Problem 1 (30 points)

a) (4 points)
A system is fully controllable and asymptotically stable. Can the system be destabilized by control? State reasons.

Yes, if the system is fully controllable the eigenvalues can be shifted arbitrarily.

b) (8 points)
The dynamical behavior of a machine is described by the matrices

$$A = \begin{bmatrix} 0 & a \\ -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The plant is controlled by a state feedbackcontroller with the matrix $K = \begin{bmatrix} 0 & k_1 \end{bmatrix}$. The parameters are given by $a = -5$ and $k_1 = 2$.

Due to maintenance the parameters are changed to $a = 4$ and $k_1 = 3$. Furthermore the feedback is changed to positive feedback.

What can be stated about the stability of the closed-loop system (machine with controller) in the two cases? State reasons.

System a)

$$|(\lambda I - (A - BK))| \overset{!}{=} 0$$

$$\lambda^2 - 4\lambda - 21 \overset{!}{=} 0$$

$\lambda_1 = 7 \qquad \lambda_2 = -3$ $\Rightarrow$ system is unstable

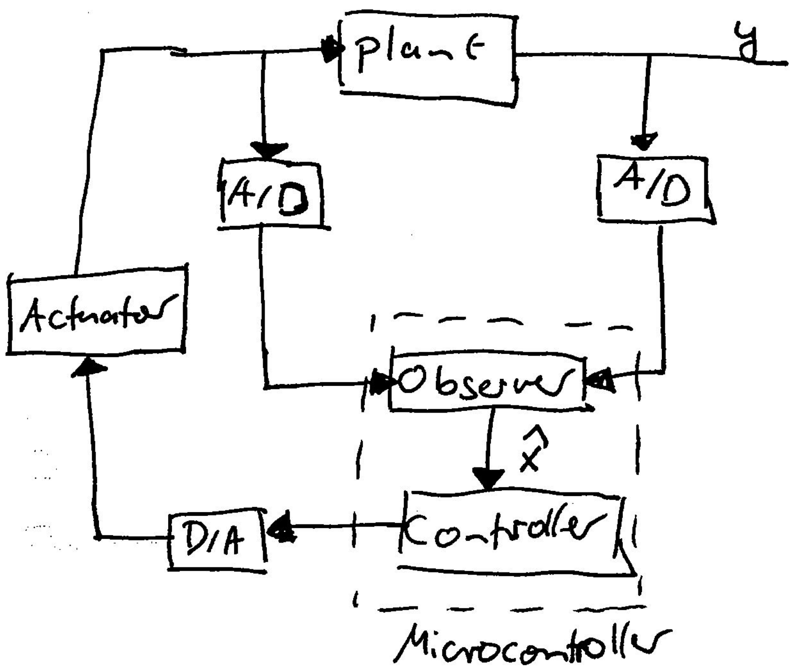
System b)

$$|(\lambda I - (A + BK))| \overset{!}{=} 0$$

$$\lambda^2 - 4\lambda + 21 = 0$$

$\lambda_{1,2} = 2 \pm \sqrt{-17}$ $\Rightarrow$ system is unstable

c) (9 points)
Declare the principal strategy for the practical realization of model-based state control. Define the processor-based observer realization using elements of A/D and D/A conversion, the micro controller, and measuring devices by drawing a sketch and denoting the relating devices.



d) (5 points)
A dynamical system is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

State the transfer function of the system and calculate the invariant zeros.

$$G(s) = C\,(sI - A)^{-1}\,B = \frac{1}{s-1}$$

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s+1 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 1 \\ 0 & -1 & s-2 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\det P(s_0) = (s_0+1)(2-s_0) \overset{!}{=} 0$$

$$s_{01} = -1$$
$$s_{02} = 2$$

e) (4 points)

The eigenvalues of a system are given by

$$\begin{aligned}
\lambda_1 &= 0,\\
\lambda_{2,3} &= \pm\ 3\ j,\\
\lambda_{4,5} &= 5\ \pm\ 2\ j,\\
\lambda_6 &= -7,\\
\lambda_7 &= 0\ +\ 0j,\\
\lambda_8 &= 88\ +\ 88j,\\
\lambda_9 &= -88\ -\ 88j,\\
\lambda_{10,11} &= -1 \pm\ 1j,\\
\lambda_{12,13} &= 10^8\ \pm\ 10^{-7}, and\\
\lambda_{14,15} &= 10^{-8} \pm\ 10^7 j.
\end{aligned}$$

What is wrong with the statements denoting the eigenvalues? State reasons.

