# Reading-up-time

For reviewing purposes of the problem statements, there is a "reading-up-time" of **10 minutes** prior to the official examination time. During this period it is **not** allowed to start solving the problems. This means explicitly that during the entire "reading-up-time" no writing utensils, e.g. pen, pencil, etc. at all are allowed to be kept on the table. Furthermore the use of carried documents, e.g. books, (electronic) translater, (electronic) dictionaries, etc. is strictly forbidden. When the supervisor refers to the end of the "reading-up-time" and thus the beginning of the official examination time, you are allowed to take your utensils and documents. Please **then**, begin with filling in the **complete** information on the titlepage and on page 3.

*Good Luck!*

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| LAST NAME | |
| FIRST NAME | |
| MATRIKEL-No. | |
| TABLE-No. | |

# Klausurunterlagen

Ich versichere hiermit, dass ich sämtliche für die Durchführung der Klausur vorgesehenen Unterlagen erhalten, und dass ich meine Arbeit ohne fremde Hilfe und ohne Verwendung unerlaubter Hilfsmittel und sonstiger unlauterer Mittel angefertigt habe. Ich weiß, dass ein Bekanntwerden solcher Umstände auch nachträglich zum Ausschluss von der Prüfung führt. Ich versichere weiter, dass ich sämtliche mir überlassenen Arbeitsunterlagen sowie meine Lösung vollständig zurück gegeben habe. Die Abgabe meiner Arbeit wurde in der Teilnehmerliste von Aufsichtsführenden schriftlich vermerkt.

Duisburg, den __________ ________________________________

(Unterschrift der/des Studierenden)

Falls Klausurunterlagen vorzeitig abgegeben: ______________Uhr

# Bewertungstabelle

| | |
|---|---|
| Aufgabe 1 | |
| Aufgabe 2 | |
| Aufgabe 3 | |
| Aufgabe 4 | |
| Gesamtpunktzahl | |
| Angehobene Punktzahl | |
| % | |
| Bewertung gem. PO<br>in Ziffern | |

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(Datum und Unterschrift 1. Prüfer, Univ.-Prof. Dr.-Ing. Söffker)

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(Datum und Unterschrift 2. Prüfer, Yan Liu, M. Eng.)

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(Datum und Unterschrift des für die Prüfung verantwortlichen Prüfers, Söffker)

Fachnote gemäß Prüfungsordnung:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ | ☐ |
| 1,0 | 1,3 | 1,7 | 2,0 | 2,3 | 2,7 | 3,0 | 3,3 | 3,7 | 4,0 | 5,0 |
| sehr gut | | gut | | | befriedigend | | | ausreichend | | mangelhaft |

Bemerkung: ________________________________________

**Attention:** Give your answers to ALL problems directly below the questions in the exam question sheet.

You are NOT allowed to use a pencil and also NOT red color (red color is used for corrections).

This exam "Control Theory" is taken by me as a
☐ mandatory (Pflichtfach)
☐ elective (Wahlfach)
☐ prerequsite (Auflage)
subject (cross ONE option according to your own situation).

| | |
|---|---|
| Maximum achievable points: | **100** |
| Minimum points for the grade 1,0: | **95%** |
| Minimum points for the grade 4,0: | **50%** |

**Problem 1** (30 Points)

1a) (4 Points)

State for the system

$$\ddot{y} + y = u$$

with the input $u$ and the output $\dot{y}$ the matrices $A, B, C$ of the related state space description.

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$\tilde{Y} = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

1b) (6 Points)

Assume a linear MIMO system with the system matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4a \end{bmatrix}.$$

For which values of $a$ is the system asymptotically stable?

Characteristic equation:
$p(\lambda) = \lambda^3 + 4a\lambda^2 + 3\lambda + 2$

Using Hurwitz criteria:
First condition: $a_i$ have identical signs
$4a > 0 \quad \Longrightarrow \quad a > 0$

Second condition: $|H_i| > 0$

$$H = \begin{bmatrix} 4a & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 4a & 2 \end{bmatrix}$$

$H_1 : D_1 = 4a \quad \Longrightarrow \quad a > 0$
$H_2 : D_2 = 12a - 2 > 0 \quad \Longrightarrow \quad a > 1/6$
$H_3 : D_3 = 2D_2 \quad \Longrightarrow \quad a > 1/6$

System is asymptotically stable for $a > 1/6$.

