Solutions of the First Two Tasks, CTH'08

- **Problem 1**

a) *Define the difference between a MIMO and a SISO control system with respect to the design of the feedback gains.*

Solution: The feedback gain of MIMO system is a $m \times n$ or $m \times r$ matrix, depending on the types of the feedback control. The feedback gain of SISO system is a 1×1 or $m \times 1$ matrix.
The design of SISO feedback control can be formed by hand in the 1×1 case, while the MIMO design is a full mathematical approach.

b) *State the state space description of the SISO system with the transfer function*

$$F(s) = \frac{Y(s)}{U(s)} = k\frac{s^2+5}{(s^2+2s+5)(s-1)}$$

in state space notation.

Solution: First,

$$F(s) = \frac{k(s^2+5)}{(s^2+2\,s+5)(s-1)} = \frac{k(s^2+5)}{s^3+s^2+3\,s-5}.$$

Denote

$$\begin{aligned} a_0 &= 1, & b_0 &= 0,\\ a_1 &= 1, & b_1 &= k,\\ a_2 &= 3, & b_2 &= 0,\\ a_3 &= -5, & b_3 &= 5\,k, \end{aligned}$$

and define the state vector $x(t) = [\,x_1 \;\; x_2 \;\; x_3\,]^T$ as

$$\begin{aligned} x_1 &= y - \beta_0\, u,\\ x_2 &= \dot{x}_1 - \beta_1\, u,\\ x_3 &= \dot{x}_2 - \beta_2\, u, \end{aligned}$$

where

$$\begin{aligned} \beta_0 &= b_0 = 0,\\ \beta_1 &= b_1 - a_1\,\beta_0 = k,\\ \beta_2 &= b_2 - a_1\,\beta_1 - a_2\,\beta_0 = -k,\\ \beta_3 &= b_3 - a_1\,\beta_2 - a_2\,\beta_1 - a_3\,\beta_0 = 3k. \end{aligned}$$

The state space expression of the system equation should be

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t))\ ,\\ y(t) &= C\,x(t) + D\,u(t)\ . \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -a_3 & -a_2 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 5 & -1 & -1 \end{bmatrix},\quad B = \begin{bmatrix} \beta_1\\ \beta_2\\ \beta_3 \end{bmatrix} = k\begin{bmatrix} 1\\ -1\\ 3 \end{bmatrix},$$

$C = [\,1 \;\; 0 \;\; 0\,]$, and $D = 0$.

c) *State the conditions for the matrices P, Q of the Lyapunov equation $A^T P + PA = -Q$ to guarantee the asymptotic stability of the system $\dot{x} = Ax(t)$.*

Solution:$A^T\,P + P\,A = Q$ should be solvable and both P and Q should be symmetric positive definite matrices.

d) *For the system $\dot{x} = Ax$ with the system matrix*

$$A = \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 4 \\ -1 & -1 & -2 \end{pmatrix}$$

and the matrices

$$P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \text{ and } Q = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0, \\ 1 & 0 & 1 \end{pmatrix}$$

where Q and P are respectively the weighting matrix and the solution matrix of the Lyapunov equation.. Check and state about the quality of the stability of A (asymptotic stable, boundary stable (stable), unstable).

Solution: The eigenvalues of P are $2 \times (0.5858,\, 2.0000,\, 3.4142)$ and greater than zero, so P is a positive definite matrix.
The eigenvalues of Q are $(0,\, 0,\, 2)$, therefore Q is a semi-positive definite matrix.
Because the first derivative of the Lyapunov function $V(t) = x^T\,P\,x$ is $\dot{V}(t) = -x^T\,Q\,x$ and Q is a semi-positive definite matrix, $\dot{V}(t) = -x^T\,Q\,x \leq 0$. According to Lyapunov stability theory, it can be concluded that the system is stable.
From the given P and Q we can't get information about asymptotic stability. The eigenvectors of A are $[1\ -1\,1]^T, [0.17 - 0.34i\ 0.85\ -0.17 + 0.34i]^T, [0.17 + 0.34i\ 0.85\ -0.17 - 0.34i]^T$ and linear independent, therefore A can be transformed into a diagonal matrix. The eigenvalues of A is $(-1, \pm i)$, two of which lie in the imaginary axes. Therefore the system is boundary stable.

e) *Is the system described with $A = \begin{bmatrix} 0 & 1 & 0 \\ a & 1 & a \\ 3 & 0 & 0 \end{bmatrix}$ stable? Calculate the stability condition(s) with respect to the parameter a?*

Solution: From $\det(\lambda\,I - A) = 0$ the characteristic polynomial

$$\lambda^3 - \lambda^2 - a\,\lambda - 3\,a \;=\; 0$$

can be calculated.
The necessary condition for Hurwitz is not fulfilled ($a_3 = 1, a_2 = -1$), so the system must be unstable.