The eigenvalues λ_8, λ_9 have to be conjugate complex eigenvalues.

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Problem 2 (25 points)
A system to be controlled is given in state space description by

$$A = \begin{bmatrix} 2 & 3 & a \\ 0 & -4 & 1 \\ 0 & 0 & b \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 0 & c \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad D = 0.$$

a) (3 points)
Calculate the eigenvalues of the system. Is the system asymptotically stable? State reasons.

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-2 & -3 & -a \\ 0 & \lambda+4 & -1 \\ 0 & 0 & \lambda-b \end{bmatrix}\right)$$

$$= (\lambda-2)(\lambda+4)(\lambda-b) \overset{!}{=} 0$$

$$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = -4, \quad \lambda_3 = b$$

$\mathrm{Re}(\lambda_1) > 0 \Rightarrow$ The system is unstable.

For the next tasks, the parameters a and b are given as

$$a = 0 \quad \text{and} \quad b = -1.$$

b) (5 points)
Give the controllability matrix Q_S and define the range of c in which the system is fully controllable.

$$Q_s = \begin{bmatrix} 1 & 3 & 14 & 6 & -20 & 12+3c \\ 4 & 0 & -16 & c & 64 & -5c \\ 0 & c & 0 & -c & 0 & c \end{bmatrix}$$

The Q_s matrix has a full rank only if $c \neq 0$.

$\Rightarrow$ The system is fully controllable when $c \neq 0$.

Use $c = 1$ for all following tasks.

c) (9 points)
The system should be controlled by full state feedback (negative feedback). The feedback gain matrix is supposed as

$$K = \begin{bmatrix} k_1 & 0 & k_2 \\ 0 & k_3 & 0 \end{bmatrix}.$$

Determine the parameters k_1, k_2, and k_3 such that the closed-loop system has the eigenvalues

$$\lambda_{1/2} = -2 \quad \text{and} \quad \lambda_3 = -1.$$

The characteristic polynomial of the desired dynamics:

$$(\lambda+2)^2(\lambda+1) = \lambda^3 + 5\lambda^2 + 8\lambda + 4 \qquad (*)$$

$$\det(\lambda I - (A - BK)) =$$

$$\lambda^3 + (3+k_1)\lambda^2 + (-6 - 12k_1k_3 + 12k_1 - 4k_2k_3)\lambda$$
$$- 8 - 2k_3 - 11k_1k_3 + 8k_2k_3 + 16k_1$$

Comparing the coefficients with $(*)$

$$3 + k_1 = 5 \Rightarrow \underline{\underline{k_1 = 2}}$$

$$\Rightarrow \begin{cases} 28 - 23k_3 - 4k_2k_3 = 8 \\ 24 - 24k_3 + 8k_2k_3 = 4 \end{cases}$$

$$\Rightarrow \underline{\underline{k_3 = \frac{6}{7}}}, \quad \underline{\underline{k_2 = \frac{1}{12}}}$$

In order to realize the control task (without measurements of all states), an identity observer has to be designed.

d) (3 points)
Give the observability matrix Q_B. Why is the system fully observable?

$$\left.\begin{aligned} c &= [1\ 0\ 0] \\ cA &= [2\ 3\ 0] \\ cA^2 &= [4\ -6\ 3] \end{aligned}\right\} \Rightarrow Q_B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & -6 & 3 \end{bmatrix}$$

$$\mathrm{Rank}(Q_B) = 3$$

$\Rightarrow$ The system is fully observable.

e) (5 points)
The dynamics of the observer should have the eigenvalues

$$\lambda_1 = -1, \quad \lambda_1 = -2, \quad \text{and} \quad \lambda_3 = -3.$$

Define the related parameters l_1, l_2, and l_3 of the observer gain matrix

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}.$$

The characteristic polynomial of the desired dynamics:

$$(\lambda+1)(\lambda+2)(\lambda+3) = \lambda^3 + 6\lambda^2 + 11\lambda + 6 \quad (*)$$

$$\det(\lambda I - (A - LC))$$

$$= \lambda^3 + (3 + l_1)\lambda^2 + (3l_2 - 6 + 5l_1)\lambda + 3l_3 - 8 + 3l_2 + 4l_1$$

