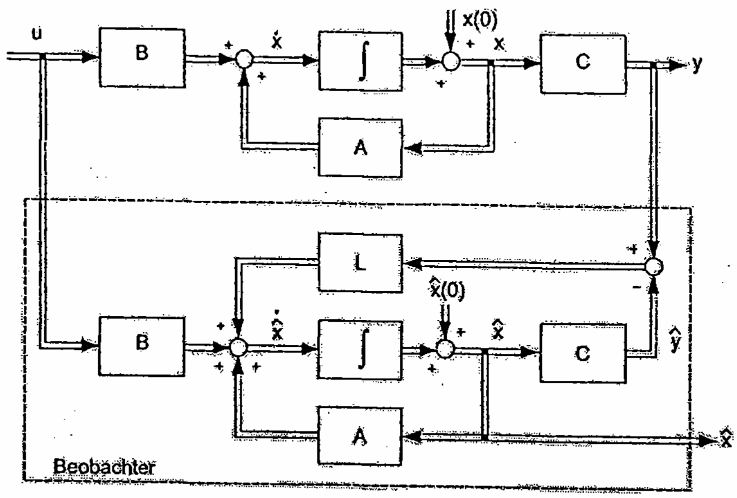
1.a)



b) Transmission zeros are those s_0, such that the transfer function matrix $G(s)$ fulfills one of the following conditions:

$$\det(G(s_0)) = 0 \qquad \text{if} \quad m = r \quad \text{or}$$

$$\operatorname{rank}(G(s_0)) < \max(\operatorname{rank}(G(s))) \quad \text{if} \quad m \neq r$$

in which m is the number of inputs and r the number of outputs.

Decoupling zeros are calculated by using the Rosenbrock matrix $P(s)$.
The output decoupling zeros are those s_0, such that the left column of $P(s)$ has a loss of rank

$$\operatorname{rank}\begin{pmatrix} s_0 I - A \\ C \end{pmatrix} < n$$

while the input decoupling zeros are those s_0, such that the upper row of $P(s)$ has a loss of rank

$$\operatorname{rank}(s_0 I - A \quad -B) < n$$

c)

$$A^* = (A - BK) = \begin{bmatrix} 0 & 1 & -bk \\ 0 & 1 & 0 \\ 3 & 0 & a \end{bmatrix}$$

$$\det(\lambda I - A^*) = \lambda^3 - (a+1)\lambda^2 + (a+3bk)\lambda - 3bk$$

First constraint: all coefficients have the same algebraic sign

$$\Rightarrow \quad -a-1 > 0 \quad \Leftrightarrow \quad a < -1$$
$$a + 3bk > 0 \quad \Leftrightarrow \quad a > -3bk$$
$$-3bk > 0 \quad \Leftrightarrow \quad 3bk < 0$$

The system is not stable due to contradiction in the conditions.

d) State-space description:

$$\begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -4/3 & -5/3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$Y = [-3 \quad 2] \begin{bmatrix} \dot{x} \\ x \end{bmatrix}$$

Transfer function matrix:

$$G(s) = C(sI - A)^{-1}B = -6 \frac{3s-2}{3s^2+4s+5}$$

e) State-stable always means BIBO-stable also.
If all eigenvalues are stable $\Rightarrow$ all poles are stable, regardless of decoupling zeros.

2a)

$$P(s) = \begin{bmatrix} s+3 & 0 & 0 & -1 & 0 \\ 0 & s+2 & 0 & 0 & 0 \\ 0 & 0 & s+1 & 0 & b \\ c & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

b) $\det(\lambda I - A) = 0 \quad \Rightarrow \lambda_1 = -3,\ \lambda_2 = -2,$ and $\lambda_3 = -1.$

The system is stable, due to $\lambda_i \leq 0 \ (i = 1,2,3)$

The system is asymptotically stable, due to $\lambda_i < 0 \ (i = 1,2,3)$

c) $m = r \quad \Rightarrow \quad \det(P(s_0)) = 0$

$$\det\left(\begin{bmatrix} s_0+3 & 0 & 0 & -1 & 0 \\ 0 & s_0+2 & 0 & 0 & 0 \\ 0 & 0 & s_0+1 & 0 & b \\ c & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}\right) = -c(s_0+2)b = 0$$

$\Rightarrow s_0 = -2$

d) Input decoupling zeros: $\operatorname{rank}(s_0 I - A \quad -B) < n; \quad n = 3$

$$\operatorname{rank}\left(\begin{bmatrix} s_0+3 & 0 & 0 & -1 & 0 \\ 0 & s_0+2 & 0 & 0 & 0 \\ 0 & 0 & s_0+1 & 0 & b \end{bmatrix}\right) = 2 \quad \text{for } s_0 = -2$$

