$\boxed{1}$

ⓐ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 2 & a \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C = [0 \quad 0 \quad c]$

Transfer function matrix ;

$$G(s) = c[sI - A]^{-1} B$$

$$= [0 \quad 0 \quad c] \left[\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 2 & a \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [0 \quad 0 \quad c] \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -3 & -2 & s-a \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to find $[\quad]^{-1} \Rightarrow$

$$[\quad]^{-1} = \frac{1}{|[\quad]|} \left[\text{cofactor matrix of } [\quad] \right]^T$$

$$= \frac{1}{s^3 - as^2 - 2s - 3} \begin{bmatrix} s^2 - as - 2 & s - a & 1 \\ 3 & s^2 - as & s \\ 3s & 2s + 3 & s^2 \end{bmatrix}$$

$$G(s) = C[sI - A]^{-1} B$$

$$= \frac{1}{s^3 - 9s^2 - 2s - 3} \begin{bmatrix} 0 & 0 & c \end{bmatrix} \begin{bmatrix} s^2 - 9s - 2 & s - 9 & 1 \\ 3 & s^2 - 9s & s \\ 3s & 2s + 3 & s^2 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{s^3 - 9s^2 - 2s - 3} \begin{bmatrix} 3cs & 2cs + 3c & cs^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow G(s) = \frac{1}{s^3 + 9s^2 - 2s - 3} \begin{bmatrix} 5cs + 3c & 2cs + 3c & cs^2 \end{bmatrix}$$

(b) For asymptotic stability the eigenvalues must be in the Left side of the S-Plane

$$|\lambda I - A| = 0$$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 2 & 9 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -3 & -2 & \lambda - 9 \end{bmatrix}$$

$$|[\quad]| = \lambda^2(\lambda - 9) - 3 - 2\lambda$$

$$= \lambda^3 - 9\lambda^2 - 2\lambda - 3 = 0$$

$$\equiv a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

Hurwitz $\Rightarrow$ a_0 and a_1 are negative for any value of a_2

$\Rightarrow$ not asymptotically stable under any conditions.

(c) Input decoupling zeros :

$$\text{rank}\begin{bmatrix} s_0 I - A & -B \end{bmatrix} \overset{!}{<} n \qquad n = 3$$

$$\text{rank}\begin{bmatrix} s_0 & -1 & 0 & -1 & 0 & 0 \\ 0 & s_0 & -1 & -1 & -1 & 0 \\ -3 & -2 & s_0-9 & 0 & 0 & -1 \end{bmatrix} \overset{!}{<} 3$$

no values of s_0 satisfy the condition

$\rightarrow$ no input decoupling zeros exist.

Output decoupling zeros :

$$\text{rank}\begin{bmatrix} s_0 I - A \\ C \end{bmatrix} \overset{!}{<} n \qquad n = 3$$

$$\text{rank}\begin{bmatrix} s_0 & -1 & 0 \\ 0 & s_0 & -1 \\ -3 & -2 & s_0-9 \\ 0 & 0 & c \end{bmatrix} \overset{!}{<} 3$$

for $c \neq 0$

$\rightarrow$ no output decoupling zeros exist.

(d) quadratic optimal regulator/controller calculation:

* Choose Q and R (as positive definite)
* Solve Ricatti equation to get P

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

* Calculate gain-matrix K as

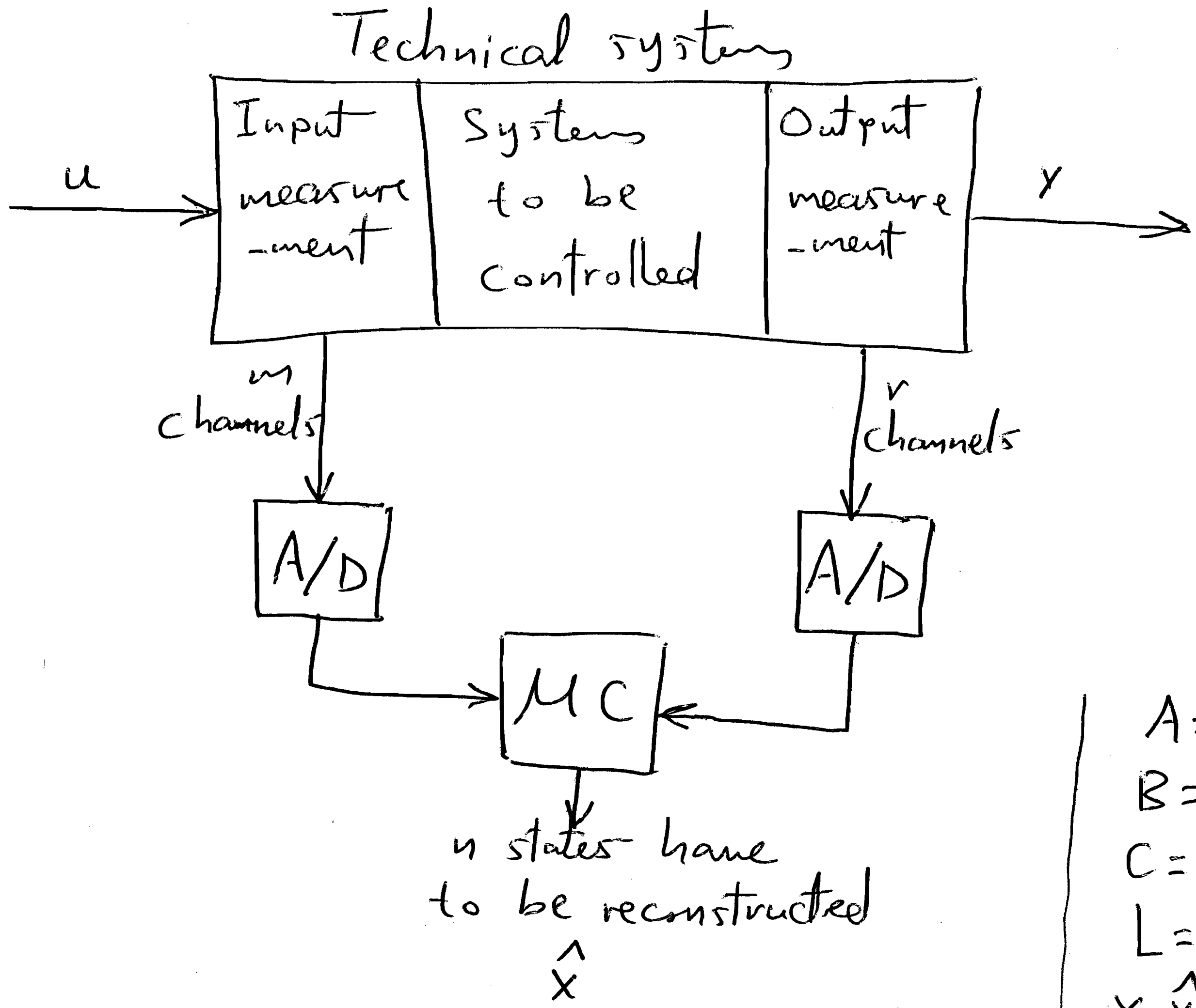
$$K = R^{-1} B^T P$$

Full-state feedback makes sense when the full-controllability is given.
If the states are not completely measurable, they have to be observed by a full-state observer. Therefore the full-observability is necessary.

For the system; no decoupling zeros exist $\rightarrow$ fully observable and fully controllable

$\rightarrow$ Conditions are satisfied.

(e)



$A = n \times n$
$B = n \times m$
$C = r \times n$
$L = n \times r$
$x, \hat{x} = n \times 1$
$y = r \times 1$
$u = m \times 1$

Additionally for state feedback:

$$u = -K \cdot \hat{x}$$

Multiplication

$$+Bu$$

Multiplication

and addition

P2

a) i) Yes, because not every state is measured.

$$\text{Rang}(C) = 2 < 3$$

ii) If (A, C) is fully observable, then the eigenvalues of $\tilde{A} = [A - LC]$ are arbitrary placeable.

