Problem 1: a)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & a & b \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ a & 1 \end{bmatrix} \qquad C = [0 \ 0 \ 1]$$

- Rosenbrock System Matrix:

$$P(s) = \begin{pmatrix} SI - A & -B \\ C & D \end{pmatrix}$$

$$SI - A = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & a & b \end{bmatrix} = \begin{bmatrix} S & -1 & 0 \\ 0 & S & -1 \\ +3 & -a & S-b \end{bmatrix}$$

$$P(S) = \begin{bmatrix} S & -1 & 0 & -1 & 0 \\ 0 & S & -1 & -1 & 0 \\ 3 & -a & S-b & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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b) The invariant zeros are those frequencies which meet one of following criteria:

1) $\det P(s_0) = 0$ für $m = r$
2) $\text{Rang}\ P(s_0) < \max \text{Rang}\ P(s)$ für $m \neq r$

Rang $P(s_0) \overset{!}{<} 4$

* Statement: The are no invarient Zeros!

* Argument: It is verifiable that the rows of $P(s_0)$ are clearly independent of each other. Simple inspection of the giben Matrix suggests the above ~~statement~~ Statment.

No invariant Zeros $\Longrightarrow$ No decoupling Zeros

b) - Part 2

c) characteristic Polynom of A

$$|SI-A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3 & -a & s-b \end{vmatrix} = s^2(s-b)+3-as$$

$$= s^3 - bs^2 - as + 3$$

According to Hurwitz $\Rightarrow$ $\begin{cases} \text{i) all cofficient } a_i \text{ positive} \\ \text{ii) } H_n = \begin{vmatrix} a_{n-1} & a_{n-3} & \cdots \\ a_n & a_{n-2} & \cdots \\ \vdots & \cdots & \end{vmatrix} \end{cases}$

$$\begin{cases} a, b < 0 \\ \Delta_2 = \begin{vmatrix} -b & 3 \\ 1 & -a \end{vmatrix} = ab-3 > 0 \Rightarrow ab > 3 \end{cases}$$

Stabile and asymptotic Stabile

If $a, b < 0$ and $ab > 3 \Rightarrow$ asymptotic Stable

If $a, b < 0$ and $ab \geq 3 \Rightarrow$ Stabile

otherwise $\Rightarrow$ Unstable !

d) Kalman $[B \quad AB \quad A^2B]$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & a & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -3+a & b \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & a & b \\ -3b & -3+ab & a+b^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & b \\ -3b+ab-3 & a+b^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & a & b \\ 0 & 1 & a-3 & b & ab-3b-3 & b^2+a \end{bmatrix}$$

The rows are independent any way.

For all $a, b \rightarrow A, B$ are fully Controllable

d) Part 2

Characteristic equation of an observer

$\det(\lambda I - A + LC)$

$$A - LC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & a & b \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -l_1 \\ 0 & 0 & 1-l_2 \\ -3 & a & b-l_3 \end{bmatrix}$$