- **Problem 2**

a) *Calculate the damping of the eigenvalue pair* $\lambda_{1,2} = -1 \pm j$.

Solution: The damping ratio is 0.707, calculated by $\zeta = \cos(\arctan(\frac{1}{1}))$.

b) *The system*

$$\dot{x} = Ax + Bu, y = Cx$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -a & a-10 \end{pmatrix}, \quad B = b\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \textit{and } C = c\begin{pmatrix} 10 & 1 \end{pmatrix}$$

is given. For which parameter a *is the uncontrolled system asymptotic stable?*

Solution: Condition for the state stability: $Re\{\lambda_i\} \leq 0$.

$$\det(\lambda I - A) = 0$$

$\Longrightarrow$

$$\lambda^2 - (a-10)\lambda + a = 0$$

$\Longrightarrow$

$$\lambda_{1,2} = \frac{(a-10) \pm \sqrt{(a-10)^2 - 4a}}{2}$$

If $a \geq 0$, $|a-10| \geq \sqrt{(a-10)^2 - 4a}$, $Re\{\lambda_i\} \leq 0$ only requires $a - 10 < 0$.
If $a < 0$, $|a-10| < \sqrt{(a-10)^2 - 4a}$ and therefore $(a-10) + \sqrt{(a-10)^2 - 4a} > 0$, which will lead to an unstable mode. So a must be greater than 0.
Therefore, when $0 < a < 10$, the uncontrolled system is stable.

c) *What are the domains of parameters* a, b *if the system described in task b) is fully controllable?*

Solution: $Q_s = [B\ AB] = b\begin{bmatrix} 1 & 2 \\ 2 & a-20 \end{bmatrix}$. Being fully controllable means $rank(Q_s) = 2$.
Therefore $a - 20 \neq 4$ and $b \neq 0$, which means $a \neq 24$ and $b \neq 0$.

d) *What are the domains of parameters* a, c *if the system described in task b) is fully observable?*

Solution: $Q_b = [C\ CA]^T = c\begin{bmatrix} 10 & 1 \\ -a & a \end{bmatrix}$. Fully observable means $rank(Q_b) = 2$.
Therefore $a \neq 0$ and $c \neq 0$ must be fulfilled.

e) *For which values of parameters* k_1, k_2 *of a state feedback* $u = -k_1x_1 - k_2x_2$ *has the controlled system described in task b) with* $a = 12$, $b = 1$ *the eigenvalues* $\lambda_{1,2} = -2$
Solution: $[k_1\ k_2] = [\frac{14}{3}\ \frac{2}{3}]$, calculated by pole placement.

Problem 31

a) $\det(\lambda I - A) = 0$

$\Leftrightarrow \lambda^3 + (-q-p)\lambda^2 + (-q-1+pq)\lambda + pq + p = 0$

$$H = \begin{bmatrix} -q-p & (q+1)p & 0 \\ 1 & -q-1+pq & 0 \\ 0 & -q-p & (q+1)p \end{bmatrix}$$

HURWITZ

1. $-q-p > 0 \Leftrightarrow -p > q$ (1.1)

$-q-1+pq > 0 \Leftrightarrow q(-1+p) > 1$ (1.2)

$(q+1)p > 0$ (1.3)

(1.3): $(q+1)p > 0 \Leftrightarrow \begin{cases} p > 0 \wedge q > -1 & (1.3a) \\ p < 0 \wedge q < -1 & (1.3b) \end{cases}$

(1.2): a) $q = 0 \Rightarrow -1 > 0$ (f) $\Rightarrow q \neq 0$

b) $p < 0 \Rightarrow q < \frac{1}{-1+p} < -1$ (cf. (1.3b))

c) $0 < p < 1 \Rightarrow q < \frac{1}{-1+p} < -1$ but $q > -1$ (1.3a) (f)

d) $p > 1 \underset{(1.2)}{\Rightarrow} -1+p > 0 \underset{(1.2)}{\Rightarrow} q > 0$ but $0 > -p > q$ (1.1) (f)

Problem 3)

$(1.1), (1.2), (1.3) \Rightarrow p < 0$ and $q < -1$ (necessary condition)