1c) (6 Points)

An unstable linear MIMO system should be stabilized. Choose the necessary condition(s) of the system $(A, B, C)$ to be fulfilled.

- ◯ $A$ must be stable.
- ⊗ $A, B$ must be stabilizable.
- ◯ $A, C$ must be fully observable.
- ◯ $BC = 0$ must be fulfilled.
- ◯ An unstable system can never be stable respectively stabilized.
- ◯ rank $\begin{bmatrix} A & -B \\ C & D \end{bmatrix} = n + r$ for all $\lambda_\mathrm{i}$ of $A$.
- ◯ rank $\begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix} = n$ for all $\lambda_\mathrm{i}$ of $A$ with Re $\{\lambda_i\} \geq 0$.
- ⊗ rank $\begin{bmatrix} \lambda I - A & -B \end{bmatrix} = n$ for all $\lambda_\mathrm{i}$ of $A$ with Re $\{\lambda_i\} \geq 0$.

1d) (8 Points)

The system

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \text{ and } \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ should be controlled using } b > 0,\ k > 0,$$

and $K = \begin{bmatrix} k & 0 \end{bmatrix}$.

Can the controlled system become unstable (Re $\{\lambda_i\} > 0$)? State the reason upon to the calculation of $b$ and $k$.

$det[\lambda I - (A - BK)] = det \begin{bmatrix} \lambda & -1 \\ -1 + bk & \lambda + 1 \end{bmatrix}$

$P(\lambda) = \lambda^2 + \lambda - 1 + bk$

Stodola: if $a_i$ have identical signs, the system is asymptotically stable.

$-1 + bk > 0 \implies bk > 1$ If this condition is fulfilled, the System is asyptotically stable.
The system would be unstable for $bk < 1$ .

1e) (6 Points)

Use Hautus criteria to check the observability of the system given with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}, \text{ and } \quad D = 0.$$

The system is fully observable if rank $\begin{bmatrix} \lambda_i I - A \\ c \end{bmatrix} = n$, for all $\lambda_i$ of $A$ holds.

$\lambda_1 = -1,\ \lambda_2 = 0,\ \lambda_3 = 4$

$$\lambda_1 = -1 : rank \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ -1 & -4 & -4 \\ 1 & 3 & 4 \end{bmatrix} = 3$$

$$\lambda_2 = 0 : rank \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & -4 & -3 \\ 1 & 3 & 4 \end{bmatrix} = 3$$

$$\lambda_3 = 4 : rank \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & -1 \\ -1 & -4 & 1 \\ 1 & 3 & 4 \end{bmatrix} = 3$$

No losses of rank for all eigenvalues $\Longrightarrow$ System is fully observable.

$\sum \square$

**Problem 2** (15 Points)
A system is described by

$$A = \begin{bmatrix} 0 & 3 & 0 \\ -4 & a & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad D = 0.$$

2a) (3 Points)
For which values of the parameter $a$ ($a \in \mathbb{R}$) is the system asymptotically stable?

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Characteristic polynomial:

$$\det(\lambda I - A) = \lambda^3 + (1-a)\lambda^2 + (12-a)\lambda + 12$$

According to *Hurwitz* criterion $\Rightarrow$

1. All coefficients possess the same sign:

$$\left\{ \begin{matrix} 1-a > 0 \\ 12-a > 0 \end{matrix} \right\} \Rightarrow a < 1$$

2. Determinants of the *Hurwitz* matrix:

$$\begin{vmatrix} 1-a & 12 & 0 \\ 1 & 12-a & 0 \\ 0 & 1-a & 12 \end{vmatrix} \Rightarrow \left\{ \begin{matrix} 1-a > 0 \\ \begin{vmatrix} 1-a & 12 \\ 1 & 12-a \end{vmatrix} = a^2 - 13a > 0 \\ 12 \begin{vmatrix} 1-a & 12 \\ 1 & 12-a \end{vmatrix} = 12(a^2 - 13a) > 0 \end{matrix} \right\} \Rightarrow a < 0$$

For $a < 0$ the system is asymptotically stable.