$$\det(\lambda I - A + Ic) =$$

$$= \lambda^3 + (l_3 - b)\lambda^2 + (al_2 - 3l_2 - a)\lambda - 3(l_2 - 1)$$

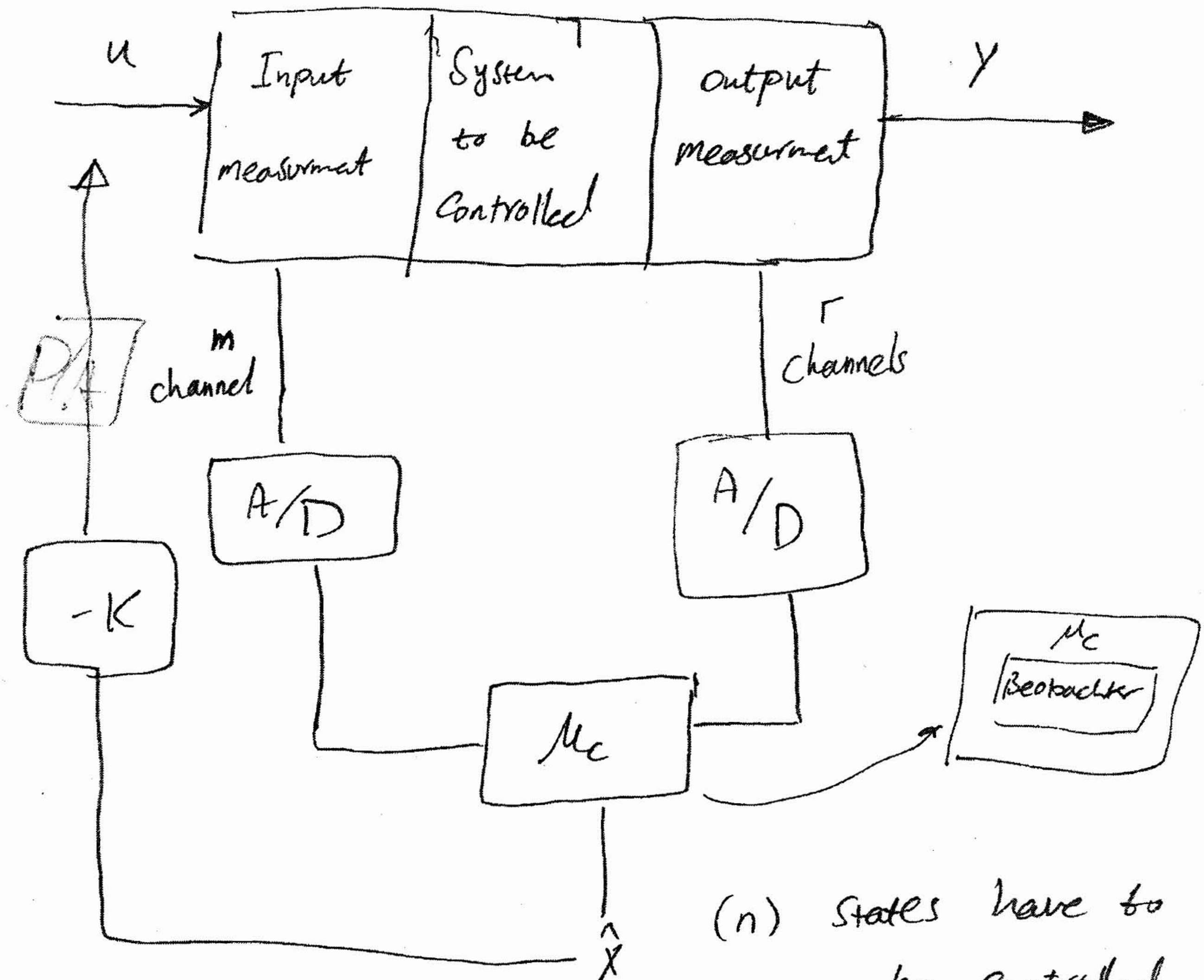
Desired characteristic equation: $(\lambda+5)(\lambda-5)(\lambda+10)$

$$= \lambda^3 + 10\lambda^2 - 25\lambda - 250$$

Comparison of the ~~eq~~ Coefficients

$$\left.\begin{aligned} l_3 - b &= 10 \\ al_2 - 3l_1 - a &= -25 \\ -3(l_2 - 1) &= -250 \end{aligned}\right\} \Rightarrow \begin{bmatrix} l_1 = \dfrac{250a + 75}{9} \\ l_2 = \dfrac{253}{3} \\ l_3 = b + 10 \end{bmatrix}$$

e)



(n) states have to be controlled

For State Feedback $\boxed{u = -K\hat{x}}$

$A = n \times m$

$B = n \times m$

$C = r \times n$

$L = n \times r$

$x, \hat{x} = n \times 1$

$y = r \times 1$

$u = m \times 1$

$\Rightarrow$ Rang C

the number of independent matrix rows express number of measurement.

