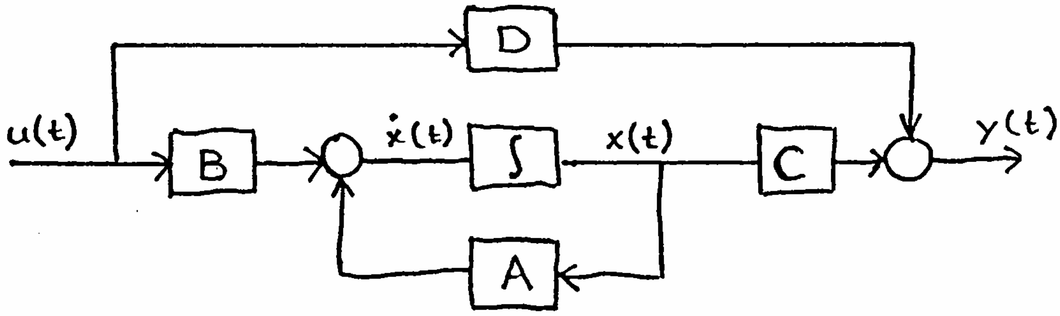
1a)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = x_0$$

x – state vector
u – Input vector
y – Output vector
A – System matrix
B – Input matrix
C – Output matrix
D – Direct transmission matrix

Illustration:



1b) Transfer function:

$$G(s) = C(sI-A)^{-1}B + D$$

Rosenbrock System matrix:

$$P(s) = \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} \quad ; \quad P(s)\begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ Y(s) \end{bmatrix}$$

1c)

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda-1 & 0 \\ -3 & 0 & \lambda-a \end{vmatrix} = (\lambda-a)\cdot\begin{vmatrix} \lambda & -1 \\ 0 & \lambda-1 \end{vmatrix} =$$

$$= \lambda(\lambda-a)(\lambda-1) \Rightarrow \lambda_1 = 0 \quad \lambda_2 = a \quad \lambda_3 = 1$$

positive eigenvalue $\lambda_3 = 1$ independent of $a \Rightarrow$ not stable.

1 d)

$$G(s) = C(sI-A)^{-1}B + D =$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & -2 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \frac{1}{s(s+2)} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 2 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{2}{s} & -\frac{1}{s} \end{bmatrix} \Rightarrow$$

$$G(j\omega) = \begin{bmatrix} \frac{2}{j\omega} & -\frac{1}{j\omega} \end{bmatrix}$$

1 e)
* If decoupling zeros exist, it means that there exist more eigenvalues than poles. Hence, BIBO stable does not mean state stable.
* If decoupling zeros do <u>not</u> exist, it means that all eigenvalues are poles. Hence, BIBO stable means state stable.

2a)

$$P(s) = \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s+1 & 0 & 0 & -1 & 0 \\ 0 & s-2 & 0 & 0 & 0 \\ 0 & 0 & s+3 & 0 & -b \\ c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

b) <u>State stability:</u>

Since system is in diagonal canonical form, the eigenvalues can be seen in the system matrix directly:

$\lambda_1 = -1 \quad \lambda_2 = 2 \quad \lambda_3 = -3.$

System is not stable due to positive eigenvalue $\lambda_2 = 2$

<u>BIBO-stability:</u>

$$G(s) = C(sI-A)^{-1}B + D = \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s-2 & 0 \\ 0 & 0 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & b \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{c}{s+1} & 0 \\ 0 & \frac{b}{s+3} \end{bmatrix}$$

No positive poles $\Rightarrow$ System is BIBO-stable.

c)

$$P(z) = \begin{bmatrix} zI-A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} z+1 & 0 & 0 & -1 & 0 \\ 0 & z-2 & 0 & 0 & 0 \\ 0 & 0 & z+3 & 0 & -b \\ c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$\mathrm{rank}(P(z)) = 4 < n$ for $z = 2$ $(c \neq 0$ and $b \neq 0) \Rightarrow$

$\Rightarrow$ Invariant zero $z_0 = 2$.

d) Since the state space representation is in diagonal canonical form, it can be seen directly that the "second" eigenvalue $\lambda_2 = 2$ is not controllable (2^{nd} row in the B-matrix = 0) and not observable (2^{nd} column in the C-matrix = 0). This means there exists an input/output decoupling zero $z_0 = 2$.

e) The positive eigenvalue $\lambda_2 = 2$ should be controllable in order to stabilize the system. From d) it can be seen that this is not the case. Hence, system is not stabilizable.

