Reading-up-time

For reviewing purposes of the problem statements, there is a "reading-up-time" of **10 minutes** prior to the official examination time. During this period it is **not** allowed to start solving the problems. This means explicitly that during the entire "reading-up-time" no writing utensils, e.g. pen, pencil, etc. at all are allowed to be kept on the table. Furthermore the use of carried documents, e.g. books, (electronic) translater, (electronic) dictionaries, etc. is strictly forbidden. When the supervisor refers to the end of the "reading-up-time" and thus the beginning of the official examination time, you are allowed to take your utensils and documents. Please **then**, begin with filling in the **complete** information on the titlepage and on page 3.

Good Luck!

LAST NAME	
FIRST NAME	
MATRIKEL-NO.	
TABLE-NO.	

Klausurunterlagen

Ich versichere hiermit, dass ich sämtliche für die Durchführung der Klausur vorgesehenen Unterlagen erhalten, und dass ich meine Arbeit ohne fremde Hilfe und ohne Verwendung unerlaubter Hilfsmittel und sonstiger unlauterer Mittel angefertigt habe. Ich weiß, dass ein Bekanntwerden solcher Umstände auch nachträglich zum Ausschluss von der Prüfung führt. Ich versichere weiter, dass ich sämtliche mir überlassenen Arbeitsunterlagen sowie meine Lösung vollständig zurück gegeben habe. Die Abgabe meiner Arbeit wurde in der Teilnehmerliste von Aufsichtsführenden schriftlich vermerkt.

Duisburg, den ________ ______________________________
(Unterschrift der/des Studierenden)

Falls Klausurunterlagen vorzeitig abgegeben: ____________Uhr

Bewertungstabelle

Aufgabe 1	
Aufgabe 2	
Aufgabe 3	
Aufgabe 4	
Gesamtpunktzahl	
Angehobene Punktzahl	
%	
Bewertung gem. PO in Ziffern	

____

(Datum und Unterschrift 1. Prüfer, Univ.-Prof. Dr.-Ing. Dirk Söffker)

____

(Datum und Unterschrift 2. Prüfer, Dr.-Ing. Yan Liu)

____

(Datum und Unterschrift des für die Prüfung verantwortlichen Prüfers, Söffker)

Fachnote gemäß Prüfungsordnung:

☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
1,0	1,3	1,7	2,0	2,3	2,7	3,0	3,3	3,7	4,0	5,0
sehr gut		gut			befriedigend			ausreichend		mangelhaft

Bemerkung: ____________________________________

Attention: Give your answers to ALL problems directly below the questions in the exam question sheet.

You are NOT allowed to use a pencil and also NOT red color (red color is used for corrections).

This exam is taken by me as a
☐ mandatory (Pflichtfach)
☐ elective (Wahlfach)
☐ prerequsite (Auflage)
subject (cross ONE option according to your own situation).

Maximum achievable points:	**100**
Minimum points for the grade 1,0:	**95%**
Minimum points for the grade 4,0:	**50%**

Problem 1 (30 Points)

1a) (3 Points)
State the difference between a SISO and a MIMO control system with respect to the number of actuators/inputs.

Solution:
A SISO system has one input/actuator. The number of actuators in MIMO systems is (theoretically) arbitrary (at least one).

For the following tasks, a system description is given with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & a & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} \quad \text{with} \quad a \neq 0 \text{ and } b \neq 0.$$

It is noted, that no direct transmission between the inputs and the outputs/measurements is given.

1b) (5 Points)
Calculate the transfer function matrix of the system depending on the parameters a and b.

Solution:

$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = \frac{1}{s^2(s+3) - sa - a} \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} s(s+3) - a & s+3 & 1 \\ a & s(s+3) & s \\ as & a + as & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}$$

$$G(s) = \frac{2bs^2}{s^3 + 3s^2 - as - a}$$

1c) (4 Points)
State the Rosenbrock matrix of the system depending on the parameters a and b. Determine the invariant zeros, the transmission zeros, and the decoupling zeros of the system.