Problem 2:

a) i: Check for stability:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \right)$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 3 & 0 \\ -1 & -1 & \lambda - 2 \end{vmatrix}$$

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 3)(\lambda - 2) \overset{!}{=} 0$$

$$\Rightarrow \lambda_1 = +1 \,, \quad \lambda_2 = +3 \,, \quad \lambda_3 = +2 \Rightarrow$$

The system is unstable, therefore, it is necessary to control the system with respect to the time behavior of the system to guarantee ~~stability~~ stability.

a) ii: Check for controllability

$Q_S = [B \;\; AB \; \cdots \; A^{n-1}B]$

$\text{rank}\, Q_S \overset{!}{=} n$

$$A \cdot B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A^2 \cdot B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 3 & 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \\ -5 & 3 \end{bmatrix}$$

$$Q_S = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -3 & 0 & -9 & 0 \\ 0 & 0 & -1 & 1 & -5 & 3 \end{array}\right]$$

$\text{rank}\, Q_S = 3 = n \Rightarrow$ System is full contollable $\Rightarrow$

Yes, it is possible to place the eigenvalues of the system to arbitrary locations.

b) Desired eigenvalues:

$\lambda_1 = -1.0,\ \lambda_2 = -2.0,\ \lambda_3 = -3.0$

$(\lambda+1)(\lambda+2)(\lambda+3) = \lambda^3 + 6\lambda^2 + 11\lambda + 6 \overset{!}{=} 0$

$$(A-BK) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 & K_3 \\ K_4 & K_5 & K_6 \end{bmatrix}$$

$$(A-BK) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} K_4 & K_5 & K_6 \\ -K_1 & -K_2 & -K_3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-BK) = \begin{bmatrix} 1-K_4 & -K_5 & -K_6 \\ +K_1 & 3+K_2 & +K_3 \\ 1 & 1 & 2 \end{bmatrix} ;\ K_3 = K_4 = K_5 = 0.$$

$$(A-BK) = \begin{bmatrix} 1 & 0 & -K_6 \\ K_1 & 3+K_2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$(A-BK)-\lambda I = \begin{bmatrix} 1 & 0 & -K_6 \\ K_1 & 3+K_2 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$(A-BK)-\lambda I = \begin{bmatrix} 1-\lambda & 0 & -K_6 \\ K_1 & 3+K_2-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix}$$

$$\det[(A-BK)-\lambda I] = \begin{vmatrix} 1-\lambda & 0 & -K_6 \\ K_1 & 3+K_2-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} \quad (4)$$

$$= (1-\lambda)\begin{vmatrix} 3+K_2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} - 0 \begin{vmatrix} K_1 & 0 \\ 1 & 2-\lambda \end{vmatrix}$$

$$-K_6 \begin{vmatrix} K_1 & 3+K_2-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (1-\lambda)(3+K_2-\lambda)(2-\lambda) - K_6(K_1-(3+K_2-\lambda))$$

$$\det[(A-BK)-\lambda I] = \lambda^3 - (6+K_2)\lambda^2 - (-11-3K_2-K_6)\lambda - 6 - 2K_2 + K_6K_1$$
$$-3K_6 - K_6K_2 \overset{!}{=} 0$$

Canparison of Coefficients,

λ^3: $1 \overset{!}{=} 1$ ✓

λ^2: $6 \overset{!}{=} -(6+K_2) \Rightarrow \boxed{K_2 = -12}$

λ^1: $11 = -(-11-3K_2-K_6) \Rightarrow \cancel{-11} = \cancel{-11} - 3K_2 - K_6$

$\Rightarrow 0 = -3K_2 - K_6 \Rightarrow K_6 = -3K_2 = -3(-12)$

$\boxed{K_6 = +36}$

λ^0: $6 = -6 - 2K_2 + K_6K_1 - 3K_6 - K_6K_2 \Rightarrow$

$12 = -2(-12) + 36K_1 - 3(36) - (36)(-12)$

$12 = +24 + 36K_1 - 108 + 432 \Rightarrow$

$36K_1 = -336 \Rightarrow \boxed{K_1 = \frac{-336}{36}}$

$$K = \begin{bmatrix} \frac{-336}{36} & -12 & 0 \\ 0 & 0 & +36 \end{bmatrix}$$

c) The formulas are:

* $AP + PA^T - PC^T R^{-1} C P + Q = 0.$
* $L = (R^{-1} C P)^T = P C^T R^{-1}.$

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d)

$$G(s) = C\cdot(sI - A)^{-1} \cdot B + \cancel{D}^{\,0}$$

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \cdot \operatorname{adj}(sI - A)$$

$$(sI - A) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s-3 & 0 \\ -1 & -1 & s-2 \end{bmatrix}$$

$$\det(sI - A) = (s-1)\begin{vmatrix} s-3 & 0 \\ -1 & s-2 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ -1 & s-2 \end{vmatrix} + 0\begin{vmatrix} 0 & s-3 \\ -1 & -1 \end{vmatrix}$$

$$\det(sI - A) = (s-1)(s-3)(s-2)$$

$$\operatorname{adj}(sI - A) = \begin{pmatrix} (s-3)(s-2) & 0 & 0 \\ 0 & (s-1)(s-2) & 0 \\ (s-3) & (s-1) & (s-1)(s-3) \end{pmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (sI - A)^{-1} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

6

$$G(s) = \frac{[1\ 0\ 0]}{(s-1)(s-3)(s-2)} \begin{bmatrix} 0 & (s-3)(s-2) \\ -(s-1)(s-2) & 0 \\ -(s-1) & (s-3) \end{bmatrix}$$

$$G(s) = \frac{1}{(s-1)(s-3)(s-2)} \begin{bmatrix} 0 & (s-3)(s-2) \end{bmatrix}$$

$$\boxed{G(s) = \begin{bmatrix} 0 & \frac{1}{(s-1)} \end{bmatrix}}$$

e) the related integral $J \to$ Min.

$$J = \frac{1}{2}\int_0^{\infty} [x^T Q x + y^T R y]\, dt \to \text{Min}$$

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow J = \int_0^{\infty} x^T Q x \,. dt$$

$$J = \frac{1}{2}\int_0^{\infty} \left([x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \right) dt$$

$$J = \frac{1}{2}\int_0^{\infty} \left([x_1 \;\; 2x_2 \;\; 3x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) dt$$

$$J = \frac{1}{2}\int_0^{\infty} ([x_1^2 + 2x_2^2 + 3x_3^2]) dt$$

the dimensions:

x: $n \times 1 = 3 \times 1$

B: $n \times 2 = 3 \times 2$

C: $2 \times n = 2 \times 3$

$\boxed{3}$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K_f}{s(s+2)}$$

ⓐ

$$s(s+2)Y(s) = K_f U(s)$$
$$[s^2+2s]\,Y(s) = K_f U(s)$$
$$\ddot{y}(t) + 2\dot{y}(t) = K_f u(t)$$
$$\ddot{x}_2 + 2\dot{x}_2 = K_f u$$

$y = x_2$, $\dot{x}_2 = x_1$

$\ddot{x}_2 = \ddot{y} = \dot{x}_1$

$$\longrightarrow \boxed{\begin{aligned} \dot{x}_1 &= -2x_1 + K_f u \\ \dot{x}_2 &= x_1 \end{aligned}}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} K_f \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} K_f \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

ⓑ To find the eigenvalues ;

$$\det[\lambda I - A] = 0$$

$$[\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda+2 & 0 \\ -1 & \lambda \end{bmatrix}$$

$$\det[\lambda I - A] = (\lambda+2)\lambda = 0$$

$\Rightarrow$ eigenvalues $\boxed{\lambda_1 = -2}$, $\boxed{\lambda_2 = 0}$

the system is stable (boundary , not asymptotic stable ($\lambda_2 = 0$).

(c) controllability check ;