Output decoupling zeros: $\operatorname{rank}\begin{pmatrix} s_0 I - A \\ C \end{pmatrix} < n;\ n = 3$

$$\operatorname{rank}\left(\begin{bmatrix} s_0+3 & 0 & 0 \\ 0 & s_0+2 & 0 \\ 0 & 0 & s_0+1 \\ c & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) = 3$$

$\Rightarrow$ no output decoupling zeros.

e) Kalman criteria for observability

$$\operatorname{rank}(Q_B) = \operatorname{rank}\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 1 \\ -3c & 0 & 0 \\ 0 & -2 & -1 \\ 9c & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix} = 3 = n$$

$\Rightarrow$ The system is fully observable, independent of b and c.

For this reason, it is possible to realize an identity observer.

Problem 3

a) $X=[X_1\ \ X_2\ \ X_3\ \ X_4]^T=[x_1\ \ x_2\ \ \dot{x}_1\ \ \dot{x}_2]^T$

$$\dot{X}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-\frac{k_1+k_2}{m_a}&\frac{k_2}{m_a}&-\frac{d_1+d_2}{m_a}&\frac{d_2}{m_a}\\\frac{k_2}{m_r}&-\frac{k_2}{m_r}&\frac{d_2}{m_r}&-\frac{d_2}{m_r}\end{bmatrix}X+\begin{bmatrix}0\\0\\-\frac{1}{m_a}\\\frac{1}{m_a}\end{bmatrix}f_u+\begin{bmatrix}0\\0\\\frac{d_1}{m_a}\dot{x}_d+\frac{k_1}{m_a}x_d\\0\end{bmatrix}$$

$$=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-4&2&-4&1\\2&-2&1&-1\end{bmatrix}X+\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}f_u$$

$$y=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}X$$

b) $\det(\lambda I-A)$

$$=\begin{vmatrix}\lambda&0&-1&0\\0&\lambda&0&-1\\4&-2&\lambda+4&-1\\-2&2&-1&\lambda+1\end{vmatrix}=\lambda\cdot[\lambda\cdot((\lambda+4)(\lambda+1)-1)-1\cdot(2-2\cdot(\lambda+4))]$$
$$-1\cdot[-\lambda\cdot(4(\lambda+1)-2)-1\cdot(4\times2-4)]$$
$$=\lambda^4+5\lambda^3+9\lambda^2+8\lambda+4=0$$

$\lambda_1=-2,\ \lambda_2=-2\ \Rightarrow\ \lambda^2+4\lambda+4=0$

$$\det(\lambda I-A)=(\lambda^2+4\lambda+4)\cdot\frac{\lambda^4+5\lambda^3+9\lambda^2+8\lambda+4}{\lambda^2+4\lambda+4}$$

$$=(\lambda^2+4\lambda+4)\cdot\frac{\lambda^2(\lambda^2+4\lambda+4)+\lambda\cdot(\lambda^2+4\lambda+4)+1\cdot(\lambda^2+4\lambda+4)}{\lambda^2+4\lambda+4}$$

$$=(\lambda^2+4\lambda+4)(\lambda^2+\lambda+1)=0$$

$\lambda_{1,2}=-2\qquad \lambda_{3,4}=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}j$

$Re\{\lambda_i\}<0$ for $i=1,\cdots,4$ $\Rightarrow$ The system is asymptotically stable.

c) Kalman

$$Q_B = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \qquad CA = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline & CA^2 & & \\ & CA^3 & & \end{bmatrix} = \begin{bmatrix} I \\ CA^2 \\ CA^3 \end{bmatrix}$$

Rank $Q_B = 4 = n$ $\Rightarrow$ The system is fully observable.

d) i) desired characteristic polynomial

$$(\lambda - \lambda_{1,des})(\lambda - \lambda_{2,des})(\lambda - \lambda_{3,des})(\lambda - \lambda_{4,des})$$

$$= (\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)$$

$$= \lambda^4 + 10\lambda^3 + 35\lambda^2 + 50\lambda + 24$$

ii) characteristic polynomial of the controlled system

$$\det(\lambda I - A + BK)$$

$$= \left| \begin{bmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ 4 & -2 & \lambda+4 & -1 \\ -2 & 2 & -1 & \lambda+1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -K_1 & -K_2 & -K_3 & -K_4 \\ K_1 & K_2 & K_3 & K_4 \end{bmatrix} \right| = \begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ 4-K_1 & -2-K_2 & \lambda+4-K_3 & -1-K_4 \\ -2+K_1 & 2+K_2 & K_3-1 & \lambda+1+K_4 \end{vmatrix}$$