Comparing the coefficients with $(*)$

$$\Rightarrow 3 + l_1 = 6 \Rightarrow \underline{\underline{l_1 = 3}}$$

$$\Rightarrow \begin{cases} 3l_2 + 9 = 11 \\ 3l_3 + 4 + 3l_2 = 6 \end{cases} \Rightarrow \begin{aligned} &\underline{\underline{l_2 = \tfrac{2}{3}}} \\ &\underline{\underline{l_3 = 0}} \end{aligned}$$

Problem 3 (20 points)
A control plant is given with the state equations

$$\begin{aligned}\dot{x}_1(t) + ax_1(t) &= u(t),\\ \dot{x}_2(t) &= bu(t),\\ \dot{x}_3(t) + cx_3(t) &= x_1(t) + x_2(t), \quad \text{and}\\ y &= x_3(t)\end{aligned}$$

in time domain.

a) (2 points)
Develop the state space model with the state vector $x = [x_1(t) \quad x_2(t) \quad x_3(t)]^T$ and give the matrices A, B, C, and D.

$$\dot{x} = \underbrace{\begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -c \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ b \\ 0 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} x + \underbrace{[0]}_{D} u$$

□

b) (1 point)
For a modified system the matrix A is given by

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix}.$$

Is the system asymptotically stable?

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+3 & 0 & 0 \\ 0 & \lambda & 0 \\ -1 & -1 & \lambda+2 \end{vmatrix} \stackrel{!}{=} 0$$

$\Rightarrow$ eigenvalues $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = -3$

The system is not asymptotically stable because $\lambda_1 = 0$ (boundary stable).

For the following subtasks c) - g) an unstable system with the state space representation

$$\dot{x}(t) = \begin{bmatrix} -2 & 4 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u(t)$$

is given.
The system should be stablized with the help of a state feedback controller $u = -Kx$, where

$$K = [k_1 \quad k_2 \quad k_3] \,.$$

c) (4 points)
State the eigenvalues and the corresponding eigenvectors of the uncontrolled system.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+2 & -4 & 1 \\ 0 & \lambda-1 & -5 \\ 0 & 0 & \lambda+3 \end{vmatrix} = (\lambda+2)(\lambda-1)(\lambda+3) \overset{!}{=} 0$$

eigenvalues: $\lambda_1 = -2, \quad \lambda_2 = 1, \quad \lambda_3 = -3$

eigenvectors: $A v_i = \lambda_i v_i \Rightarrow (A - \lambda_i I) v_i = 0$

$$\lambda_1 = -2, \quad \begin{bmatrix} 0 & -4 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1, \quad \begin{bmatrix} 3 & -4 & 1 \\ 0 & 0 & -5 \\ 0 & 0 & 4 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$$\lambda_3 = -3, \quad \begin{bmatrix} -1 & -4 & 1 \\ 0 & -4 & -5 \\ 0 & 0 & 0 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_3 = \begin{bmatrix} 6 \\ -\frac{5}{4} \\ 1 \end{bmatrix}$$

d) (1 point)
Why is the uncontrolled system unstable? State reasons.

The system is unstable because of $\mathrm{Re}(\lambda_2)=1>0$.

e) (4 points)
Is the given system fully controllable? If not, which eigenvalues are not controllable?

Controllability

Kalman: $Q_s = [B \quad AB \quad A^2B]$

$$= \begin{bmatrix} 0 & 8 & -8 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank $Q_s = 2 < 3 \Rightarrow$ not fully controllable

Hautus:

λ_1: Rank $[\lambda_1 I - A \quad B]$

$$= \text{Rank} \begin{bmatrix} 0 & -4 & 1 & 0 \\ 0 & -3 & -5 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} = 3 \Rightarrow \lambda_1 \text{ controllable}$$

λ_2: Rank $[\lambda_2 I - A \quad B]$

$$= \text{Rank} \begin{bmatrix} 3 & -4 & 1 & 0 \\ 0 & 0 & -5 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix} = 3 \Rightarrow \lambda_2 \text{ controllable}$$