Solution:

$$P(s) = \begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ -a & -a & s+3 & -b \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\det(P(s)) = 2bs^2 \overset{!}{=} 0$$

Invariant zero: $s_0 = 0$
Transmission zero: $s_0 = 0$
Decoupling zeros: none

1d) (3 Points)
For which parameters a and b is the system described by A, B, C fully controllable?

Solution:

$$Q_\text{S} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$
$$Q_\text{S} = \begin{bmatrix} 0 & 0 & b \\ 0 & b & -3b \\ b & -3b & ab+9b \end{bmatrix}$$

Requirement: $\det(Q_\text{S}) \overset{!}{\neq} 0$ for full controllability

For $b \neq 0$ the system is fully controllable.

1e) (4 Points)
Assuming full observability, a state observer should be designed for the previously given system A, B, C with $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = 1$. For this subtask it can be assumed that $a = 1$. Calculate the gains of the observer matrix $L = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T$.

Solution:

$$\det(\lambda I - (A - LC)) = \det \begin{bmatrix} \lambda & -1 & 2l_1 \\ 0 & \lambda & 2l_2 - 1 \\ -1 & -1 & \lambda + 3 + 2l_3 \end{bmatrix}$$

Characteristic polynomial: $\lambda^3 + \lambda^2(3 + 2l_3) + \lambda(2l_2 + 2l_1 - 1) + 2l_2 - 1$
comparing the coefficients with: $\lambda^3 + 2\lambda^2 - \lambda - 2$
leads to the results: $l_1 = 0.5$, $l_2 = -0.5$, and $l_3 = -0.5$.

1f) (6 Points)
Assume $b = 1$. Calculate the gain matrix $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ (depending on a) by using the pole placement method to realize state feedback control. The desired eigenvalues of the closed-loop system are given as $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = 1$.

Solution:

$$\det(\lambda I - (A - BK)) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ k_1 - a & k_2 - a & \lambda + k_3 + 3 \end{bmatrix}$$

Characteristic polynomial: $\lambda^3 + \lambda^2(k_3 + 3) + \lambda(k_2 - a) + k_1 - a$
comparing the coefficients with: $\lambda^3 + 2\lambda^2 - \lambda - 2$
leads to the results: $k_1 = -2 + a$, $k_2 = -1 + a$, and $k_3 = -1$.

$\sum \square$

Problem 2 (20 Points)
Given is the standard description of a linear MIMO system. The system described by

$$A = \begin{bmatrix} 0 & 0 & 0.5 \\ 2 & -1 & 1 \\ 2 & 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 0 & 5 \\ 3 & 5 & 9 \\ 0 & 12 & 0 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

should be controlled using the state feedback matrix $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$.

2a) (2 Points)

Is it necessary for practical purposes to build an observer for the given system to realize full state feedback control? State reason.

Solution:

No,
All states are measured.

Rank $C = 3 = n$

2b) (3 Points)

For which b_1, b_2 (b_1, $b_2 \neq 0$) is the system fully controllable?

Solution:

$$Q_\mathrm{s} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} b_1 & 0 & b_1 + 1.5b_2 \\ b_2 & 2b_1 - b_2 & 4b_2 \\ 0 & 2b_1 + 3b_2 & 6b_1 - 3b_2 \end{bmatrix}$$

Rank $Q_\mathrm{B} \overset{!}{=} n = 3$

$\Rightarrow$ det $Q_\mathrm{B} \neq 0$

Full controllability for
$12b_1^3 - 18b_1^2 b_2 - 3b_1 b_2^2 + 4.5b_2^3 \neq 0$ or $3(2b_1 + b_2)(2b_1 - b_2)(b_1 - 1.5b_2) \neq 0$

$\Rightarrow b_2 \neq 2b_1 \ \vee \ b_2 \neq -2b_1 \ \vee \ b_1 \neq 1.5b_2$

2c) (1 Point)

For task 2b) is it possible to design the controller eigenvalues arbitrarily?

Solution:

Yes, if the conclusion for full controllability is fulfilled.

2d) (5 Points)

A modified system is given with $A = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 1 \\ a & 0 & a \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & b \end{bmatrix}^T$ with $b \neq 0$. Calculate k_1, k_2, and k_3 so that the eigenvalues of the closed-loop system are $\lambda_1 = -3$, $\lambda_2 = 1$, and $\lambda_3 = 2$.

