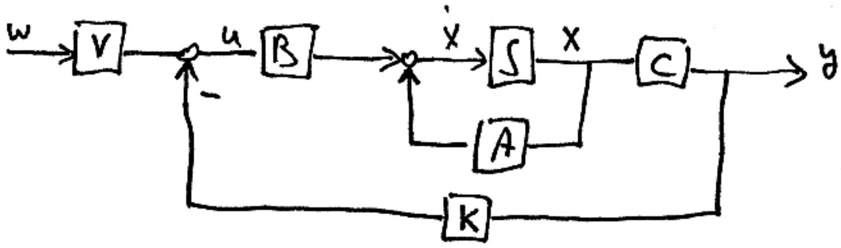
Problem 1

a)



Control law: $y = -Kw + Vw$

A: System matrix
B: Input matrix
C: Output matrix
K: feedback matrix

x: state vector
y: output
u: input
w: reference

b) The poles are the eigenvalues of the controlable/observable part of the system. All poles are eigenvalues but not all eigenvalues are necessary poles.

No. eigenvalues ≥ No. poles

c)
$$\det\begin{pmatrix}\lambda+1 & 0 & 0\\ 0 & \lambda-1 & 0\\ 0 & 1 & \lambda-2\end{pmatrix} = (\lambda+1)(\lambda-1)(\lambda-2) = 0$$

$\lambda_1 = -1$
$\lambda_2 = 1$
$\lambda_3 = 2$ $\Rightarrow$ System is unstable

Hautus

$$\lambda_2: \ \operatorname{rank}\begin{pmatrix}2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & -1 & 0\end{pmatrix} = 3 = n \qquad \lambda_3: \ \operatorname{rank}\begin{pmatrix}3 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 0\end{pmatrix} = 3 = n$$

$\Rightarrow$ can be stabilized

d)
$$G(s) = C(Is-A)^{-1}B = [0 \ \ 1 \ \ 0]\begin{bmatrix}s+1 & 0 & 0\\ 0 & s-1 & 0\\ 0 & -1 & s-2\end{bmatrix}^{-1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{1}{s-1}$$

by definition pole is observable and contr. pole: $p = 1$

pole $\lambda_1 = -1$ is not controlable and not observable

$\lambda_3 = 2$ is controlable and not observable

e) Because of $Re\{\lambda_2, \lambda_3\} > 0$ $\Rightarrow$ system is unstable.

Problem 2

a)

	check whole system	check stability of each mode	numerical calc. of large systems
Hautus	x	+	+
Kalman	+	−	−
Gilbert	+	+	+

b)

$$\det\begin{pmatrix}\lambda-2 & 2 & 0\\ 1 & \lambda-a & 0\\ 0 & 0 & \lambda+1\end{pmatrix} = \left((\lambda-2)(\lambda-a)-2\right)(\lambda+1) = 0$$

$$\lambda_{1,2} = \frac{2+a}{2} \pm \sqrt{\left(\frac{2+a}{2}\right)^2 - 2a + 2}$$

$$\lambda_3 = -1$$

for $a=1$: $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = -1 \Rightarrow$ unstable

Charac. polynom.

$$\lambda^2 - (2+a)\lambda + 2a - 2 = 0$$

all coef. >0:
$-(2+a) > 0 \Rightarrow a < -2$
$2a-2 > 0 \Rightarrow a > 1$
↯ never asympt. stable

$\lambda_1 = 0$ Lyapunov stable, boundary stable
$\lambda_2 = 3$ unstable
$\lambda_3 = -1$ Lyapunov asymptotic stable, stable

c) Controlability

$$\operatorname{rank}(B \;\; AB \;\; A^2B) = \operatorname{rank}\begin{pmatrix} b & 2b & 6b\\ 0 & -b & -3b\\ 0 & 0 & 0\end{pmatrix} = 2$$

for all b not controllable

Observability

$$\operatorname{rank}\begin{pmatrix} C\\ CA\\ CA^2\end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & c & 0\\ 0 & 0 & 2c\\ -c & c & 0\\ 0 & 0 & -2c\\ -3c & 3c & 0\\ 0 & 0 & 2c\end{pmatrix} = 3 = n$$

