Problem 1

a) $\dot{x} = Ax + Bu$ x state vector A system ~~matrix~~

$y = Cx + Du$ u input vector B input matrix

y output vector C output matrix

D direct transmission matrix

$x(0) = x_0$

b) $Y(s) = \underbrace{[C(sI - A)^{-1}B + D]}_{G(s)} U(s)$ $G(s)$ transfer function matrix

c) $G(s) = [C(sI-A)^{-1}B + D]$

$$= [2 \;\; 0] \begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + [0 \;\; 0]$$

$$= \frac{1}{s(s+1)} [2 \;\; 0] \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{s(s+1)} [2s+2 \;\; 2] \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{s(s+1)} [2s+2+2 \quad -2]$$

$$= \begin{bmatrix} \frac{2(s+2)}{s(s+1)} & \frac{-2}{s(s+1)} \end{bmatrix}$$

d) characteristic equation $\det[\lambda I - A]$

$$= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda-1 & -a \\ -3 & 0 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda-1 & -a \\ 0 & \lambda \end{vmatrix} - 3 \begin{vmatrix} -1 & 0 \\ \lambda-1 & -a \end{vmatrix} = \lambda^2(\lambda-1) - 3a = 0$$

$$\lambda^3 - \lambda^2 - 3a = 0$$

according to HURWITZ necessary condition: all $a_i > 0$ $\Rightarrow$ not stable for all a

e) $C\tilde{x}_i \neq 0$

$$[a \;\; -2] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = a - 4 \neq 0 \Rightarrow a \neq 4$$

$$[a \;\; -2] \begin{bmatrix} -3 \\ a \end{bmatrix} = -3a - 2a = -5a \neq 0 \Rightarrow a \neq 0$$

$\Rightarrow$ If $a \neq \{4, 0\}$, the eigenmotion is observable

Problem 2

a) ROSENBROCK ~~system matrix~~: $P(s) = \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}$

$$P(s) = \begin{bmatrix} s & -1 & 0 & -1 & 0 & 0 \\ 0 & s-1 & -a & -1 & 1 & 0 \\ -3 & 0 & s & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) max rank $P(s) = 4$

$P(s)$ has full rank for all s and a as it can be seen from the 4 rows

$\Rightarrow$ no invarient zeros

c) output decoupling zeros (n is $\dim(x)$ and s_0 eigenvalues of A)

rank $\begin{bmatrix} s_0 I - A \\ C \end{bmatrix} < n \quad \Rightarrow$ s_0 output decoupling zero

input decoupling zeros

rank $\begin{bmatrix} s_0 I - A & -B \end{bmatrix} < n \quad \Rightarrow$ s_0 input decoupling zero

d) Design of L:

i) choose Q and R

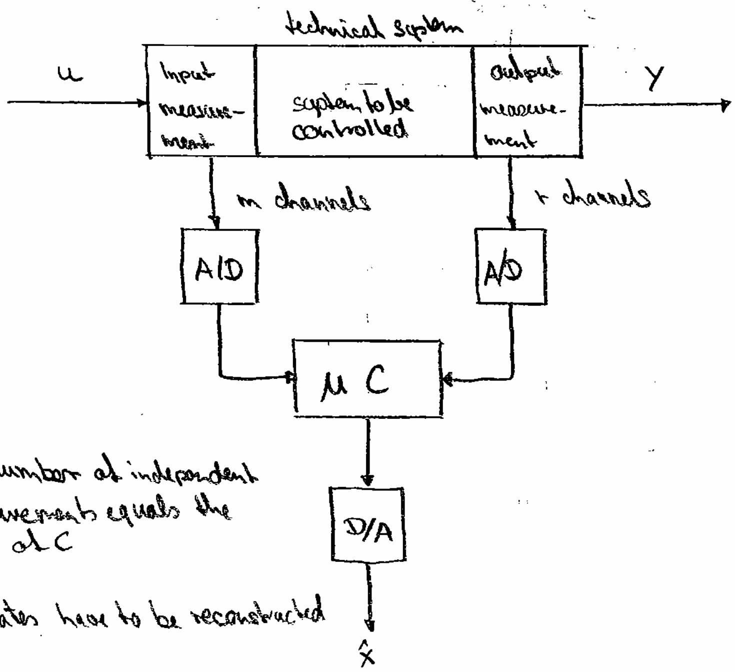
ii) solve Ricatti equation: $AP + PA^T - PC^T R^{-1} C P + Q = 0$

for P

iii) $L^T = R^{-1} C P$

Condition: the system has to be observable, e.g. check with KALMAN: rank $\begin{bmatrix} C \\ CA \\ \vdots \\ C A^{n-1} \end{bmatrix} = n$

e)



The number of independent measurements equals the rank of C

n states have to be reconstructed

μC simulate:

$$\dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x})$$
$$= (A - LC)\hat{x} + Bu + LCx$$