$$Q_s = [B \quad AB] = \begin{bmatrix} K_f & -2K_f \\ 0 & K_f \end{bmatrix}$$

$$\det[Q_s] = K_f^2 \neq 0$$

$\rightarrow$ fully controllable for all $K_f \neq 0$

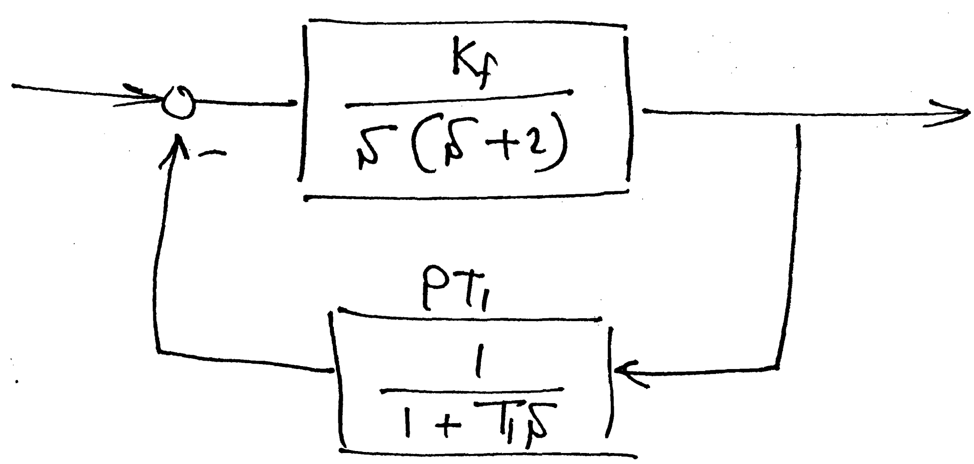
observability check ;

$$Q_b = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det[Q_b] = -1 \neq f(K_f)$$

$\rightarrow$ fully observable for all K_f

(d)



$$G_r = \frac{G_1}{1+G_1 G_2} = \frac{\frac{K_f}{s(s+2)}}{1 + \frac{K_f}{s(s+2)} \frac{1}{(1+T_1 s)}}$$

$$= \frac{K_f(1+T_1S)}{S(S+2)(1+T_1S)+K} = \frac{Y}{U}$$

$$K_f(1+T_1S)U = [S(S+2)(1+T_1S)+K_f]Y$$

$$K_fU + K_fT_1US = S^2Y + T_1YS^3 + 2T_1YS^2 + 2SY + K_fY$$

Inv. laplace ;

$$K_fu + T_1K_f\dot{u} = \ddot{y} + T_1\dddot{y} + 2T_1\ddot{y} + 2\dot{y} + K_fy$$

$$T_1\dddot{y} + (2T_1+1)\ddot{y} + 2\dot{y} + K_fy = K_fT_1\dot{u} + K_fu$$

$$\Rightarrow \dddot{y} + (T_1 + \frac{1}{T_1})\ddot{y} + \frac{2}{T_1}\dot{y} + \frac{K_f}{T_1}y = K_f\dot{u} + \frac{K_f}{T_1}u$$

$$\dot{x} = Ax + B_1u + B_2\dot{u}$$

$$x = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} \Longrightarrow A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{K_f}{T_1} & \frac{-2}{T_1} & -(T_1+\frac{1}{T_1}) \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{K_f}{T_1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ K_f \end{bmatrix}, \quad C_1 = [0 \quad 0 \quad 1]$$

(e)

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_f & -\frac{1}{T_1} & -1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_n = A_3 - B_3K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_f & -\frac{1}{T_1} & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [K_1 \quad K_2 \quad K_3]$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_f-K_1 & -\frac{1}{T_1}-K_2 & -1-K_3 \end{bmatrix}$$

$$\lambda I - A_n = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ K_f+K_1 & \frac{1}{T_1}+K_2 & \lambda+1+K_3 \end{bmatrix}$$

$$\det[\lambda I - A_n] = \lambda\left[\lambda(\lambda+1+K_3) + \left(\frac{1}{T_1}+K_2\right)\right] + [K_f+K_1]$$

$$= \lambda^3 + [1+K_3]\lambda^2 + \left[\frac{1}{T_1}+K_2\right]\lambda + [K_f+K_1] = 0$$

$$\longrightarrow *$$

Hurwitz cr. ;

I) $1+K_3 > 0 \longrightarrow K_3 > -1$

$\frac{1}{T}+K_2 > 0 \longrightarrow K_2 > -\frac{1}{T_1}$

$K_f+K_1 > 0 \longrightarrow K_1 > -K_f$

II) Hurwitz det. $= \begin{bmatrix} 1+K_3 & K_f+K_1 & 0 \\ 1 & \frac{1}{T_1}+K_2 & 0 \\ 0 & 1+K_3 & K_f+K_1 \end{bmatrix}$

$H_1:\ 1+K_3 > 0$

$H_2:\ (1+K_3)\left(\frac{1}{T_1}+K_2\right) - K_f - K_1 > 0$

$\rightarrow K_2 + \frac{1}{T_1}K_3 + K_2K_3 - K_1 > K_f - \frac{1}{T_1}$

→ conditions ;