(2) $\det H_1 = -q - p > 0$ ✓ (cf. (1.1))

$$\det H_2 = q^2 + q - pq^2 - p^2 q \overset{!}{>} 0$$

$$\det H_3 = (q+1)p \cdot \underbrace{\det H_2}_{>0} > 0 \quad ✓ \ (\text{cf. } (1.3))$$

Calculate $\det H_2$ for $q < -1$, $p < 0$

$$q + 1 - pq - p^2 < 0$$

$$\Leftrightarrow q(1-p) + 1 - p^2 < 0$$

$$\Leftrightarrow \underbrace{(1-p)}_{>0}\underbrace{(q+p+1)}_{\overset{!}{<}0} < 0$$

$$q + p + 1 < 0 \Leftrightarrow q < -1 - p$$

This holds true for $q < -1$

$\Rightarrow \{q < -1, p < 0\}$ is the necessary and sufficient condition for a state stable system.

Problem 3 |

b) $A = \begin{bmatrix} 0 & -1 & 0 \\ -2 & -1 & 2 \\ 2 & -1 & -2 \end{bmatrix}$

$$\det(\lambda I - A) = \lambda^3 + 3\lambda^2 + 2\lambda = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = -2$$

HAUTUS for $\lambda_1 = 0$:

$$\text{Rank}\begin{bmatrix} 0 & 1 & 0 & 1 & 1+b \\ 2 & 1 & -2 & 1 & 1 \\ -2 & 1 & 2 & 1 & 1+b \end{bmatrix} = \text{Rank}\begin{bmatrix} 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Full rank for $b \neq 0$ $\Rightarrow$ For $b \neq 0$ the system can be stabilized by state feedback.

c) $\det(\lambda I - (A - BK)) \overset{!}{=} (\lambda+1)(\lambda+2)(\lambda+16)$

$$\Leftrightarrow \lambda^3 + (19 + k_{12} + k_{22})\lambda^2 + (50 + 18k_{12} + 18k_{22})\lambda + 32k_{12} + 32k_{22} + 32$$

$$\overset{!}{=} \lambda^3 + 19\lambda^2 + 50\lambda + 32$$

$\Leftrightarrow k_{12} = -k_{22}$

d) HAUTUS for $\lambda_3 = -2$:

$$\text{Rank}\begin{bmatrix} -2 & 1 & 0 & 1 & 2 \\ 2 & -1 & -2 & 1 & 1 \\ -2 & 1 & 0 & 1 & 2 \end{bmatrix} = \text{Rank}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = 2$$

$\Rightarrow \lambda_3 = -2$ is not controllable $\Rightarrow$ the eigenvalues $\{0, -1, -2)$ cannot be shifted to $\{0, -1, -4\}$.

Problem 3

e) HAUTUS for $\lambda_2 = -1$:

$$\text{Rank}\begin{bmatrix} -1 & 1 & 0 & 1 & 2 \\ 2 & 0 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 & 2 \end{bmatrix} = \text{Rank}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = 3$$

$\Rightarrow \lambda_2 = -1$ is controllable

The system is fully observable (assumption given in the problem description) and the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -1$ are controllable. So the poles are $\{0, -1\}$.

$\lambda_3 = -2$ is no pole.

a) $G(s) = C\,(sI-A)^{-1}B + D$

$$(sI-A)^{-1} = \frac{1}{\det(sI-A)}\begin{bmatrix} s+20 & -8 \\ -8 & s+8 \end{bmatrix}$$

$$= \frac{1}{(s+8)(s+20)-64}\begin{bmatrix} s+20 & -8 \\ -8 & s+8 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{(s+8)(s+20)-64}\begin{bmatrix} s+20 & -8 \\ -8 & s+8 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{(s+8)(s+20)-64}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s+20+8 \\ -8-s-8 \end{bmatrix} = \frac{1}{(s+8)(s+20)-64}\begin{bmatrix} -s-16 \\ s+28 \end{bmatrix}$$

poles: $s^2 + 28s + 96 = 0$

$s_{12} = -14 \pm \sqrt{196-96} = -14 \pm 10$

$s_1 = -24 \qquad s_2 = -4$

zeros: $-s_{01} - 16 = 0 \qquad s_{02} + 28 = 0$

$s_{01} = -16 \qquad s_{02} = -28$

b) $Q_S = [B, AB] = \begin{bmatrix} 1 & 0 \\ -1 & 12 \end{bmatrix}$; rank $Q_S = 2 \Rightarrow$ fully contr.