Solution:

Characteristic equation of the closed-loop system: $\det(\lambda I - A + BK)=0$

$$\lambda I - A + BK = \begin{bmatrix} \lambda & -4 & 0 \\ -3 & \lambda & -1 \\ bk_1 - a & bk_2 & \lambda - a + bk_3 \end{bmatrix}$$

$$\Longrightarrow \lambda^3 + (bk_3 - a)\lambda^2 + (bk_2 - 12)\lambda + 4bk_1 - 12bk_3 + 8a = 0$$

$$\lambda_1 = -3 \Rightarrow 4bk_1 - 3bk_2 - 3bk_3 - a + 9 = 0$$

$$\lambda_2 = 1 \Rightarrow 4bk_1 + bk_2 - 11bk_3 + 7a - 11 = 0$$

$$\lambda_3 = 2 \Rightarrow 4bk_1 + 2bk_2 - 8bk_3 + 4a - 16 = 0$$

$$\Longrightarrow k_1 = (2a + 3)/2b \qquad k_2 = 5/b \qquad k_3 = a/b$$

2e) (1 Point)

State the equations to calculate left and right eigenvectors.

Solution:

Left eigenvectors w_i: $w_i^T A = \lambda_i w_i^T$
Right eigenvectors v_i: $Av_i = \lambda_i v_i$

2f) (5 Points)

A modified system is given with $A = \begin{bmatrix} 0 & 1 \\ a & -2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Is the system stabilizable (use Hautus criterion with $\tilde{\tilde{x}}_i = \begin{bmatrix} \tilde{\tilde{x}}_{i1} \\ 1 \end{bmatrix}$ as the left eigenvector of A)? State conditions.

Solution:

$\lambda_1 = -1 - \sqrt{a+1}$, $\lambda_2 = -1 + \sqrt{a+1}$

The eigenvalues λ_1 is stable for all values of a, but the stability of λ_2 depends on the value of a. Therefore it is necessary to check the controllability of λ_2. If it is fully controllable, the system is stabilizable.

Using Hautus criterion, the controllability of the eigenvalue is guaranteed if $\tilde{\tilde{x}}_2^T B \neq 0$.

$\tilde{\tilde{x}}_2^T(\lambda_2 I - A)$=0

$$\Rightarrow \tilde{\tilde{x}}_2^T \begin{bmatrix} \sqrt{a+1} - 1 & -1 \\ -a & \sqrt{a+1} + 1 \end{bmatrix} = 0$$

$$\Rightarrow \tilde{\tilde{x}}_2 = \begin{bmatrix} \sqrt{a+1} + 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \tilde{\tilde{x}}_2^T B = \begin{bmatrix} \sqrt{a+1} + 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b_1\sqrt{a+1} + b_1 + b_2$$

The system is stabilizable,

$\Longrightarrow$ if $b_1\sqrt{a+1} + b_1 + b_2 \neq 0$.

2g) (3 Points)

An optimal controller should be realized using standard routines solving Algebraic Riccati Equation. The system matrices are given with A, B; additionally $\mathrm{Re}\{\lambda_i(A)\} < 0$ is known. After long discussions the weighting matrices have been chosen as

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; R = \begin{bmatrix} -r & 0 \\ 0 & r \end{bmatrix}, \; r > 0.$$

Which statement(s) is/are true (multiple choices possible: wrong answers are subtracted)?

⊗ The open-loop system is not boundary stable.

☐ The design process depends not on A, B.

⊗ If A, B is not fully controllable, the controlled system can still be stable.

⊗ The chosen design process (LQR) is principally always establishing stable systems dynamics, if A, B is fully controllable.