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2b) (2 Points)
For the system given in 2a), the parameter $a$ is assumed as $a = -7$. Determine the eigenvalues of the system.

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Characteristic equation:

$$\det(\lambda I - A) = (\lambda^2 + 7\lambda + 12)(\lambda + 1) \stackrel{!}{=} 0$$

$\Rightarrow$ The eigenvalues of the system are

$$\lambda_1 = -3, \quad \lambda_2 = -4, \quad \text{and} \quad \lambda_3 = -1.$$

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2c) (3 Points)
Calculate the invariant zero(s) of the system in 2b).

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Rosenbrock system matrix:

$$P(s_0) = \begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s_0 & -3 & 0 & 0 & -1 \\ 4 & s_0 + 7 & 0 & -2 & 0 \\ 0 & 0 & s_0 + 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$

$$\det(P) \overset{!}{=} 0 \quad \Rightarrow \quad -2(s_0 + 1) = 0$$

$\Rightarrow$ The invariant zero of the system is

$$s_0 = -1.$$

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2d) (2 Points)
Determine the decoupling zero(s) for the system given in 2b).

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From 2b) and 2c), it is known that there is only one invariant zero $s_0 = -1$. The detected invariant zero is identical to the eigenvalue $\lambda_3$, therefore the decoupling zero of the system is

$$s_0 = -1.$$

**Alternative solution:**

$$rank(s_0 I - A - B) \stackrel{!}{=} n$$

and

$$rank(\frac{s_0 I - A}{C}) \stackrel{!}{=} n$$

$\Rightarrow$ Decoupling zero:

$$s_0 = -1.$$

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2e) (3 Points)
In general: What can be concluded for a system that has no decoupling zeros? Check the correct statement/statements.

- ◯ The system is unstable.
- ⊗ The system is fully observable.
- ⊗ The system is fully controllable.
- ⊗ The canonical input matrix $\tilde{B}$ and canonical output matrix $\tilde{C}$ possess no zero rows and zero columns, respectively.
- ◯ A controller may affect the observability of the system.
- ⊗ A stable system can be destabilized by controller.
- ⊗ An unstable system can be stablized by controller.
- ◯ The dynamics of the system can only be affected if an observer is used.

2f) (2 Points)
A feedback control should be realized for the system given by

$$\dot{x} = Ax + \underbrace{Bu}_{\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}} \quad \text{with} \quad x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \quad \text{and} \quad B = \begin{bmatrix} b_1 & 0 & 0 \end{bmatrix}^T,$$

by using negative state feedback with the controller gain

$$K = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}.$$

State explicitly three formulas to be programmed for $n_1$, $n_2$, and $n_3$ by using the detailed numbers and state variables mentioned above.

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$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = Bu = B(-Kx) = \begin{bmatrix} -b_1(x_1 + 2x_2 - 3x_3) \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$

$$\begin{aligned} n_1 &= -b_1(x_1 + 2x_2 - 3x_3) \\ n_2 &= 0 \\ n_3 &= 0 \end{aligned}$$

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$\sum$ ☐

**Problem 3** (15 Points)

3a) (1 Point)
State the Rosenbrock matrix of a system given by

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} -1 & 0 \end{bmatrix},\ \text{and } D = \begin{bmatrix} 0 \end{bmatrix}.$$

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s & -1 & -1 \\ 5 & s+6 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

3b) (4 Points)
Calculate the eigenvalues, the invariant zeros, the transmission, and the decoupling zeros of the system given in 3a).

Eigenvalues:

$$|\lambda I - A| \overset{!}{=} 0$$

$$\begin{vmatrix} \lambda & -1 \\ 5 & \lambda + 6 \end{vmatrix} = \lambda^2 + 6\lambda + 5 \overset{!}{=} 0$$

$$\lambda_{1,2} = -3 \pm \sqrt{9 - 5}$$

$$\lambda_1 = -5,\ \lambda_2 = -1$$

Invariant zeros:

$$|P(s_0)| \overset{!}{=} 0$$

$$\begin{vmatrix} s_0 & -1 & -1 \\ 5 & s_0 + 6 & 0 \\ -1 & 0 & 0 \end{vmatrix} = -(s_0 + 6) \overset{!}{=} 0$$

$$s_0 = -6$$

Transmission zeros:

$$G = C(s_0 I - A)^{-1}B + D$$
$$= \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} s_0+6 & 1 \\ 5 & s_0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{s_0^2+6s_0+5} + 0$$

$$- \frac{s_0+6}{s_0^2+6s_0+5}$$

$$s_0 = -6$$

Decoupling zeros:

The invariant zeros are not identical to the eigenvalues of the system $\Rightarrow$ No decoupling zeros.