Kalman $S_{obs} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$

$\text{Rang}(S_{obs}) \overset{!}{=} n = 3$ for fully observability.

$$S_{obs} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -2 & -3 & 0 \\ 0 & 2 & 3 \\ -6 & -13 & -6 \\ 7 & 9 & 0 \end{bmatrix}$$

$$\text{Rang}(S_{obs}) = \underline{\underline{3}}$$

$\Rightarrow$ fully observable

$\Rightarrow$ arbitrary placeable

$\Rightarrow$ $\underline{\underline{\text{Yes.}}}$

P2 b) (2)

$$\underbrace{[A-LC]}_{\tilde{A}} = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2l_1 \\ l_2 & l_1 \\ 0 & l_1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2l_1 & 2 & 3 \\ 2-l_1 & 3+l_2 & 0 \\ 1-l_1 & 1 & 0 \end{bmatrix}$$

$$|\lambda I - \tilde{A}| \overset{!}{=} 0$$

$$\Rightarrow \begin{vmatrix} \lambda + 2l_1 & -2 & -3 \\ l_1 - 2 & \lambda - 3 - l_2 & 0 \\ l_1 - 1 & -1 & \lambda \end{vmatrix} \overset{!}{=} 0$$

$$\Rightarrow -3 \cdot \left[(l_1-2)(-1) - (l_1-1)(\lambda-3-l_2) \right] +$$

$$\lambda \cdot \left[(\lambda+2l_1)(\lambda-3-l_2) - (l_1-2)(-2) \right] \overset{!}{=} 0$$

$$\Rightarrow \lambda^3 + (2l_1 - 3 - l_2)\lambda^2 + (-l_1 - 2l_1 l_2 - 7)\lambda +$$

$$+ (3 - 6l_1 + 3l_2 - 3l_1 l_2) \overset{!}{=} 0$$

P2/ b) ...

given eigenvalues: $\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3.$

$$\Rightarrow (\lambda + 1)(\lambda + 2)(\lambda + 3)$$

$$\Rightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

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comparison of coefficients:

$$\lambda^2: \quad 6 = 2l_1 - l_2 - 3 \qquad (1)$$

$$\lambda^1: \quad 11 = -l_1 - 2l_1 l_2 - 7 \qquad (2)$$

$$\lambda^0: \quad 6 = 3 - 6l_1 + 3l_2 - 3l_1 l_2 \qquad (3)$$

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from (1): $l_2 = 2l_1 - 9$

in (2): $0 = l_1^2 - \frac{17}{4} l_1 + 4.5$

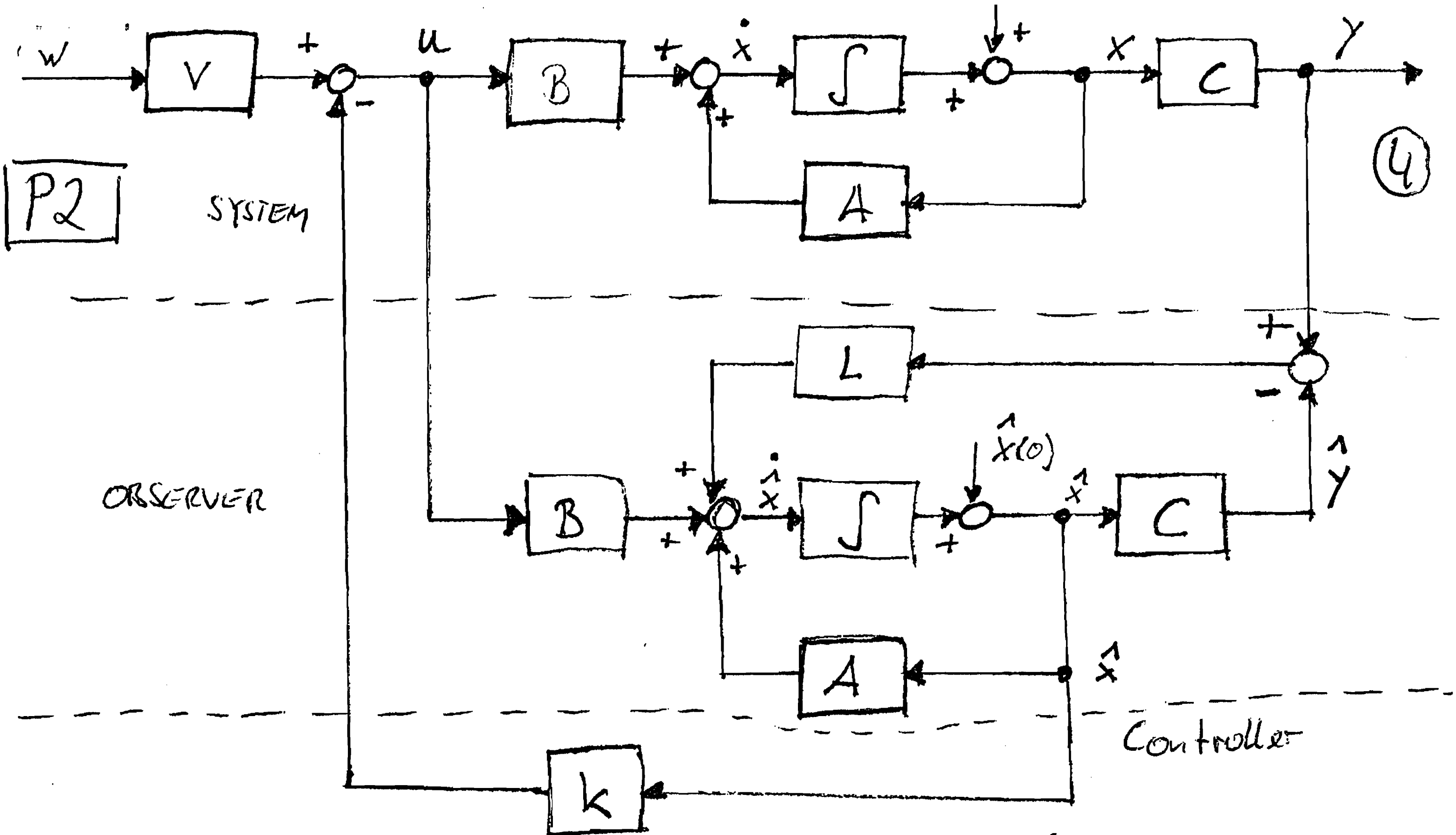
$$\Rightarrow l_1 = \frac{17}{8} \pm \sqrt{\underbrace{\frac{17^2}{4^3} - 4.5}_{= 0.125}}$$