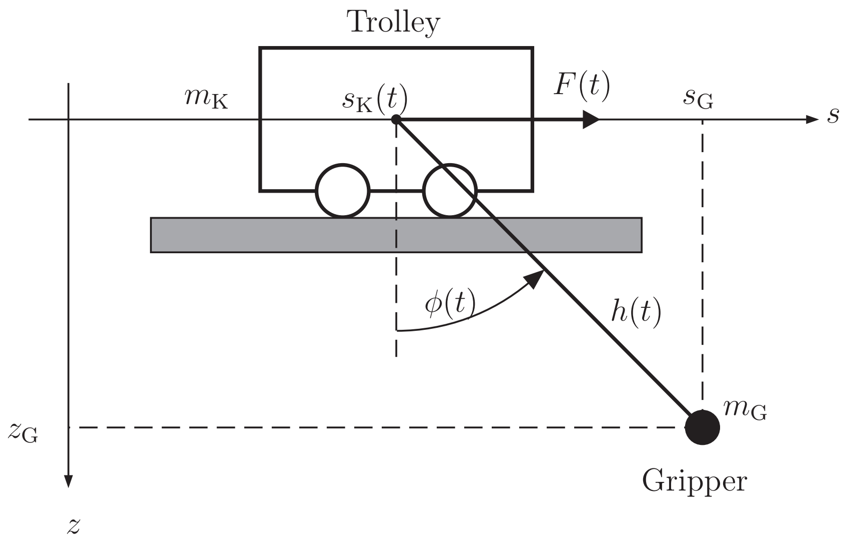
☐ The chosen design parameters always guarantees perfect dynamical behavior with respect to stablity and dynamical properties.

☐

$\sum$ ☐

Problem 3 (25 Points)
The mechanical system of a crane bridge with adjustable gripper length $h(t)$ is shown below. The differential equations for the position of the trolley $s_\mathrm{K}(t)$ and the gripper $\theta(t)$ ($\theta(t) \approx h(t) \cdot \phi(t)$) can be described by

$$\begin{aligned}\ddot{s}_\mathrm{K}(t) &= \frac{m_\mathrm{G}(g-\ddot{h}(t))}{m_\mathrm{K}h(t)}\theta(t) + \frac{1}{m_\mathrm{K}}F(t) \quad \text{and} \\ \ddot{\theta}(t) &= -\frac{(m_\mathrm{K}+m_\mathrm{G})(g-\ddot{h}(t))}{m_\mathrm{K}h(t)}\theta(t) - \frac{1}{m_\mathrm{K}}F(t)\,.\end{aligned}$$



3a) (5 Points)
Because of a misoperation the length of the gripper is not adjustable ($h(t)$ is constant with $h(t) = H$). Derive for this case a model description in the form

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Vf(t) \quad \text{with } q(t) = \begin{bmatrix} s_\mathrm{K}(t) \\ \theta(t) \end{bmatrix},$$

and also the state space model with $F(t)$ as input, $s_\mathrm{K}(t)$ as output, and the state vector

$$x(t) = \begin{bmatrix} s_\mathrm{K}(t) \\ \dot{s}_\mathrm{K}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix}.$$

Solution:

$$h(t) = H \Rightarrow \ddot{h}(t) = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{s}_\mathrm{K} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -\dfrac{m_\mathrm{G} g}{m_\mathrm{K} H} \\ 0 & \dfrac{(m_\mathrm{K} + m_\mathrm{K}) g}{m_\mathrm{K} H} \end{bmatrix} \begin{bmatrix} s_\mathrm{K} \\ \theta \end{bmatrix} = \begin{bmatrix} \dfrac{1}{m_\mathrm{K}} \\ -\dfrac{1}{m_\mathrm{K}} \end{bmatrix} F$$

or:

$$\begin{bmatrix} m_\mathrm{K} H & 0 \\ 0 & m_\mathrm{K} H \end{bmatrix} \begin{bmatrix} \ddot{s}_\mathrm{K} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -m_\mathrm{G} g \\ 0 & (m_\mathrm{K} + m_\mathrm{G}) g \end{bmatrix} \begin{bmatrix} s_\mathrm{K} \\ \theta \end{bmatrix} = \begin{bmatrix} H \\ -H \end{bmatrix} F$$

State space representation:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{m_\mathrm{G} g}{m_\mathrm{K} H} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{(m_\mathrm{K} + m_\mathrm{G}) g}{m_\mathrm{K} H} & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{m_\mathrm{K}} \\ 0 \\ -\frac{1}{m_\mathrm{K}} \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$D = 0\,.$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\square$

3b) (4 Points)

Assume $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -15 & 0 \\ 1 & -2 & 20 & 0 \end{pmatrix}$. Determine the characteristic polynomial of the system and check the asymptotic stability of the system using the Hurwitz criterion.