observable for all $c \neq 0$

Problem 2

d)

$$\det(A - BK) = \begin{vmatrix} \lambda - 2 + k_1 & 2 + k_2 & k_3 \\ 1 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 1 \end{vmatrix}$$

$$= \underbrace{\left((\lambda - 2 + k_1)(\lambda - 1) - (2 + k_2) \right)} \; (\lambda + 1)$$

charact. polynom part 1: 1 part

$$\lambda^2 - (3 - k_1)\lambda - k_2 - k_1 = 0$$

should be

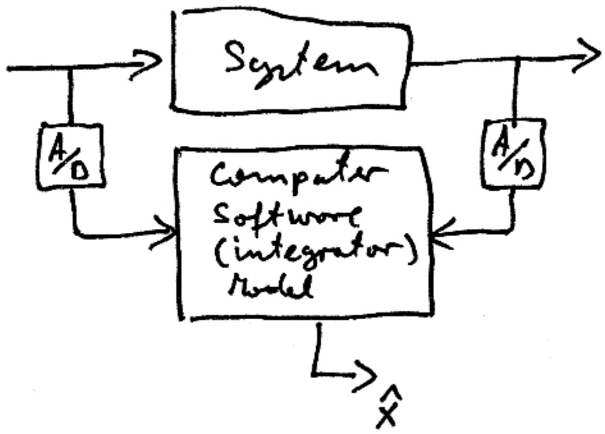
$$(\lambda + 2)(\lambda + 3) = \lambda^2 + 5\lambda + 6$$

$$-(3 - k_1) = 5 \quad \Rightarrow \quad \underline{\underline{k_1 = 8}}$$

$$-k_2 - k_1 = 6 \quad \Rightarrow \quad \underline{\underline{k_2 = -14}}$$

k_3 arbitrary

e)



to realize an observer
- computer (processor)
  A/D converters
- software, operating syst., integrator runtime
- (Integrator must be faster then realtime)

Problem 3

a)

$$m_1 \ddot{z}_1 = c(z_2 - z_1) + d(\dot{z}_2 - \dot{z}_1) + F$$
$$m_2 \ddot{z}_2 = c(z_1 - z_2) + d(\dot{z}_1 - \dot{z}_2)$$

$$z_r = z_1 - z_2$$

$$\ddot{z}_r = -c\left(\frac{1}{m_1} + \frac{1}{m_2}\right) z_r - d\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\dot{z}_r + \frac{1}{m_1} F$$

$$x = \begin{bmatrix} z_r \\ \dot{z}_r \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} 0 & 1 \\ -c\left(\frac{1}{m_1} + \frac{1}{m_2}\right) & -d\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m_1} \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

b)

$$\det \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 1 & 1 & \lambda+1 \end{pmatrix} = \lambda(\lambda(\lambda+1)+1)+1$$
$$= (\lambda^2+1)(\lambda+1) = 0$$
$$\lambda_1 = -1$$
$$\lambda_{2,3} = \pm j$$

Eigenvector: $\lambda_1 = -1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} V_{1a} \\ V_{1b} \\ V_{1c} \end{bmatrix} = \begin{bmatrix} V_{1a} \\ V_{1b} \\ V_{1c} \end{bmatrix} \cdot (-1)$$

$$V_{1b} = -V_{1a}$$
$$V_{1c} = -V_{1b} \qquad V_1 = \begin{bmatrix} a \\ -a \\ a \end{bmatrix}$$
$$-V_{1a} - V_{1b} - V_{1c} = -V_{1c}$$

Eigenvector: $\lambda_2 = j$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} V_{2a} \\ V_{2b} \\ V_{2c} \end{bmatrix} = \begin{bmatrix} V_{2a} \\ V_{2b} \\ V_{2c} \end{bmatrix} j$$

$$V_{2b} = j V_{2a}$$
$$V_{2c} = j V_{2b} \qquad V_2 = \begin{bmatrix} b \\ jb \\ -b \end{bmatrix}$$
$$-V_{2a} - V_{2b} - V_{2c} = j V_{2c}$$