$$\underline{\underline{l_1 = 2 \in \mathbb{Z}}}$$

$$(\vee\ l_1 = 2.25 \notin \mathbb{Z})$$

from (3): $6 = 6$ ✓

$$\underline{\underline{l_2 = -5}}$$



$\hat{x}$: observed state vector $n \times 1$

y : output vector $m \times 1$

u : input " $r \times 1$

w : reference " $k \times 1$

V : filter matrix $r \times k$

A : system " $n \times n$

B : input " $n \times r$

C : measurement " $m \times n$

L : observer feedback matrix $n \times m$

K : state feedback matrix $r \times n$

d) left eigenvectors: $w_i^T A = \lambda_i \, w_i^T$

right " : $A v_i = \lambda_i \, v_i$

3

a)
$$\begin{bmatrix} \dot{s} \\ \dot{M} \\ \ddot{s} \\ \ddot{M} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{g\ell^2}{r^2} & \frac{-d}{m} - \frac{d\ell^2}{mr^2} & 0 \\ 0 & -\frac{g\ell}{r^2} & \frac{d\ell}{mr^2} & 0 \end{bmatrix} \begin{bmatrix} s \\ M \\ \dot{s} \\ \dot{M} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m} - \frac{\ell^2}{mr^2} \\ \frac{\ell}{mr^2} \end{bmatrix} F_B$$

$$\begin{bmatrix} m\ell & m(\ell^2+r^2) \\ m & m\ell \end{bmatrix} \begin{bmatrix} \ddot{s} \\ \ddot{M} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix} \begin{bmatrix} \dot{s} \\ \dot{M} \end{bmatrix} + \begin{bmatrix} 0 & mg\ell \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ M \end{bmatrix} = \begin{bmatrix} 2\ell \\ 1 \end{bmatrix} F_B$$

b) $\det(\lambda I - A) = \lambda^4 + 20\lambda^3 + \lambda^2 + 10\lambda$

HURWITZ: $a_0 = 0 \Rightarrow$ not asymptotically stable

c) $\det(\lambda I - (A - Bk)) = \lambda^4 + 21\lambda^3 + 21\lambda^2 + 20\lambda + k_1$

HURWITZ: 1. $k_1 > 0$

2. $$H = \begin{bmatrix} 21 & 20 & 0 & 0 \\ 1 & 21 & k_1 & 0 \\ 0 & 21 & 20 & 0 \\ 0 & 1 & 21 & k_1 \end{bmatrix}$$