$\lambda_2 = 2$ is not observable (see d)) $\Rightarrow$

Full state feedback using estimated states. not possible.

Problem 3.

a) Eigenvalues: $\det(\lambda I - A) = 0$

$$\det \begin{vmatrix} \lambda+1 & 0 & -1 \\ 0 & \lambda+1 & -1 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda+1)(\lambda+1)(\lambda-1) = 0$$

$\lambda_1 = -1 \qquad \lambda_2 = -1 \qquad \lambda_3 = 1$

$\because \ Re\{\lambda_3\} = 1 > 0 \quad \Rightarrow$ The system is unstable.

b) transfer function matrix:

$$G(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 2 & 4 & -2 \end{bmatrix} \begin{bmatrix} s+1 & 0 & -1 \\ 0 & s+1 & -1 \\ 0 & 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$= \frac{1}{(s+1)^2(s-1)} \begin{bmatrix} 2 & 4 & -2 \end{bmatrix} \begin{bmatrix} (s+1)(s-1) & 0 & (s+1) \\ 0 & (s+1)(s-1) & +(s+1) \\ 0 & 0 & (s+1)^2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{(s+1)^2(s-1)} \begin{bmatrix} 2 & 4 & -2 \end{bmatrix} \begin{bmatrix} (s+1) & 0 \\ +(s+1) & s^2-1 \\ (s+1)^2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2(s-2)(s+1)}{(s+1)^2(s-1)} & \frac{4(s+1)(s-1)}{(s+1)^2(s-1)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2(s-2)}{(s+1)(s-1)} & \frac{4}{(s+1)} \end{bmatrix}$$

$\Rightarrow$ There are two poles $\quad s_1 = -1 \quad s_2 = 1$

The system has 3 eigenvalues.

~~$\Rightarrow$ The system has more eigenvalues.~~

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c) Kalman:

$$Q_S = [B \quad AB \quad A^2B]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$\text{rank } Q_S = 3 = n$

$\Rightarrow$ The system is fully controllable.

d) from b) the system has more eigenvalues than poles.

i) $\Rightarrow$ the system has decoupling zeros.

from c) the system ~~has~~ is fully controllable.

ii) $\Rightarrow$ the system has no input-decoupling zeros.

from i) and ii) $\Rightarrow$ the system has output-decoupling zeros.

$\Rightarrow$ The system is not fully observable.

e) $C_{new} = \begin{bmatrix} 2 & 4 & -2 \\ 2 & c & -3 \end{bmatrix}$

Gilbert: calculate eigenvectors: $(\lambda_i I - A) v_i = 0$

for $\lambda_{1,2} = -1$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix} v_i = 0 \Rightarrow v_{11} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

the system is diagonalizable.

for $\lambda_3 = 1$

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} v_3 = 0 \Rightarrow v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow V = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\tilde{C}_{new} = C_{new} V = \begin{bmatrix} 2 & 4 & -2 \\ 2 & c & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 2 & c & c-4 \end{bmatrix}$$

Gilbert:

For $\lambda_{1,2} = -1$ (The multi-eigenvalues is observable when the system matrix is diagonalizable and the corresponding columns in $\tilde{C}$ of the multi-eigenvalue ~~should be~~ is linear independent.)

In this case, the system is diagonalizable.

$\lambda_{1,2} = -1$ is observable for $c \neq 4$

~~For~~ $\lambda_3 = 1$ is observable for any c

$\Rightarrow$ The system is fully observable for $c \neq 4$.

f) $Q = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Design the gain matrix L for an optimal observer:

i) choose Q, R

ii) get P from Ricatti equation (P is positive.)