λ_3: Rank $[\lambda_3 I - A \quad B]$

$$= \text{Rank} \begin{bmatrix} -1 & -4 & 1 & 0 \\ 0 & -4 & -5 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2 < 3 \Rightarrow \lambda_3 \text{ not controllable}$$

$\Rightarrow \lambda_1 = -2, \lambda_2 = 1$ controllable, $\lambda_3 = -3$ not controllable

f) (4 points)
Determine the characteristic polynomial of the controlled system. Determine the gain matrix K so that the poles are placed at $s_{1,2} = -4 \pm j$ and $s_3 = -3$.

characteristic Polynomial:

$$\det(\lambda I - A + BK) = (\lambda+3)(\lambda^2 + (2k_2+1)\lambda + 8k_1 + 4k_2 - 2)$$

$$= \lambda^3 + (2k_2+4)\lambda^2 + (8k_1 + 10k_2 + 1)\lambda + 3(8k_1 + 4k_2 - 2)$$

Polynomial of the desired dynamics:

$$(\lambda+3)(\lambda+4+j)(\lambda+4-j)$$

$$=(\lambda+3)(\lambda^2 + 8\lambda + 17)$$

coefficients comparing:

$$\left.\begin{aligned} 2k_2 + 1 &= 8 \\ 8k_1 + 4k_2 - 2 &= 17 \end{aligned}\right\} \Rightarrow \begin{aligned} k_1 &= \frac{5}{8} \\ k_2 &= \frac{7}{2} \end{aligned}$$

k_3 can be arbitrarily chosen.

$$\Rightarrow K = \left[\frac{5}{8} \quad \frac{7}{2} \quad k_3\right]$$

g) (4 points)
For cost reasons only one state of the system can be measured. In order to perform state feedback control, which state should be measured? State reasons.
(Hint: Check the three possibilities $C_1 = [1 \quad 0 \quad 0]$, $C_2 = [0 \quad 1 \quad 0]$, and $C_3 = [0 \quad 0 \quad 1]$).

Observability $Q_B = [C^T \ (CA)^T \ (CA^2)^T]^T$

with C_1: $\text{Rank } Q_B = \text{Rank} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & -1 \\ 4 & -4 & 25 \end{bmatrix} = 3$

$\Rightarrow$ with C_1 the system is fully observable.

with C_2: $\text{Rank } Q_B = \text{Rank} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 5 \\ 0 & 1 & -10 \end{bmatrix} = 2 < 3$

$\Rightarrow$ with C_2 the sys. is not fully observable.

with C_3: $\text{Rank } Q_B = \text{Rank} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 9 \end{bmatrix} = 1 < 3$

$\Rightarrow$ with C_3 the sys. is not fully observable.

$\Rightarrow$ So the first state should be measured to get a fully observable system for realizing state feedback control.

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The linearized system equations yield the following set structurally

$$a_1\ddot{z} = a_2\dot{z} \quad \text{and}$$
$$b_1\dot{T}_g = -b_2(\dot{z} + T_g) + \dot{q}_i.$$

If not stated differently use the equations above to complete the following tasks.

a) (1 point)
What is the physical difference of using either the state vector $x_1 = \begin{bmatrix} z & \dot{z} & T_g \end{bmatrix}^T$ or the state vector $x_2 = \begin{bmatrix} \dot{z} & T_g \end{bmatrix}^T$ to describe the balloon system?

Without z in the state vector, there is no global (absolute) reference.
=> no information about absolute altitude

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b) (4 points)
Give the state space representation of the vertical dynamics of the balloon system using the state vector $x_1 = \begin{bmatrix} z & \dot{z} & T_g \end{bmatrix}^T$ by stating the A, B, C, and D matrices with the rate of heat $\dot{q}_i$ as the input. The position z of the balloon and the temperature T_g of the filling gas are measured.

Additionally, define the type of the system (SISO, SIMO, MISO, or MIMO).