$K_1 > -K_f$, $K_2 > -\frac{1}{T_1}$, $K_3 > -1$,

$K_2 + \frac{1}{T} K_3 + K_2 K_3 - K_1 > K_f - \frac{1}{T_1}$

(f) $\lambda_{1,2,3} = -1$

$(\lambda + 1)(\lambda + 1)(\lambda + 1) = 0$

$\rightarrow \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

comparison with $*$

$1 + K_3 = 3 \longrightarrow K_3 = 3 - 1 = 2$

$\frac{1}{T} + K_2 = 3 \longrightarrow K_2 = 3 - \frac{1}{T_1}$

$K_f + K_1 = 1 \longrightarrow K_1 = 1 - K_f$

$\rightarrow K = \begin{bmatrix} 1-k & 3-\frac{1}{T} & 2 \end{bmatrix}$

Control Theory WiSe 08/09

Problem 4

a) Calculate the eigenvalues

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{d}{m} \end{bmatrix}$$

$$\det |A - \lambda I| \overset{!}{=} 0$$

$$\det \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{m} & -\frac{d}{m} - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (-\lambda)\left(-\frac{d}{m} - \lambda\right) + \frac{c}{m} = 0$$

$$\Leftrightarrow \quad \frac{d}{m}\lambda + \lambda^2 + \frac{c}{m} = 0$$

$$\Rightarrow \quad \lambda_{1,2} = -\frac{d}{2m} \pm \frac{1}{2}\sqrt{\frac{d^2}{m^2} - \frac{4c}{m}}$$

$$= -\frac{d}{2m} \pm \sqrt{\frac{d^2}{4m^2} - \frac{c}{m}}$$

b)

Stodola

$$\lambda^2 + \frac{d}{m}\lambda + \frac{c}{m} = 0 \qquad \text{All } a_i \text{ must be } > 0!$$

$$\left.\begin{aligned} a_2 &= 1 \\ a_1 &= \frac{d}{m} \\ a_0 &= \frac{c}{m} \end{aligned}\right\} \text{ all } > 0 \qquad \text{for } c, m, d > 0$$

$\rightarrow$ For c, m, and $d > 0$ the system is asymptotically stable!

c) $m = 2kg$; $c = 0.5\ kg/s^2$; $d = 0.25\ kg/s$ ②

$$\Rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & -\frac{1}{8} \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ \frac{1}{4} & \lambda + \frac{1}{8} \end{bmatrix}$$

$b_1 = 1$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & b_2 \end{bmatrix}$$

Hautus criterion for controllability

$$\text{Rank} \begin{bmatrix} \lambda & -1 & 1 & 0 \\ \frac{1}{4} & \lambda + \frac{1}{8} & 0 & b_2 \end{bmatrix} = \underline{\underline{2}}$$

$\Rightarrow$ system is controllable for $b_1 = 1$.

Observability for $c_2 = 1$

$$C = \begin{bmatrix} c_1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Rank} \begin{bmatrix} \lambda & -1 \\ \frac{1}{4} & \lambda + \frac{1}{8} \\ c_1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{2}}$$

$\Rightarrow$ system is observable for $c_2 = 1$

d) Create state feedback

$$A - BK = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & -\frac{1}{8} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -k_1 & 1 - k_2 \\ -\frac{5}{4} & -\frac{9}{8} \end{bmatrix}$$

$$\det(\lambda I - (A - BK)) = 0$$

$$\det \begin{vmatrix} \lambda + k_1 & k_2 - 1 \\ \frac{5}{4} & \lambda + \frac{9}{8} \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + (k_1 + \tfrac{9}{8})\lambda + \tfrac{9}{8}k_1 + \tfrac{5}{4} - \tfrac{5}{4}k_2 = 0$$

desired :

$$(\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2$$

comparing coefficients

$$1 = 1 \quad \checkmark$$

$$k_1 + \frac{9}{8} = 3 \quad \Rightarrow \quad \underline{\underline{k_1 = \frac{15}{8}}}$$