A	$n \times n$	$x, \hat{x}$	$n \times 1$
B	$n \times m$		
C	$r \times n$	y	$r \times 1$
		u	$m \times 1$
L	$n \times r$		

Aufgabe 3

a) $\ddot{y}(t) + 0{,}32b\,\dot{y}(t) + 9y(t) = \int_0^t u(\tau)\,d\tau$

$\Leftrightarrow \dddot{y}(t) + 0{,}32b\,\ddot{y}(t) + 9\dot{y}(t) = u(t)$

$\Rightarrow$ controllable standard form with $x_1 = y$, $x_2 = \dot{y}$ and $x_3 = \ddot{y}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & -0{,}32b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Eigenvaloes:

charac. Polynom: $\lambda^3 + 0{,}32b\,\lambda^2 + 9\lambda = 0$

$\Leftrightarrow \lambda(\lambda^2 + 0{,}32b\,\lambda + 9) = 0$

$\Rightarrow \lambda_1 = 0 \wedge \lambda_{2,3} = -\frac{0{,}32b}{2} \pm \sqrt{\frac{0{,}32^2 b^2}{4} - 9}$

b) System is stable but not asymptotic stable, if $\mathrm{Re}\{\lambda_i\} \leq 0$

$\Rightarrow \mathrm{Re}\{\lambda_1\} \leq 0$, okay ✓

$\mathrm{Re}\{\lambda_{2,3}\} \leq 0$, depends on b

$\Rightarrow \mathrm{Re}\{\lambda_{2,3}\} = -\frac{0{,}32b}{2} \leq 0$

$\Leftrightarrow b \geq 0$

Real Eigenvalues:

$\Rightarrow \frac{0{,}32^2 b^2}{4} - 9 \geq 0 \Leftrightarrow b \cdot 0{,}32 \geq \sqrt{36}$

$\Leftrightarrow b > +\frac{6}{0{,}32} \vee b < -\frac{6}{0{,}32}$

$\Leftrightarrow b > +18{,}75 \vee b < -18{,}75$

c) Controllability

GILBERT: $\tilde{B} = V^{-1} \cdot B$

· controllable, if all rows are unequal zero

Eigenvalues:

charac. Polynom: $\lambda^3 - 0{,}5\lambda^2 - 14\lambda = 0$

$\Leftrightarrow \lambda(\lambda^2 - 0{,}5\lambda^2 - 14) = 0$

$\Leftrightarrow \lambda(\lambda - 4)(\lambda + 3{,}5) = 0$

$\Rightarrow \lambda_1 = 0 \,,\; \lambda_2 = 4 \,,\; \lambda_3 = -3{,}5$

Eigenvector matrix:

$V = [v_1 \; v_2 \; v_3]$ with Eigenvectors v_i

Eigenvectors:

$A v_i = \lambda_i v_i \Leftrightarrow (\lambda_i I - A) v_i = 0$

$$(\lambda I - A) = \begin{bmatrix} \lambda_i & -1 & 0 \\ 0 & \lambda_i & -1 \\ 0 & -14 & \lambda_i - 0{,}5 \end{bmatrix}$$

$$\lambda_1 = 0 \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & -14 & -0{,}5 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 4 \Rightarrow \begin{bmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & -14 & 3{,}5 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1/4 \\ 1 \\ 4 \end{bmatrix}$$

$$\lambda_3 = -3{,}5 \Rightarrow \begin{bmatrix} -3{,}5 & -1 & 0 \\ 0 & -3{,}5 & -1 \\ 0 & -14 & -4 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_3 = \begin{bmatrix} 2/7 \\ -1 \\ 3{,}5 \end{bmatrix}$$

$$\Rightarrow V = \begin{bmatrix} 1 & 1/4 & 2/7 \\ 0 & 1 & -1 \\ 0 & 4 & 3{,}5 \end{bmatrix} \Rightarrow V^{-1} = \begin{bmatrix} 1 & 0{,}036 & -0{,}071 \\ 0 & 0{,}467 & 0{,}133 \\ 0 & -0{,}533 & 0{,}133 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} 1 & 0{,}036 & -0{,}071 \\ 0 & 0{,}467 & 0{,}133 \\ 0 & -0{,}533 & 0{,}133 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -0{,}286 \\ 0{,}533 \\ 0{,}533 \end{bmatrix} \Rightarrow \text{controllable}$$

d) Observability

GILBERT: $\tilde{C} = C \cdot V$

observable, if all columns are unequal zero

$$\tilde{C} = [0 \; 1 \; 0] \begin{bmatrix} 1 & 1/4 & 2/7 \\ 0 & 1 & -1 \\ 0 & 4 & 3{,}5 \end{bmatrix} = [0 \; 1 \; -1]$$

=> not observable

e) State control is principally possible, because all eigenvalues are controllable, so the system is controllable.
State control cannot be realized, because full state is not measurable.

A state observer cannot be used, because the system is not fully observable.