$H_1 = 21 > 0$ ✓

$H_2 = 21^2 - 20 = 421 > 0$ ✓

$H_3 = 0 \cdot |\dots| - k_1 \begin{vmatrix} 21 & 20 \\ 0 & 21 \end{vmatrix} + 20 H_1 \overset{!}{>} 0$

$\Leftrightarrow k_1 < \frac{8420}{441}$

3) c) $H_4 = k_1 \cdot H_3 \overset{!}{>} 0$

$\Leftrightarrow k_1 > 0 \quad (H_3 > 0)$

$$\Rightarrow \quad \underline{\underline{0 < k_1 < \frac{8420}{441}}}$$

d) $\det(\lambda I - (A - L\,C)) \overset{!}{=} (\lambda+1)^2 (\lambda+3)(\lambda+40)$

$\Leftrightarrow \lambda^4 + (62 + L_{33})\lambda^3 + (921 + 42 L_{33})\lambda^2 + (1660 + 81 L_{33})\lambda +$
$+ 800 + 40 L_{33} \overset{!}{=} (\lambda+1)^2 (\lambda+3)(\lambda+40)$

$\Leftrightarrow \underline{\underline{L_{33} = -17}}$

e) i) Never $(\lambda_{2/3}, \lambda_6)$
ii) Always $(\lambda_{2/3}, \lambda_6)$
iii) Never (λ_6)

Problem 4

a) $\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & a & \lambda \end{vmatrix} = (\lambda+1)(\lambda^2+a)$

$= (\lambda+1)(\lambda - i\sqrt{a})(\lambda + i\sqrt{a})$

Eigenvalues: $\lambda_1 = -1$

$\lambda_{2,3} = \pm i\sqrt{a}$

The system is not asymptotically stable, because

for $a>0$ $\quad Re\{\lambda_{2,3}\} = 0 \Rightarrow$ not asymptotically stable

for $a<0$ $\quad$ one unstable eigenvalue $\Rightarrow$ unstable

Eigenvectors: $\quad A v_i = \lambda_i v_i \Rightarrow (A - \lambda_i I) v_i = 0$

$\lambda_1 = -1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -a & 1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_2 = i\sqrt{a}$

$$\begin{bmatrix} -1-i\sqrt{a} & 0 & 0 \\ 0 & -i\sqrt{a} & 1 \\ 0 & -a & -i\sqrt{a} \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ i\sqrt{a} \end{bmatrix}$$

$\lambda_3 = -i\sqrt{a}$

$$\begin{bmatrix} -1+i\sqrt{a} & 0 & 0 \\ 0 & i\sqrt{a} & 1 \\ 0 & -a & i\sqrt{a} \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_3 = \begin{bmatrix} 0 \\ 1 \\ -i\sqrt{a} \end{bmatrix}$$

b) Controllability (Kalman)

$Q_s = [\,B \quad AB \quad A^2B\,]$

$$= \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & -ab \\ 0 & -ab & 0 \end{bmatrix}$$

Rank $Q_s < 3 = n$

$\Rightarrow$ not fully controllable

b) HAUTUS

rank $[\lambda_i I - A \quad B] = n \Rightarrow \lambda_i$ controllable

$\lambda_1 = -1$

$$\text{rank}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & b \\ 0 & a & -1 & 0 \end{bmatrix} < 3$$

$\Rightarrow \lambda_1$ is not controllable.

$\lambda_2 = i\sqrt{a}$

$$\text{rank}\begin{bmatrix} i\sqrt{a}+1 & 0 & 0 & 0 \\ 0 & i\sqrt{a} & -1 & b \\ 0 & a & i\sqrt{a} & 0 \end{bmatrix} = 3 \qquad (a \neq 0 \text{ and } b \neq 0)$$

$\Rightarrow \lambda_2$ is controllable.