$$AP + PA^T - PC^T R^{-1} CP + Q = 0$$

iii) calculate L:

$$L = (R^{-1} C P)^T = P C^T R^{-1}$$

a) 2 inputs, 1 output

b) $\det[A - \lambda I]$

$$\begin{vmatrix} -\lambda & 2 & 3 \\ 2 & 3-\lambda & 0 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 3-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= -\lambda \cdot (-3\lambda + \lambda^2 - 4) + 3 \cdot (2 - 3 + \lambda)$$

$$= 3\lambda^2 - \lambda^3 + 4\lambda - 3 + 3\lambda$$

$$= -\lambda^3 + 3\lambda^2 + 7\lambda - 3$$

according to HURWITZ necessary condition:

all a_i same sign e.g. > 0

NOT fulfilled $\Rightarrow$ NOT STABLE

Check for controllability $Q_s = [B \quad AB \quad \dots \quad A^{n-1}B]$

$\text{rank } Q_s \overset{!}{=} n$

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{rank } A = 3 =: n$$

$$A \cdot B = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -3 & 2 \\ -1 & 1 \end{bmatrix}$$

$$A \cdot AB = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 \\ -3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 7 \\ -13 & 6 \\ -5 & 2 \end{bmatrix}$$

b) ... rank $Q_S = \begin{bmatrix} 0 & 1 & -2 & \\ -1 & 0 & -3 & \cdots \\ 0 & 0 & -1 & \end{bmatrix} = 3$

those are independent

$\Rightarrow$ System is CONTROLLABLE.

Yes, it is possible to place the eigenvalues arbitrary.

c) Check for observability (KALMAN)

$Q_B = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$, rank $Q_B \overset{!}{=} 3$ for observability.

$C = [1 \quad 1 \quad 1]$

$C \cdot A = [1 \quad 1 \quad 1] \cdot \begin{bmatrix} 0 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} = [3 \quad 6 \quad 3]$

$CA \cdot A = [3 \quad 6 \quad 3] \cdot \ldots = [15 \quad 27 \quad 9]$

rank $Q_B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 6 & 3 \\ 15 & 27 & 9 \end{bmatrix} = 3$

$\Rightarrow$ System is OBSERVABLE.

Yes, all states can be reconstructed from the output.

d) x has to be controlled by using $\dot{x}$.

e) Desired eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$

$$(\lambda + 1)(\lambda + 2)(\lambda + 3)$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$(A - Bk) = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & 0 \\ 0 & 0 & k_6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 3-k_6 \\ 2+k_1 & 3+k_2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\det\left[(A-Bk) - \lambda I\right] = \begin{vmatrix} -\lambda & 2 & 3-k_6 \\ 2+k_1 & 3+k_2-\lambda & 0 \\ 1 & 1 & -\lambda \end{vmatrix}$$

$$= \lambda^3 + (-3-k_2)\lambda^2 + (-7-2k_1+k_6)\lambda + (3-3k_1-k_6+\ldots$$

$$\ldots + k_1k_6 + 3k_2 - k_2k_6)$$

Comparison of coefficients:

$$\lambda^3 : \quad 1 \overset{!}{=} 1 \checkmark$$

$$\lambda^2 : \quad 6 \overset{!}{=} -3-k_2 \Rightarrow \underline{k_2 = -9} \checkmark$$