3c) (1 Points)
Is the system given in task 3a) fully controllable? (A suitable check has to be given!)

$$Q_S = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$$

$|Q_S| = -5 \neq 0$
$\Rightarrow \text{rank}(Q_S) = 2$

$\Rightarrow$ Fully controllable

3d) (1 Points)
Is the system given in task 3a) fully observable? (A suitable check has to be given!)

$$Q_B = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$|Q_B| = 1 \neq 0$
$\Rightarrow \text{rank}(Q_B) = 2$

$\Rightarrow$ Fully observable

3e) (2 Points)
Calculate the feedback gains for state control of the system given in task 3a) using the pole placement approach. The desired eigenvalues of the controlled system should be $\lambda_1 = -4$ and $\lambda_2 = -1$.

$$|\lambda I - (A - Bk)| \stackrel{!}{=} 0$$

$$(A - Bk) = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1-k_2 \\ -5 & -6 \end{bmatrix}$$

$$|\lambda I - (A - Bk)| = \begin{vmatrix} \lambda + k_1 & k_2 - 1 \\ 5 & \lambda + 6 \end{vmatrix} =$$

$$\lambda^2 + (6 + k_1)\lambda + 5 + 6k_1 - 5k_2$$

$$\stackrel{!}{=} (\lambda + 4)(\lambda + 1) = \lambda^2 + 5\lambda + 4$$

$$k_1 = -1;\; k_2 = -1;\; k = [-1; -1]$$

3f) (2 Points)
Is it necessary for a system given by

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du$$

to use an observer for realizing a state feedback by

$$u = -kx?$$

State reason(s).

$\Rightarrow$ Not necessary, if all states are measurable.

3g) (4 Points)
Why the observer and the controller design can be separately performed? State the reason based on the underlying principle.

According to the separation principle
the dynamical behavior of a system with observer and state feedback is described by

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - Bk & Bk \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}.$$

Due to the structure of the matrix of the closed-loop system it can be seen, that the eigenvalues of the controller and those of the observer are independent and thus don't have an influence to each other. So also the design processes are independent of each other.

$\sum \square$

**Problem 4** (40 Points)

4a) (3 Points)
A mechanical system is modeled using the following differential equation

$$m\,\ddot{x} + d\,\dot{x} + k\,x = 3 \cdot 1(t)\,,$$

with scalars $m$, $d$, $k > 0$. The states $x$ and $\dot{x}$ are measured. Taking the vector $z = [z_1,\, z_2]^T = [x,\, \dot{x}]^T$ as the state vector, give the matrices $A$, $B$, and $C$ of the state space description of this system.

**Answers 4a)** The equation $m\ddot{x} + d\dot{x} + kx = 3 \cdot 1(t)$ can be transformed as

$$\ddot{x} + \frac{d}{m}\dot{x} + \frac{k}{m} = \frac{3}{m} \cdot 1(t).$$

Substituting the state vector $z$ defined as

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

into the above equation, one can obtain that

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\frac{d}{m} z_2 - \frac{k}{m} z_1 + \frac{3}{m} \cdot 1(t) \end{aligned}$$

Correspondingly, the state space form of the system can be written as

$$\begin{aligned} \dot{z} &= Az + Bu \\ y &= Cz \end{aligned}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{3}{m} \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4b) (6 Points)
An electromechanical system is described in the following state-space form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & a & b \\ 0 & -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u\,, \quad y = \begin{bmatrix} m_1 & m_2 & 0 & 0 \end{bmatrix}.$$

For which values of the parameters $a$ and $b$ is the system stable?