$K_1 = 2$
$K_3 = 1$

$$= \begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ 2 & -2-K_2 & \lambda+3 & -1-K_4 \\ 0 & 2+K_2 & 0 & \lambda+1+K_4 \end{vmatrix} = \lambda \left\{ \lambda[(\lambda+3)(\lambda+1+K_4)] - 1[-(\lambda+3)(2+K_2)] \right\}$$

$$-1\left\{ -\lambda[2(\lambda+1+K_4)] - 1[2(2+K_2)] \right\}$$

$$= \lambda^4 + (4+K_4)\lambda^3 + (7+3K_4+K_2)\lambda^2 + (8+3K_2+2K_4)\lambda + 4+2K_2$$

d) iii) Compare the coefficients :

$$\left.\begin{aligned} 4+K_4 &\overset{!}{=} 10 \\ 7+3K_4+K_2 &\overset{!}{=} 35 \\ 8+3K_2+2K_4 &\overset{!}{=} 50 \\ 4+2K_2 &\overset{!}{=} 24 \end{aligned}\right\} \Rightarrow \quad K_2=10 \qquad K_4=6$$

$$\Rightarrow K = [\ 2 \quad 10 \quad 1 \quad 6\ ]$$

e) The original eigenvalues have a pair of complex number.
The desired eigenvalues in d) are all real numbers.
⇒ The system behavior will have no more oscillation with the new eigenvalues.

f) New state space model

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix}}_{A_{new}} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{B} f_u + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}}_{E} x_d$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{C} x$$

Transfer function matrix (from $x_d \rightarrow y$)

$$G(s) = C\,(sI - A_{new})^{-1} E$$

assume $(sI - A_{new})^{-1} = \begin{pmatrix} a_{11} & \cdots & a_{14} \\ \vdots & \ddots & \vdots \\ a_{41} & \cdots & a_{44} \end{pmatrix}$

f)
$$G(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{14} \\ \vdots & \ddots & \vdots \\ a_{41} & \cdots & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2a_{13} \\ 2a_{23} \end{bmatrix}$$

from the Hint

$$= \begin{bmatrix} \dfrac{2(s^2+s+2)}{s^4+2s^3+6s^2+2s+4} \\ \dfrac{2(s+2)}{s^4+2s^3+6s^2+2s+4} \end{bmatrix}$$

Transfer function from $X_d \to y_1$

$$G_{X_d \to y_1}(s) = \frac{2(s^2+s+2)}{s^4+2s^3+6s^2+2s+4}$$

Problem 4)

a) $$G(s) = C\cdot(sI-A)^{-1}\cdot B + \cancel{D}^{0}$$

$$(sI-A)^{-1} = \frac{1}{\det(sI-A)}\cdot \operatorname{adj}(sI-A)$$

$$(sI-A) = \begin{bmatrix} s+1 & -12 & +3 \\ 0 & s-2 & 0 \\ 0 & -10 & s+3 \end{bmatrix}$$

$$\det(sI-A) = (s+1)(s-2)(s+3)$$

$$\operatorname{adj}(sI-A) = \begin{pmatrix} (s-2)(s+3) & +(-12(s+3)+30) & -3(s-2) \\ 0 & (s+1)(s+3) & 0 \\ 0 & -(-10(s+1)) & (s+1)(s-2) \end{pmatrix}$$

$$= \begin{pmatrix} (s-2)(s+3) & 12s+6 & -3s+6 \\ 0 & (s+1)(s+3) & 0 \\ 0 & 10s+10 & (s+1)(s-2) \end{pmatrix}$$

$$G(s) = [1\,0\,0]\,(sI-A)^{-1}\cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$= \frac{[(s-2)(s+3) \quad 12s+6 \quad -3s+6]\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \end{bmatrix}}{\det(sI-A)}$$

$$= \left[\frac{-9s-12}{(s+1)(s-2)(s+3)} \quad \frac{1}{s+1} \right]$$

$s_1 = \lambda_1 = -1$ $\qquad s_{01} = -1\frac{1}{3}$

$s_2 = \lambda_2 = 2$

$s_3 = \lambda_3 = -3$

Problem 4)

b)

$$J = \int_0^\infty (x' Q x + y' R y)\, dt$$

$$= \int_0^\infty \left([x_1\ x_2\ x_3] \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + y_1 \cdot 2 \cdot y_1 \right) dt$$

$$= \int_0^\infty \left([x_1 - x_2 \quad -x_1 + 3x_2 \quad +x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2y_1^2 \right) dt$$

$$= \int_0^\infty \left([x_1^2 - x_1 x_2 - x_1 x_2 + 3x_2^2 + x_3^2] + 2y_1^2 \right) dt$$

$$= \int_0^\infty \left(x_1^2 - 2x_1 x_2 + 3x_2^2 + x_3^2 + 2y_1^2 \right) dt$$

Problem 4)

c) $\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$

$$= \begin{bmatrix} 0 & 12 & -3 \\ -2 & 2 & 0 \\ +2 & 10 & -3 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \end{bmatrix} u + \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} y$$