$$\lambda^1 : \quad 11 \overset{!}{=} -7-2k_1+k_6 \Rightarrow k_6 = 18+2k_1$$

$$\lambda^0 : \quad 6 \overset{!}{=} 3-3k_1-(18+2k_1)+k_1(18+2k_1)+\ldots$$

$$\ldots + 3(-9) - (-9)(18+2k_1)$$

$$\Rightarrow \quad k_1^2 + \frac{31}{2}k_1 + 57 = 0$$

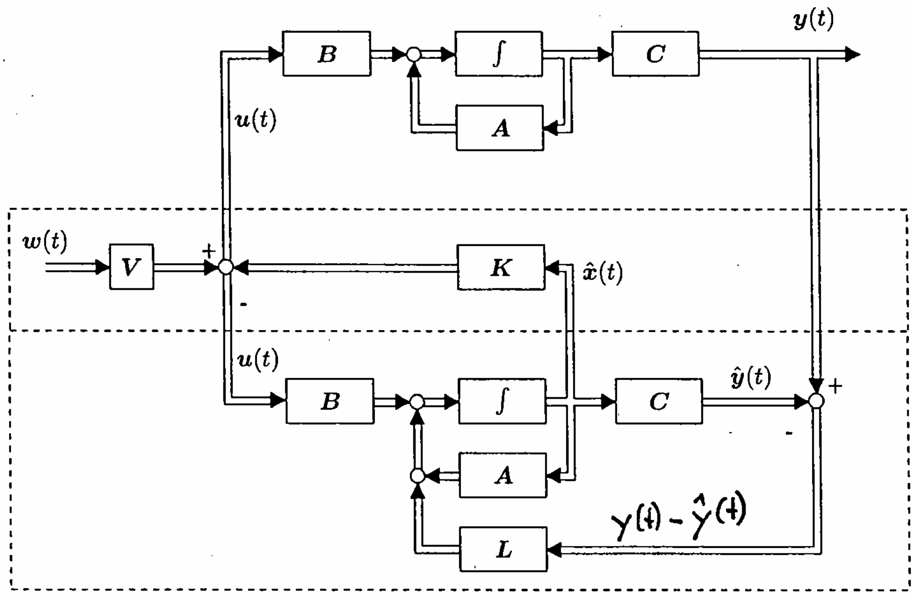
e) ... $k_{1,1/2} = -\frac{31}{4} \pm \frac{7}{4}$

$k_{1,1} = -9.5 \quad \wedge \quad k_{1,2} = -6$

$k_{6,1} = -1 \quad \wedge \quad k_{6,2} = 6$

$$k = \begin{bmatrix} -9.5 & -9 & 0 \\ 0 & 0 & -1 \end{bmatrix} \vee k = \begin{bmatrix} -6 & -9 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

f)



g) Yes, because the system is observable ⇒ Check with KALMAN

$$J = \int_{t=0}^{\infty} x^T Q x + \underbrace{y^T R y}\, dt \longrightarrow \min.$$

for observer use y

for controller use u

$$= \int_{t=0}^{\infty} [x_1 \; x_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [y_1 \; y_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} dt$$

$$= \int_{t=0}^{\infty} [x_1 \; x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [y_1 \; y_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} dt$$

$$= \int_{t=0}^{\infty} x_1^2 + x_2^2 + y_1^2 + y_2^2 \, dt$$

observer matrix $L^T = (R^{-1})^T C P^T$

RICCATI-Matrix $AP + PA^T - PC^T (R^{-1})^T C P + Q = 0$

[Compare $A - Bk \longleftrightarrow A^T - C^T L^T$]

$$P = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}, \quad (R^{-1})^T = R, \quad C^T = C$$

$$\Rightarrow \underbrace{\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}}_{A^T} - \dots$$

$$\dots - \underbrace{\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C^T} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{(R^{-1})^T} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q} \overset{!}{=} 0$$

$$\Rightarrow \begin{bmatrix} -2p & 0 \\ 0 & -2p \end{bmatrix} + \begin{bmatrix} -2p & 0 \\ 0 & -2p \end{bmatrix} - \begin{bmatrix} p^2 & 0 \\ 0 & p^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{!}{=} 0$$

g) ...

$$\Rightarrow \begin{bmatrix} -p^2-4p+1 & 0 \\ 0 & -p^2-4p+1 \end{bmatrix} \stackrel{!}{=} 0$$

$$-p^2-4p+1 = 0 \Rightarrow p^2+4p-1 = 0$$

$$p_{1/2} = -2 \pm \sqrt{5}$$

$$\Rightarrow P = \begin{bmatrix} -2+\sqrt{5} & 0 \\ 0 & -2+\sqrt{5} \end{bmatrix}$$

$$L^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2+\sqrt{5} & 0 \\ 0 & -2+\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} -2+\sqrt{5} & 0 \\ 0 & -2+\sqrt{5} \end{bmatrix} = L$$

h) $$\det \left| (A^T - C^T L^T) - \lambda I \right|$$

$$= \begin{vmatrix} -2-(-2+\sqrt{5})-\lambda & 0 \\ 0 & -2-(-2+\sqrt{5})-\lambda \end{vmatrix}$$

$$= \lambda_{1/2} = -\sqrt{5}$$

i)
- observes to unobservabel system senseless
- (all) states cannot be measured → observer states used for controlling X_i (system states)