$$\frac{9}{8} k_1 + \frac{5}{4} - \frac{5}{4} k_2 = 2$$

$$\vdots$$

$$\frac{5}{4} k_2 = \frac{87}{64}$$

$$\Rightarrow \quad \underline{\underline{k_2 = \frac{87}{80}}}$$

$$\Rightarrow \quad k = \begin{bmatrix} \frac{15}{8} & \frac{87}{80} \\ 1 & 1 \end{bmatrix}$$

e) **Observer** $\qquad c_1 = c_2 = 1$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & l_1 \\ 1 & l_2 \end{bmatrix}$$

$$A - LC = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & -\frac{1}{8} \end{bmatrix} - \begin{bmatrix} 1 & l_1 \\ 1 & l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 - l_1 \\ -\frac{1}{4} - 1 & -\frac{1}{8} - l_2 \end{bmatrix}$$

$$\det |\lambda I - (A - LC)| \overset{!}{=} 0$$

$$\det \begin{vmatrix} \lambda + 1 & l_1 - 1 \\ \frac{5}{4} & \lambda - (-\frac{1}{8} - l_2) \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + \left(\frac{1}{8} + l_2 + 1\right)\lambda + \frac{1}{8} + l_2 - \frac{5}{4} l_1 + \frac{5}{4} = 0$$

comparing coefficients

desired

$$(\lambda+3)(\lambda+4) = \lambda^2 + 7\lambda + 12$$

① $1 = 1$

② $\frac{1}{8} + l_2 + 1 = 7 \quad \Rightarrow \quad l_2 = \frac{47}{8}$

③ $\frac{1}{8} + l_2 - \frac{5}{4} l_1 + \frac{5}{4} = 0 \quad \Rightarrow \quad l_1 = -\frac{19}{5}$

$$L = \begin{bmatrix} 1 & -\frac{19}{5} \\ 1 & \frac{47}{8} \end{bmatrix}$$

f) $\dot{e} = \dot{x} - \dot{\hat{x}}$

$= Ax + Bu - (A\hat{x} + Bu + L(y - \hat{y}))$

$= (A - LC) \cdot e$

$$\dot{e} = \begin{bmatrix} -1 & \frac{24}{5} \\ -\frac{5}{4} & -6 \end{bmatrix} e$$

$\lambda_{o_1} = -3 \quad , \quad \lambda_{o_2} = -4$

Yes, due to the asymptotic stability.

g)

$$\begin{bmatrix} \dot{x} \\ e \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

$$\dot{x} = Ax + Bu \qquad\qquad u = -K\hat{x}$$

$$\dot{x} = Ax - BK\hat{x}$$

$$= (A-BK)x + BK(x-\hat{x})$$

$$\begin{bmatrix} \dot{x} \\ e \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ e \end{bmatrix} = \begin{bmatrix} -\frac{15}{8} & -\frac{7}{80} & \frac{15}{8} & \frac{87}{80} \\ -\frac{5}{4} & -\frac{9}{8} & 1 & 1 \\ 0 & 0 & -1 & \frac{24}{5} \\ 0 & 0 & -\frac{5}{4} & -6 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

eigenvalues of the extended system

$\lambda_1 = -1$; $\lambda_2 = -2$; $\lambda_3 = -3$; $\lambda_4 = -4$

$\rightarrow$ Separation principle

4 h) 5 – 10 (other authors 2–6) times faster than the dynamics of the controlled system

i) transfer function matrix:

$$G(s) = C\,[sI - A]^{-1} B + D$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{1}{4} & s+\frac{1}{8} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{s(s+\frac{1}{8}) + \frac{1}{4}} \begin{bmatrix} s+\frac{1}{8} & 1 \\ -\frac{1}{4} & s \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+\frac{1}{8}}{s^2+\frac{1}{8}s+\frac{1}{4}} & \dfrac{1}{s^2+\frac{1}{8}s+\frac{1}{4}} \\ \dfrac{-\frac{1}{4}}{s^2+\frac{1}{8}s+\frac{1}{4}} & \dfrac{s}{s^2+\frac{1}{8}s+\frac{1}{4}} \end{bmatrix}$$

j)

$$y_2(s) = g_{21}(s)\, u_1(s)$$

$$g_{21}(s) = \frac{-\frac{1}{4}}{s^2 + \frac{1}{8}s + \frac{1}{4}}$$