Eigenvalues:

$$\det((A-LC) - \lambda E) \overset{!}{=} 0$$

$$\Leftrightarrow \begin{vmatrix} -\lambda & 12 & -3 \\ -2 & 2-\lambda & 0 \\ 2 & 10 & -3-\lambda \end{vmatrix} \overset{!}{=} 0$$

$$\Leftrightarrow \lambda^3 + \lambda^2 + 24\lambda \overset{!}{=} 0 \qquad \lambda_1 = 0$$

$$\Rightarrow \lambda^2 + \lambda + 24 = 0$$

$$\overset{pq}{\Rightarrow} \lambda_{2,3} = -\frac{1}{2} \pm \underbrace{\sqrt{23{,}75}}_{\approx 4{,}87}\, i$$

Problem 4)

d) damping of the system

$$d_1 = d_2 = d_3$$

damping of the observer

$$d_{o_1} = 1$$

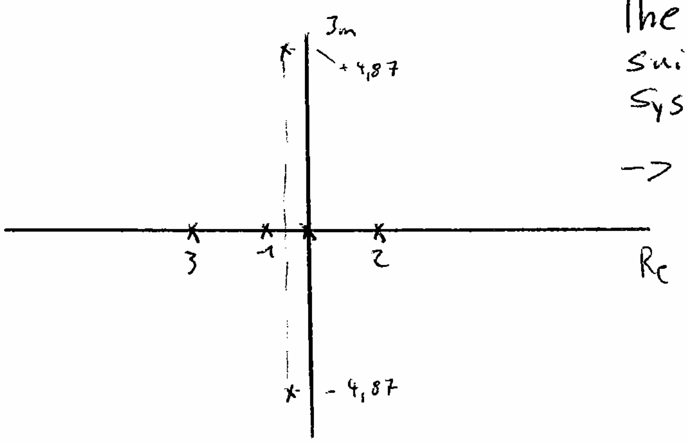
d_{o_2}/d_{o_3} :
$$\left| \frac{Re\{\lambda_i\}}{Im\{\lambda_i\}} \right| = \frac{d}{\sqrt{1-d^2}} = \cot \phi_d$$

$$\Rightarrow \quad \frac{0{,}5}{\sqrt{23{,}75}} = \frac{d}{\sqrt{1-d^2}} \quad |^2$$

$$\Leftrightarrow \quad \frac{0{,}25}{23{,}75} = \frac{d}{1-d^2}$$

$$\Leftrightarrow \quad \frac{0{,}25}{23{,}75} = \left(\frac{0{,}25}{23{,}75} + 1 \right) d^2$$

$$\Rightarrow \quad \underline{\underline{d_{o_{2/3}} = 0{,}102}}$$



The observer is not suitable for the given system.

-> Eigenvalues of the system are faster than eigenvalues of the observer

Problem 4)

e) $(A - B \cdot k) = \begin{bmatrix} -1 & 12 & -3 \\ 0 & 2 & 0 \\ 0 & 10 & -3 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -4 \\ 1 & 4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 12 & -3 \\ 0 & 2 & 0 \\ 0 & 10 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 2 \\ -2 & 1 & 4 \\ -2 & 1 & 4 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -2 & 8 & -5 \\ 2 & 1 & -4 \\ 2 & 9 & -7 \end{bmatrix}}_{A_k}$$

$$\det(A_k - \lambda E) \overset{!}{=} 0$$

$$\Leftrightarrow \begin{vmatrix} -2-\lambda & 8 & -5 \\ 2 & 1-\lambda & -4 \\ 2 & 9 & -7-\lambda \end{vmatrix} = \lambda^3 + 8\lambda^2 + 35\lambda + 90 \overset{!}{=} 0$$

Hurwitz: $a_i > 0$ ✓

$$H = \begin{pmatrix} 8 & 90 & 0 \\ 1 & 35 & 0 \\ 0 & 8 & 90 \end{pmatrix}$$

$|H_1| = 8 > 0$

$|H_2| = 190 > 0$

$\Rightarrow$ The controlled system is stable.

Problem 4)

f)