Solution:
Characteristic polynomial:

$$\lambda^4 + 15\lambda^3 + 3\lambda^2 + 10\lambda + 1$$

i) All coefficents $a_i > 0$.

ii)

$$H = \begin{bmatrix} 15 & 10 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 15 & 10 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$$H_1 = 15 > 0$$
$$H_2 = 35 > 0$$
$$H_3 = 125 > 0$$
$$H_4 = H_3 > 0$$

$\Rightarrow$ The system is asymptotic stable.

3c) (6 Points)

Assume measurements are available with $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$. The matrices A, B are given as $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. An output feedback is realized by $u(t) = -K_\mathrm{y} y(t) + V w(t)$ with $K_\mathrm{y} = [1 \quad 0 \quad 1]$ and $V = [3 \quad 2 \quad 1]$. Determine the transfer function matrix $G_\mathrm{w}(s)$ of the closed loop system. Assume $x_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.

Solution:

$$G_\mathrm{w}(s) = C(sI - A + BK_\mathrm{y}C)^{-1}BV$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{1}{s^3} & \frac{1}{s^2} \\ 0 & \frac{1}{s} & 0 \\ 0 & \frac{1}{s^2} & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{s^2} & \frac{2}{s^2} & \frac{1}{s^2} \\ \frac{3}{s} & \frac{2}{s} & \frac{1}{s} \\ -\frac{3}{s^2} & -\frac{2}{s^2} & -\frac{1}{s^2} \end{bmatrix}$$

$$= \frac{1}{s} \begin{bmatrix} \frac{3}{s} & \frac{2}{s} & \frac{1}{s} \\ 3 & 2 & 1 \\ -\frac{3}{s} & -\frac{2}{s} & -\frac{1}{s} \end{bmatrix}$$

3d) (2 Points)

A system is given with $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and $D = 0$.

Determine the pole(s) and invariant zero(s) of the system.

Solution:

$$
\begin{aligned}
G(s) &= C(sI - A)^{-1}B + D \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & \frac{2}{s^2-1} \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \\
&= \frac{3(s+1)}{s^2-1} \\
&= \frac{3(s+1)}{(s+1)(s-1)}
\end{aligned}
$$

$\Rightarrow$ The pole of the system: $s = 1$

The invariant zero of the system: $s_0 = -1$

3e) (2 Points)

Assume a system has three poles $s_1 = 1$, $s_2 = 2$, $s_3 = 3$, one transmission zero $s_{o1} = 0$, one output decoupling zero $s_{o2} = -1$, and no input decoupling zero. Is the system fully controllable? Is the system fully observable? State also the minimum order number of the system. State reasons.

Solution:

The system has no input decoupling zeros. $\Rightarrow$ fully controllable
The system has one output decoupling zero. $\Rightarrow$ not fully observable
Minimum order number = 4.

3f) (6 Points)
A system is given with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -2 & a \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ b \end{bmatrix}, \; C = \begin{bmatrix} c & 0 & 1 \end{bmatrix}, \text{ and } D = 0.$$

For which values of a, b, and c is the system

i) asymptotic stable?
ii) fully controllable?
iii) fully observable?

Solution:

i) The eigenvalues of the system can be calculated by $\det(\lambda I - A) \stackrel{!}{=} 0$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 1 & 2 & \lambda - a \end{bmatrix} \stackrel{!}{=} 0$$

$\Rightarrow (\lambda^2 - 1)(\lambda - a) = 0$

$\Rightarrow \lambda_1 = -1 \;, \; \lambda_2 = 1 \;, \; \lambda_1 = a \;.$

$\Rightarrow$Because there is a positive eigenvalue $\lambda_1 = 1$,
the system is NOT asymptotic stable.

ii) Using Kalman criterion to check the controllability:

$$Q_S = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ b & ab - 2 & a^2b - 2a - 1 \end{bmatrix}$$