$\lambda_3 = -i\sqrt{a}$

$$\text{rank}\begin{bmatrix} -i\sqrt{a}+1 & 0 & 0 & 0 \\ 0 & -i\sqrt{a} & -1 & b \\ 0 & a & -i\sqrt{a} & 0 \end{bmatrix} = 3 \qquad (a \neq 0 \text{ and } b \neq 0)$$

$\Rightarrow \lambda_3$ is controllable.

c) Observability (KALMAN)

$$Q_B = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & c \\ 0 & -ac & c \\ 0 & -ac & -ac \end{bmatrix}$$

rank $Q_B < 3 = n \Rightarrow$ not fully observable

HAUTUS

$$\text{rank}\begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = n \Rightarrow \lambda_i \text{ observable}$$

$\lambda_1 = -1$

$$\text{rank}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & a & -1 \\ 0 & 0 & c \end{bmatrix} < 3 \Rightarrow \lambda_1 \text{ is not observable}$$

c) $\lambda_2 = i\sqrt{a}$

$$\text{rank}\begin{bmatrix} i\sqrt{a} & 0 & 0 \\ 0 & i\sqrt{a} & -1 \\ 0 & a & i\sqrt{a} \\ 0 & 0 & c \end{bmatrix} = 3 \qquad (a \neq 0 \text{ and } c \neq 0)$$

$\Rightarrow \lambda_2$ is observable.

$\lambda_3 = -i\sqrt{a}$

$$\text{rank}\begin{bmatrix} -i\sqrt{a} & 0 & 0 \\ 0 & -i\sqrt{a} & -1 \\ 0 & a & -i\sqrt{a} \\ 0 & 0 & c \end{bmatrix} = 3 \qquad (a \neq 0 \text{ and } c \neq 0)$$

$\Rightarrow \lambda_3$ is observable.

d) Controllability (KALMAN)

$$Q_s = [\, B \quad AB \quad AB^2 \,]$$

$$= \begin{bmatrix} b_1 & -b_1 & b_1 \\ b_2 & 0 & -ab_2 \\ 0 & -ab_2 & 0 \end{bmatrix}$$

$$\det Q_s = ab_2(-ab_1b_2 - b_1b_2) = -ab_1b_2^2(a+1) \neq 0$$

if $a \neq -1$

$\Rightarrow$ An arbitrary desired dynamics can be achieved by state control, if $a \neq -1$

e) Characteristic polynomial

$$(A - BK) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -a & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ b_2 \\ 0 \end{bmatrix} [\, 0 \quad k_1 \quad k_2 \,]$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -k_1b_2 & 1-k_2b_2 \\ 0 & -a & 0 \end{bmatrix}$$

e) $\det(\lambda I - A + BK)$

$$= \begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda+k_1 b_2 & k_2 b_2 - 1 \\ 0 & a & \lambda \end{vmatrix} = (\lambda+1)\left[\lambda(\lambda+k_1 b_2) - a(k_2 b_2 - 1)\right]$$

$$= (\lambda+1)\left[\lambda^2 + k_1 b_2 \lambda - a(k_2 b_2 - 1)\right]$$

desired characteristic polynomial

$$(\lambda+1)\left(\lambda+\tfrac{10}{3}\right)^2 = (\lambda+1)\left(\lambda^2 + \tfrac{20}{3}\lambda + \tfrac{100}{9}\right)$$

Comparison of coefficients

$$\left.\begin{aligned} k_1 b_2 &= \tfrac{20}{3} \\ -a(k_2 b_2 - 1) &= \tfrac{100}{9} \end{aligned}\right\} \Rightarrow k_1 = \frac{20}{3 b_2} \qquad k_2 = \left(-\frac{100}{9a} + 1\right)\frac{1}{b_2}$$

f) $G(s) = C(sI - A)^{-1}B + D$

$$= \begin{bmatrix} 0 & 0 & c \end{bmatrix} \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s & -1 \\ 0 & a & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 0 & 0 & c \end{bmatrix} \frac{1}{(s+1)(s^2+a)} \begin{bmatrix} s^2+a & 0 & 0 \\ 0 & s(s+1) & s+1 \\ 0 & -a(s+1) & s(s+1) \end{bmatrix} \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

$$= \frac{-abc}{s^2+a}$$

Poles $\quad s_{1,2} = \pm i\sqrt{a}$

g) All poles are eigenvalues, but not all eigenvalues must be poles.

Poles are the eigenvalues, which are controllable and observable.

In this problem, eigenvalues $\lambda_{2,3}$ are poles and they are both controllable and observable. The eigenvalue λ_1 is not a pole and it is not controllable and not observable.