Eigenvector: $\lambda_3 = -j$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} V_{3a} \\ V_{3b} \\ V_{3c} \end{bmatrix} = \begin{bmatrix} V_{3a} \\ V_{3b} \\ V_{3c} \end{bmatrix} \cdot (-j)$$

$$\ldots \qquad V_3 = \begin{bmatrix} b \\ -jb \\ -b \end{bmatrix}$$

Problem 3

5

transformation matrix

$$V = \begin{bmatrix} a & b & 0 \\ -a & 0 & b \\ a & -b & 0 \end{bmatrix} \qquad V^{-1} = \frac{1}{\det V} \begin{bmatrix} b & 0 & b^2 \\ ab & 0 & -ab \\ ab & -2ab & ab \end{bmatrix} = \begin{bmatrix} \frac{1}{2a} & 0 & \frac{b}{2a} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$B_{mod} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2a} & \dots & \\ \frac{1}{2} & & \\ \frac{1}{2} & & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad a = 2$$

~~b arbitrary~~

b arbitrary $\neq 0$

$$V = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

c) controllable (B_{mod} all rows $\neq 0$)

$$C_{mod} = [0 \ -1 \ \ 1] \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} = [4 \ -1 \ -1]$$

observable
all coul. $\neq 0$

d)

$$\lambda I - A + BKC = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} + \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} a [0 \ -1 \ -1]$$

$$\begin{vmatrix} \lambda & -1-ab & ab \\ 0 & \lambda & -1 \\ 1 & 1 & \lambda+1 \end{vmatrix} = \lambda(\lambda^2+\lambda+1) + 1 + ab - 2ab$$

Charact. polyn. $\quad \lambda^3 + \lambda^2 + \lambda(1-ab) + 1 + ab = 0$

$$H = \begin{pmatrix} 1 & 1+ab & 0 \\ 1 & 1-ab & 0 \\ 0 & 1 & 1+ab \end{pmatrix}$$

i) $1-ab > 0 \Rightarrow ab < 1$

ii) $1+ab > 0 \Rightarrow ab > -1$

$\det H_1 > 0$ ✓

$\det H_2 > 0$

$\Rightarrow (1-ab) - (1+ab) > 0$

$-2ab > 0$

$ab < 0$

$\det H_3 > 0$ ✓

for all $\underline{\underline{-1 < ab < 0}}$ stab.

Problem 3

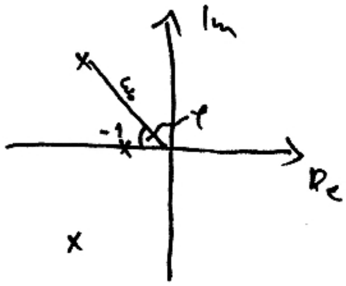
e)

$\det(\lambda I - A + BK$

$$= \det\begin{pmatrix} \lambda + K_1 & -1 + K_2 & K_3 \\ 0 & \lambda & -1 \\ 1 & 1 & \lambda + 1 \end{pmatrix} = (\lambda + K_1)(\lambda^2 + \lambda + 1) + (1 - K_2)\lambda K_3$$

Charact. polyn.

$$\lambda^3 + \lambda^2(1 + K_1) + \lambda(1 + K_1 - K_3) + K_1 + 1 - K_2 = 0$$



$\cos\varphi = D$

$D = \frac{\sqrt{2}}{2} \Rightarrow \varphi = 45°$

$\omega_0 = \sqrt{Re^2 + Im^2}$

$\lambda_{1,2} = -2 \pm 2j$

$\lambda_3 = -1$

$$(\lambda + 1)(\lambda + 2 + 2j)(\lambda + 2 - 2j) = \lambda^3 + 5\lambda^2 + 12\lambda + 8$$

coeff.

$1 + K_1 = 5 \Rightarrow K_1 = 4$

$5 - K_3 = 12 \Rightarrow K_3 = -7$

$5 - K_2 = 8 \Rightarrow K_2 = -3$

$K = [4 \; -3 \; -7]$

f) Resonance $f_r = \frac{\omega}{2\pi} = \frac{\sqrt{8}}{2\pi}$ Hz

To avoid large oscillation the frequency f should be

$f > f_r$ or $f < f_r$