**Answers to 4b)** The characteristic polynomial can be calculated as

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ 3 & 0 & \lambda - a & b \\ 0 & 4 & 0 & \lambda \end{bmatrix} = \lambda^4 - a\lambda^3 + 7\lambda^2 - 4\lambda + 12.$$

Let $a_0 = 12$, $a_1 = -4a$, $a_2 = 7$, $a_3 = -a$ and $a_4 = 1$, the Hurwitz matrix can be calculated as

$$H = \begin{bmatrix} a_3 & a_4 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & a_0 \end{bmatrix} = \begin{bmatrix} -a & 1 & 0 & 0 \\ -4a & 7 & -a & 1 \\ 0 & 12 & -4a & a7 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

According to the Hurwitz criterion, the system is asymptotically stable, if

$$\begin{aligned} -a &> 0 \\ -4a &> 0 \\ D_1 &= \det \begin{bmatrix} -a \end{bmatrix} > 0 \\ D_2 &= \det \begin{bmatrix} -a & 1 \\ -4a & 7 \end{bmatrix} > 0 \\ D_3 &= \det \begin{bmatrix} -a & 1 & 0 \\ -4a & 7 & -a \\ 0 & 12 & -4a \end{bmatrix} > 0, \text{ and} \\ D_4 &= \det \begin{bmatrix} -a & 1 & 0 & 0 \\ -4a & 7 & -a & 1 \\ 0 & 12 & -4a & a7 \\ 0 & 0 & 0 & 12 \end{bmatrix} > 0. \end{aligned}$$

The solution to the above inequality array is $a < 0$. Therefore, when $a < 0$ and $b$ is arbitrary the system is asymptotically stable.

4c) (4 Points)
Suppose $c_1 = c_2 = 0$. Check whether an observer feedback gain matrix $L$ exists, so that the real parts of the eigenvalues of the observer system $\dot{x} = (A - LC)x$ with

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -8 & 0 & -8 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} c_1 & c_2 & 1 \end{bmatrix}$$

are less than 2 (i.e., $Re(\lambda_i) < 2$ for $i = 1, \dots n$, where $\lambda_i$ is the $i$-th eigenvalue of the observer system, and $n = 3$ is the system order).

**Answers 4c)** If $(A,\ C)$ is fully observable, the observer system $\dot{x} = (A - LC)x$ is fully controllable, and it is possible to allocate eigenvalues of the observer arbitrarily to obtain $\mathrm{Re}(\lambda_i) < 2$.

Use the Kalman criterion to check the observability.

$$Q_b = \begin{bmatrix} C \\ C\,A \\ C\,A^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -8 & 1 \\ 62 & 0 & 6 \end{bmatrix}.$$

$$\mathrm{rank}(Q_b) = 3$$

Therefore, the system is fully observerable and it is possible to allocate eigenvalues of the observer system to obtain $\mathrm{Re}(\lambda_i) < 2$.

4d) (6 Points)
Use KALMAN observability criterion to check the observability of a state-space system with the system matrix

$$A = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and the output matrix

$$C = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}.$$

If the system is not fully observable, use Hautus observability criterion to determine which eigenvalues are observable.

**Answers to 4d)** *Using Kalman criterion to check observability:*

$$Q_b = \begin{bmatrix} C \\ C\,A \\ C\,A^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 3 & 0 \\ -3 & -11 & 6 \end{bmatrix}.$$

Because $\det(Q_b) = 0$, the matrix $Q_b$ is not full of rank. Thus the system is not fully observable.

*Using Hautus criterion to find unobservable eigenvalues:*

$$\det(\lambda I - A) = 0 \Rightarrow \lambda = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$$

For $\lambda_1 = -3$:

$$\text{rank}\left(\begin{bmatrix} \lambda_1 I - A \\ C \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} 3 & 11 & 6 \\ -1 & -3 & 0 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{bmatrix}\right) = 2 < 3$$

Therefore, $\lambda_1 = -3$ is not observable.

For $\lambda_2 = -2$:

$$\text{rank}\left(\begin{bmatrix} \lambda_2 I - A \\ C \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} 4 & 11 & 6 \\ -1 & -2 & 0 \\ 0 & -1 & -2 \\ 0 & 1 & 3 \end{bmatrix}\right) = 3 = 3$$

Therefore, $\lambda_2 = -2$ is observable.