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & a_2/a_1 & 0 \\ 0 & -b_2/b_1 & -b_2/b_1 \end{bmatrix}$$

$$\left.\begin{aligned} B &= \begin{bmatrix} 0 \\ 0 \\ 1/b_1 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}\right\} \Rightarrow \text{SIMO}$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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At a certain ambient condition and several assumptions on the caloric equation, which can be assumed for specific locations on Titan, the state space representation can be approximated by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ c & -1 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \qquad D = 0.$$

If not stated differently use the matrices above to complete the following tasks.

c) (1 point)
For which parameters of c is the system completely controllable? State your reason refering to the rank of the controllability matrix Q_S.

$$Q_S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -8 & 32 \end{bmatrix}$$

$\text{rank}(Q_S) = 1 < n \Rightarrow$ not completely controllable for all c

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d) (2 points)
For which parameters of c is the system completely observable? State your reason refering to the rank of the observability matrix Q_B.

$$Q_B = \begin{bmatrix} 0 & 0 & 1 \\ c & -1 & -4 \\ -4c & c+6 & -16 \end{bmatrix}$$

$$\det(Q_B) = c(c+2)$$

$\text{rank}(Q_B) < n$ iff $c=0 \vee c=-2$

$\Rightarrow$ not completely observable for $c=0 \vee c=-2$

e) (2 points)
For which values of the parameter c is the system asymptotically stable?

$$\det(\lambda I - A) = \lambda(\lambda+2)(\lambda+4) \stackrel{!}{=} 0$$

$$\lambda_1 = 0, \quad \lambda_2 = -2, \quad \lambda_3 = -4$$

$Re\{\lambda_i\} < 0$ not true for λ_1

$\Rightarrow$ System is not asymptotically stable for all c.

The system is transferred to the reduced state space representation with the state vector x_2 as

$$A = \begin{bmatrix} -2 & d \\ -1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \text{and} \quad D = 0.$$

If not stated differently use the matrices above with the parameter $d = 0$ to complete the following tasks.

f) (8 points)
Determine the transfer function matrix $G(s)$ of the state space representation. Determine the eigenvalues, the poles, the invariant zeros, the decoupling zeros, and the transmission zeros of the system.

In case of decoupling zeros, state the type (input/output decoupling zero) additionally.

Eigenvalues:
$$\det(\lambda I - A) = (\lambda+2)(\lambda+4) \quad \Rightarrow \quad \lambda_1 = -2, \; \lambda_2 = -4$$

Transfer function matrix:
$$G(s) = C(sI-A)^{-1}B + D = \frac{2}{s+4}$$

Poles: $s = -4$

Transmission zeros: none

Decoupling / Invariant zeros:
$$\det(P) = 2s+4 = 0 \quad \Rightarrow \quad s_d = -2$$

type: $\operatorname{rank}([s_d I - A \quad -B]) = 1 < n$
$\Rightarrow$ input decoupling zero

g) (5 points)
For the design of an optimal controller for an altitude control of the balloon, the matrices

$$Q = \begin{bmatrix} 20 & 10 \\ 10 & 20 \end{bmatrix}, \qquad P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}, \quad \text{and} \qquad R = 4$$

are given. Calculate K^* of a linear quadratic optimal controller (assume $p_2 = p_3 = 1$, $p_1, p_4 > 0$).

RICATTI equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

$$\begin{bmatrix} 17-4p_1 & 4-2p_4 \\ 4-2p_4 & 20-8p_4-p_4^2 \end{bmatrix} = 0$$

$$\Rightarrow \quad p_1 = 17/4, \quad p_4 = 2$$

$$K^* = R^{-1}B^T P = \begin{bmatrix} 1/2 & 1 \end{bmatrix}$$

The following transfer function matrix $G(s)$ of the closed loop of the balloon system is obtained by using a slightly modified input and output matrix to

$$G(s) = \begin{bmatrix} \dfrac{s+2}{s^2+4ds+4d^2} \\[2ex] \dfrac{s}{s^2+2ds+4s+8d} \end{bmatrix}.$$

h) (2 points)
Determine the values of the parameter d for which the closed loop is input/output stable.

$$G(s) = \begin{bmatrix} \dfrac{s+2}{(s+2d)^2} \\[2ex] \dfrac{s}{(s+4)(s+2d)} \end{bmatrix}$$

I/O - stable, iff $Re\{s_i\} < 0$

$\Rightarrow \quad d > 0$

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