c) $Q_B = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -8 & -20 \\ -8 & -8 \end{bmatrix}$; rank $(Q_B) = 2 \Rightarrow$ fully observable

d) $|\lambda I - (A - BK)| = \left|\lambda I - \left(\begin{bmatrix}-8 & -8\\ -8 & -20\end{bmatrix} - \begin{bmatrix}1\\ -1\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}\right)\right|$

$$= \left|\lambda I - \left(\begin{bmatrix}-8 & -8\\ -8 & -20\end{bmatrix} - \begin{bmatrix}k_1 & k_2\\ -k_1 & -k_2\end{bmatrix}\right)\right|$$

$$= \left|\lambda I - \begin{bmatrix}-8-k_1 & -8-k_2\\ -8+k_1 & -20+k_2\end{bmatrix}\right| = \begin{vmatrix}\lambda+8+k_1 & 8+k_2\\ 8-k_1 & \lambda+20-k_2\end{vmatrix}$$

$$= \lambda^2 + \lambda(k_1 - k_2 + 28) + 28k_1 - 16k_2 - 96$$

$$k_1 - k_2 + 28 > 0 \qquad (I)$$

$$28k_1 - 16k_2 - 96 > 0 \quad (II)$$

aus I: $k_1 + 28 > k_2 \iff k_1 > k_2 - 28$

aus II: $28k_1 - 96 > 16k_2$

$$\frac{28}{16}k_1 - \frac{96}{16} > k_2$$

$$k_1 > \left(k_2 + \frac{96}{16}\right)\frac{16}{28}$$

$$k_1 > \left(k_2 \cdot \frac{16}{28} + \frac{96}{28}\right)$$

$$k_1 > k_2 \cdot \frac{4}{7} + \frac{24}{7}$$

$$Q = \begin{bmatrix} 84 & 96 \\ 96 & 112 \end{bmatrix} \qquad R = 0{,}25$$

$$A = \begin{bmatrix} -8 & -8 \\ -8 & -20 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_2 \end{bmatrix}$$

$$A^T P + PA - PBR^{-1}B'P = -Q$$

$$\begin{bmatrix} -8 & -8 \\ -8 & -20 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_2 \end{bmatrix}\begin{bmatrix} -8 & -8 \\ -8 & -20 \end{bmatrix} \dots$$

$$-\begin{bmatrix} p_1 & p_2 \\ p_2 & p_2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 4 \cdot \begin{bmatrix} 1; -1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_2 \end{bmatrix}$$

$$= \begin{bmatrix} -8p_1 - 8p_2 & -8p_2 - 8p_2 \\ -8p_1 - 20p_2 & -8p_2 - 20p_2 \end{bmatrix} + \begin{bmatrix} -8p_1 - 8p_2 & -8p_1 - 20p_2 \\ -8p_2 - 8p_2 & -8p_2 - 20p_2 \end{bmatrix} - \left(\begin{bmatrix} p_1 - p_2 \\ p_2 - p_2 \end{bmatrix} \cdot 4 \cdot \begin{bmatrix} p_1 - p_2; 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -8p_1 - 8p_2 & -16p_2 \\ -8p_1 - 20p_2 & -28p_2 \end{bmatrix} + \begin{bmatrix} -8p_1 - 8p_2 & -8p_1 - 20p_2 \\ -16p_2 & -28p_2 \end{bmatrix} - 4\begin{bmatrix} (p_1 - p_2)^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 16p_1 + 16p_2 - 4(p_1 - p_2)^2 & ; 8p_1 + 36p_2 \\ 8p_1 + 36p_2 & ; 56p_2 \end{bmatrix} = \begin{bmatrix} 84 & 96 \\ 96 & 112 \end{bmatrix}$$

$$56p_2 = 112 \qquad 8p_1 + 72 = 96$$
$$p_2 = 2 \qquad 8p_1 = 24$$
$$p_1 = 3$$

$$K^* = R^{-1}B' \cdot P = 4\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 12 - 8 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \end{bmatrix}$$

f)

$$\operatorname{rank}\begin{bmatrix} C \\ C(A-BK) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \multicolumn{2}{c}{C\left(\begin{bmatrix} -8 & -8 \\ -8 & -20 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -8-k_1 & -8-k_2 \\ -8+k_1 & -20+k_2 \end{bmatrix}} \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -8+k_1 & -20+k_2 \\ -8-k_1 & -8-k_2 \end{bmatrix} = 2 \Rightarrow \text{always observable}$$