For $\lambda_3 = -1$:

$$\text{rank}\left(\begin{bmatrix} \lambda_3 I - A \\ C \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} 5 & 11 & 6 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 3 \end{bmatrix}\right) = 3 = 3$$

Therefore, $\lambda_3 = -1$ is observable.

From the tasks given in 4e) to 4i) the system

$$\begin{aligned} \dot{x} &= Ax + Bu\,, \\ y &= Cx \end{aligned}$$

with

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -2 & -1 & 2 \\ 2 & -1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} a & b \\ c & c \\ a & b \end{bmatrix}, \text{ and } C = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

has to be considered.

4e) (3 Points)
Is the open-loop system asymptotically stable? State reasons.

**Answers to 4e)** Calculate the characteristic equation:

$$\det(\lambda I - A) = \lambda^3 + 3\lambda^2 + 2\lambda = 0$$

The eigenvalues are

$$\lambda = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}.$$

Because $\lambda_1 = 0$, the open-loop system is not asymptotically stable.

4f) (3 Points)
Using $a = b = c = 0$ and for the feedback matrix

$$K = \begin{bmatrix} -6 & 7 & 1/2 \\ 8 & -2 & -2 \end{bmatrix},$$

the differences $\Delta\lambda$ between the eigenvalues of the controlled system $\lambda_c$ and the eigenvalues of the uncontrolled system $\lambda_u$, $\Delta\lambda = \lambda_c - \lambda_u$, have to be calculated.

**Answers to 4f)** Substituting the values of $a$, $b$, and $c$ into the matrix $B$, one can obtain

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus the control input cannot influence the system, which means the eigenvalues cannot be changed by the control input. Therefore $\Delta\lambda = 0$.

4g) (5 Points)
Suppose that the parameters $a$, $b$, and $c$ be chosen properly so that the system is fully controllable. The system states are estimated by a Luenberger observer with the observer gain matrix $L$. The system is controlled by a state feedback controller with feedback matrix $K$. Draw the block diagram of this closed-loop system. The matrices $A$, $B$, $C$, $L$, and $K$ should be denoted in the diagram.

Can the state feedback with different choices of $K$ influence the stability of the Luenberger observer in the closed-loop system? State reasons.

**Answers to 4g)** Different $K$ cannot influence the stability of the observer because of the SEPARATION PRINCIPLE.

The block diagram is shown in the figure :