The control cannot be improved by an observer, beause of reason mentioned above.

Problem 4

a) Linear System $\Rightarrow x_g = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Eigenvalues: $\det(\lambda I - A) = 0$

$$\Leftrightarrow \begin{vmatrix} \lambda & 0 & -125/16 \\ -8 & \lambda & -5/4 \\ 0 & -16 & \lambda+2 \end{vmatrix} = \lambda^3 + 2\lambda^2 - 20\lambda - 1000 = 0$$

$$\Leftrightarrow (\lambda^2 + 12\lambda + 100)(\lambda - 10) = 0$$

$$\Leftrightarrow \underline{\lambda_1 = 10} \wedge \lambda_2 = -6 + j8 \wedge \lambda_3 = -6 - j8$$

$\hookrightarrow x_g$ is unstable

b) KALMAN

$$Q_B = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 128 & -16 \\ 1024 & -256 & 192 \end{bmatrix}$$

As it can be seen from Q_B, Q_B has full rank, so the system is fully observable.

$\Rightarrow$ All states can be estimated (reconstructed) by the measurements.

c) • 2 inputs

• Adjust the input matrix B so that only the 1st input is used : $B^* = \begin{bmatrix} -125/128 \\ 0 \\ 0 \end{bmatrix}$

KALMAN

$$Q_S = [B^* \ AB^* \ A^2B^*] = \begin{bmatrix} -125/128 & 0 & 0 \\ 0 & -125/16 & 0 \\ 0 & 0 & -125 \end{bmatrix}$$

Q_S has full rank, so the system is fully controllable.
=> The eigenvalues can be shifted arbitrarily.

d) 1. $\lambda_{1,desired} = -10$

2. $\lambda_{2/3} = -\omega_0 d \pm \omega_0 \sqrt{d^2-1} \Rightarrow 0 < d < 1$

$\lambda_3: \tan \phi_{d,3} = \dfrac{-\omega_0 \sqrt{1-d^2}}{-\omega_0 d} = \dfrac{\sqrt{1-d^2}}{d} = \dfrac{4}{3}$

$\Leftrightarrow$ (0 < d < 1) $d_3 = \frac{3}{5} \Rightarrow d_{desired} = \frac{4}{3} \cdot \frac{3}{5} = \frac{4}{5}$

$\Rightarrow \lambda_{2/3, desired} = -\omega_0 \frac{4}{3} \cdot \frac{3}{4} d \pm \frac{3}{4} \omega_0 \sqrt{\left(\frac{4}{3} d\right)^2 - 1}$

$= -6 \pm j\frac{9}{2}$

Calculation of $K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$: only the 2nd input is used

$\Rightarrow B^{*1} = \begin{bmatrix} 0 \\ -125/16 \\ 0 \end{bmatrix}$

$|\lambda I - (A - B^{*1} K^T)| = \lambda^3 + \lambda^2 \left(2 - \frac{125}{16} k_2\right) + \lambda\left(-\frac{125}{8} k_2 - 20 - 125 k_3\right) - 1000 - \frac{125^2}{16} k_1$

desired characteristic polynom: $(\lambda+10)(\lambda+6+j\frac{9}{2})(\lambda+6-j\frac{9}{2})$

$= \lambda^3 + 22\lambda^2 + 176{,}25\lambda + 562{,}5$

comparison of coefficients : $\left.\begin{array}{l} k_1 = -1{,}6 \\ k_2 = -64/25 \\ k_3 = -1{,}25 \end{array}\right\} K = \begin{bmatrix} -1{,}6 \\ -64/25 \\ -1{,}25 \end{bmatrix}$

e) 1. $\lambda = -10$ is the fastest.

2. Eigenvalues of the observer should be "to the left of" the eigenvalues of the observed system.

3. desired characteristic polynom:

$$(\lambda+20)(\lambda+30)(\lambda+40) = \lambda^3 + 90\lambda^2 + 2600\lambda + 24000$$

$$L_0^T = \begin{bmatrix} 24000 & 2600 & 90 \end{bmatrix} - \begin{bmatrix} -1000 & -20 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 25000 & 2620 & 88 \end{bmatrix}$$

$$L = T_0 L_0 = \begin{bmatrix} 3125/128 \\ 655/32 \\ 11 \end{bmatrix}$$

f) • 3 eigenvalues

• $$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -B^{**}K^T \\ LC & A - LC - B^{**}K^T \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 125/16 & 0 & 0 & 0 \\ 8 & 0 & 5/4 & -25/2 & -20 & -625/64 \\ 0 & 16 & -2 & 0 & 0 & 0 \\ 0 & 0 & 3125/16 & 0 & 0 & -375/2 \\ 0 & 0 & 655/4 & -9/2 & -20 & -11025/64 \\ 0 & 0 & 88 & 0 & 16 & -90 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

• Eigenvalues: -40; -30; -20; -10; $-6 \pm \frac{8}{2} j$