2 m_2^2} > 0 \qquad | \cdot m_1^3 m_2^2 \quad / c_1$$

$$\boxed{\Leftrightarrow k_1^2 c_2 m_2 - k_1 k_2 (c_1+c_2) m_1 + k_1 k_2 c_2 m_2 - c_1 k_2^2 m_1 > 0}$$

$$\det H_4 = \det H_3 \; a_0 > 0$$

A3 a) $F_2(s) = \frac{Y(s)}{U(s)} = \frac{5}{(s+2)(s+a)}$

$\Leftrightarrow (s^2 + 2s + as + 2a)\, Y(s) = 5\, U(s)$

$\mathcal{L}^{-1} \Leftrightarrow \ddot{y}(t) + (2+a)\dot{y}(t) + 2a y(t) = 5 u(t)$ $\qquad y = x_1$

$\Leftrightarrow \ddot{x}_1 + (2+a)\dot{x}_1 + 2a x_1 = 5u$

$\Leftrightarrow \ddot{x}_1 = -(2+a)\dot{x}_1 - 2a x_1 + 5u$ $\qquad \dot{x}_1 = x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2a & -2-a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Eigenvalues of the system

$\det[\lambda I - A] = \begin{vmatrix} \lambda & -1 \\ 2a & \lambda+2+a \end{vmatrix} = \lambda^2 + (2+a)\lambda + 2a = (s+2)(s+a) = 0$

$\Rightarrow$ Eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -a$

c) asymptotic stable if $\mathrm{Re}\{\lambda_i\} < 0 \Rightarrow$ if $a < 0$

d) Controllability matrix $\qquad Q_c = [\,B \quad AB\,]$

$\det Q_c = \begin{vmatrix} 0 & 5 \\ 5 & -10-5a \end{vmatrix} = -25 \neq 0 \Rightarrow \mathrm{rank}\, Q_c = 2 = n$

$\Rightarrow$ fully controllable independent of a

e) Observability matrix $\qquad Q_B = \begin{bmatrix} C \\ CA \end{bmatrix}$

$\det Q_B = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \mathrm{rank}\, Q_B = 2 = n$

$\Rightarrow$ fully observable independent of a

f) $u = -k_1 \dot{y} - k_2 y - k_3 \int_0^t y\, d\tau$ $\qquad \dot{y} = \dot{x}_1 = x_2$

$= -k_1 x_2 - k_2 x_1 - k_3 x_3$ $\qquad y = x_1$, $\int_0^t y\, d\tau = x_3 \Rightarrow \dot{x}_3 = y = x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2a & -2-a & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} (-k_1 x_2 - k_2 x_1 - k_3 x_3)$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -2a-5k_2 & -2-a-5k_1 & -5k_3 \\ 1 & 0 & 0 \end{bmatrix}}_{A_c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A3 g)

$$\det(\lambda I - A_c) = \begin{vmatrix} \lambda & -1 & 0 \\ 2a+5k_2 & \lambda+2+a+5k_1 & 5k_3 \\ -1 & 0 & \lambda \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \lambda+2+a+5k_1 & 5k_3 \\ 0 & \lambda \end{vmatrix} + \begin{vmatrix} 2a+5k_2 & 5k_3 \\ -1 & \lambda \end{vmatrix}$$

$$= \lambda^2(\lambda+2+a+5k_1) + \lambda(2a+5k_2) + 5k_3$$

$$= \lambda^3 + \underbrace{(2+a+5k_1)}_{a_2}\lambda^2 + \underbrace{(2a+5k_2)}_{a_1}\lambda + \underbrace{5k_3}_{a_0}$$

HURWITZ

i) all coefficients $a_i > 0$:

$$2+a+5k_1 > 0 \Rightarrow k_1 > -\frac{2+a}{5}$$

$$2a+5k_2 > 0 \Rightarrow k_2 > -\frac{2a}{5}$$

$$5k_3 > 0 \Rightarrow k_3 > 0$$

ii)

$$H = \begin{bmatrix} 2+a+5k_1 & 5k_3 & 0 \\ 1 & 2a+5k_2 & 0 \\ 0 & 2+a+5k_1 & 5k_3 \end{bmatrix}$$

$$\det H_1 = 2+a+5k_1 > 0 \Rightarrow k_1 > -\frac{2+a}{5}$$

$$\det H_2 = (2+a+5k_1)(2a+5k_2) - 5k_3$$

$$= 4a + 10k_2 + 2a^2 + 5ak_2 + 10ak_1 + 25k_1k_2 - 5k_3$$

$$= 4a + 2a^2 + 10ak_1 + (10+5a)k_2 + 25k_1k_2 - 5k_3 > 0$$

$$\det H_3 = \det H_2 \; 5k_3 > 0 \Rightarrow k_3 > 0$$

h) $\lambda_{1,2,3} = -1$ $\quad (\lambda+1) = \lambda^3 + 3\lambda^2 + 3\lambda + 1$

$$\det(\lambda I - A_c) = \lambda^3 + (2+a+5k_1)\lambda^2 + (2a+5k_2)\lambda + 5k_3$$

Comparing coefficients:

$$3 = 2+a+5k_1 \Rightarrow k_1 = \frac{1-a}{5}$$

$$3 = 2a+5k_2 \Rightarrow k_2 = \frac{3-2a}{5}$$

$$1 = 5k_3 \Rightarrow k_3 = \frac{1}{5}$$