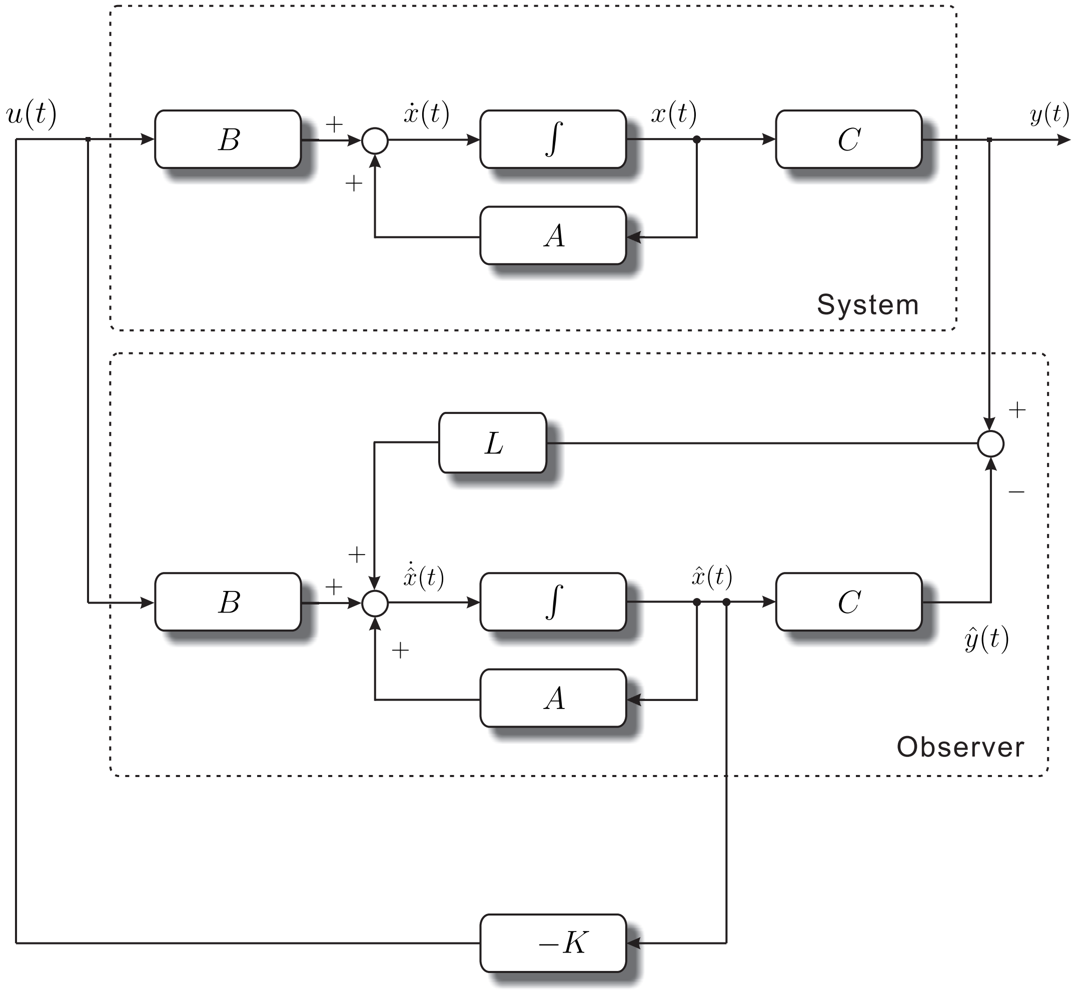


**Figure 4.1**: Block diagram of the closed-loop system with observer

4h) (4 Points)
Suppose $a = c = 1$ and $b = 2$. Is the system fully controllable?

**Answers to 4h)** Substitute $a = c = 1$ and $b = 2$ into $B$ to obtain

$$B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Calculate the observerability matrix $Q_c$

$$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 2 & -1 & -1 & 1 & 1 \end{bmatrix}$$

Because $\text{rank}(Q_c) = 2$, the system is not fully controllable.

4i) (6 Points)
If $a = c = 1$ and $b = 2$, is it possible to find a state feedback matrix $K$ so that the controlled system has the eigenvalues $\lambda_1 = -2$, $\lambda_2 = -2$, and $\lambda_3 = -2$?

**Answers to 4i)** Because the system is not fully controllable, if the uncontrollable eigenvalues are equal to $-2$, it is still possible to get $\lambda_1 = \lambda_2 = \lambda_3 = -2$.

The eigenvalues of the systems are

$$\lambda = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}.$$

Because $\lambda_3 = -2$, it is not necessary to check the controllability for $\lambda_3$.

*Using Hautus criterion to check the controllability of $\lambda_1 = 0$ and $\lambda_2 = -1$.*

For $\lambda_1 = 0$,

$$\begin{aligned} \text{rank}(Q_{c,\lambda_1}) &= \text{rank}\left(\begin{bmatrix} \lambda_1 I - A & B \end{bmatrix}\right) \\ &= \text{rank}\left(\begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 2 & 1 & -2 & 1 & 1 \\ -2 & 1 & 2 & 1 & 2 \end{bmatrix}\right) = 3 \end{aligned}$$

Thus, $\lambda_1 = 0$ is controllable.

Similarly for $\lambda_2 = -1$,

$$\begin{aligned} \text{rank}(Q_{c,\lambda_1}) &= \text{rank}\left(\begin{bmatrix} \lambda_1 I - A & B \end{bmatrix}\right) \\ &= \text{rank}\left(\begin{bmatrix} -1 & 1 & 0 & 1 & 2 \\ 2 & 0 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 & 2 \end{bmatrix}\right) = 3 \end{aligned}$$

Thus, $\lambda_2 = 0$ is also controllable.

From the above discussion it can be seen, that it is possible for the closed-loop system to have eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = -2$.

$\sum \square$