The system is fully controllable if $\det(Q_S) \neq 0$

$\Rightarrow \det(Q_S) = 2a - a^2b + b + 1 \neq 0$

iii) Using Kalman criterion to check the observability:

$$Q_B = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} c & 0 & 1 \\ -1 & c - 2 & a \\ c - a - 2 & -2a - 1 & a^2 \end{bmatrix}$$

The system is fully observable if $\det(Q_B) \neq 0$

$\Rightarrow \det(Q_B) = a^2c^2 + 2ac - c^2 + 4c - 3 \neq 0$

Σ

Problem 4 (25 Points)
Gripper systems are typical devices applied for manufacturing tasks. They enable the system to handle sensitive system components.
For a robot gripper system, as shown in Fig. 4.1, a control algorithm has to be designed. The gripper consists of three fingers with three rotational joints each. The kinematic relations of a finger are illustrated in Fig. 4.2.

Fig. 4.1: Gripper system (Source: Robotiq)

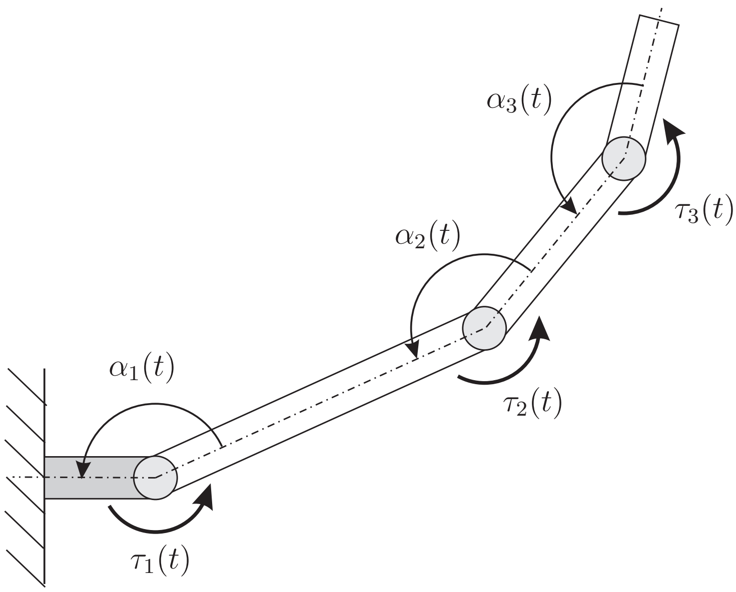


Fig. 4.2: Kinematics of the gripper system

In this context $\alpha(t) = \begin{bmatrix} \alpha_1(t) & \alpha_2(t) & \alpha_3(t) \end{bmatrix}^T$ denotes the angles of the rotational joints and $\tau(t) = \begin{bmatrix} \tau_1(t) & \tau_2(t) & \tau_3(t) \end{bmatrix}^T$ the torques applied on them by the actuator.
For the dynamical behavior of this system a simplified relation given by

$$\begin{bmatrix} J_\mathrm{a} & J_\mathrm{b} & J_\mathrm{c} \\ J_\mathrm{b} & J_\mathrm{a} & J_\mathrm{b} \\ J_\mathrm{c} & J_\mathrm{b} & J_\mathrm{a} \end{bmatrix} \ddot{\alpha}(t) + \begin{bmatrix} D_\mathrm{a} & D_\mathrm{b} & D_\mathrm{c} \\ D_\mathrm{b} & D_\mathrm{a} & D_\mathrm{b} \\ D_\mathrm{c} & D_\mathrm{b} & D_\mathrm{a} \end{bmatrix} \dot{\alpha}(t) + \begin{bmatrix} K_\mathrm{a} & K_\mathrm{b} & K_\mathrm{c} \\ K_\mathrm{b} & K_\mathrm{a} & K_\mathrm{b} \\ K_\mathrm{c} & K_\mathrm{b} & K_\mathrm{a} \end{bmatrix} \alpha(t) = \tau(t) \tag{4.1}$$

can be assumed, where $J_\mathrm{a,b,c}$ denote the inertia, $D_\mathrm{a,b,c}$ the damping, and $K_\mathrm{a,b,c}$ the stiffness coefficients of the system.

4a) (1 Point)
Give the order of the system stated above when described by a system matrix A?

Solution:

$n = 6$

4b) (6 Points)
Give the state space representation of the system defined by the equations stated in eq. 4.1 in terms of the matrices A, B, C, D.

Therefore use the state vector $x(t) = \begin{bmatrix} \alpha_1(t) & \dot{\alpha_1}(t) & \alpha_2(t) & \dot{\alpha_2}(t) & \alpha_3(t) & \dot{\alpha_3}(t) \end{bmatrix}^T$. The torques $\tau_i(t),\ i = 1, 2, 3$ of the actuators have to be applied as system inputs and the angles of the joints $\alpha_i(t),\ i = 1, 2, 3$ have to be considered as system outputs.

How can the input/output structure be classified (SISO, SIMO, MISO, or MIMO)?

For the purpose of simplification, assume $J_\mathrm{b} = J_\mathrm{c} = 0$.

Solution:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -K_\mathrm{a}/J_\mathrm{a} & -D_\mathrm{a}/J_\mathrm{a} & -K_\mathrm{b}/J_\mathrm{a} & -D_\mathrm{b}/J_\mathrm{a} & -K_\mathrm{c}/J_\mathrm{a} & -D_\mathrm{c}/J_\mathrm{a} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -K_\mathrm{b}/J_\mathrm{a} & -D_\mathrm{b}/J_\mathrm{a} & -K_\mathrm{a}/J_\mathrm{a} & -D_\mathrm{a}/J_\mathrm{a} & -K_\mathrm{b}/J_\mathrm{a} & -D_\mathrm{b}/J_\mathrm{a} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -K_\mathrm{c}/J_\mathrm{a} & -D_\mathrm{c}/J_\mathrm{a} & -K_\mathrm{b}/J_\mathrm{a} & -D_\mathrm{b}/J_\mathrm{a} & -K_\mathrm{a}/J_\mathrm{a} & -D_\mathrm{a}/J_\mathrm{a} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1/J_\mathrm{a} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/J_\mathrm{a} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/J_\mathrm{a} \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dot{x} = Ax + Bu$
$y = Cx + Du$

$\Rightarrow$ MIMO − System

4c) (2 Points)

Considering the conditions within a specific working range of the system the dynamic behavior can be simplified as

$$A = \begin{bmatrix} -1.371 & 0 & 0 \\ -(p_1 + 2p_3)^{-3/2} & 2.625p_2 & 4.373 \\ 2.157 & 0 & -2.477 \end{bmatrix}, \quad B = \begin{bmatrix} 2.783 \\ -7.886p_3 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.32 & 0 & 1.01 \end{bmatrix}, \text{ and } D = 0$$

with $p_1,\ p_2,\ p_3 > 0$.

Give the eigenvalues of the system in terms of the parameters $p_1,\ p_2,\ p_3$. What can be concluded concerning the stability [asymptotic stable, stable, unstable] of the system?

Solution:

$\lambda_1 = -1.371,\ \lambda_2 = 2.625\ p_2,\ \lambda_3 = -2.477$

$\Rightarrow$ unstable for $p_2 > 0$.

4d) (2 Points)

For a similiar system the eigenvalues are obtained as

$$\lambda_{1,2} = -1 \pm 2\ \frac{(p_1 - 4p_2)^{-1/2}}{4}, \text{ and } \lambda_3 = -1.237.$$

For which range of parameters $p_i > 0,\ i = 1, 2$ is the system asymptotic stable?

Solution:

$$\lambda_2 = -1 + \frac{2}{4\sqrt{p_1 - 4p_2}} \ < \ 0$$

$$\Rightarrow \ p_1 - 4p_2 > \tfrac{1}{4}$$

4e) (6 Points)
Using a specific type of a differential gear the dynamical behavior of the system changes to

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -d_1 & -d_2 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -d_3 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \text{ and } D = 0,$$

with $d_i > 0,\ i = 1, 2, 3.$

For $d_1 = d_2 = d_3 = 2$ the eigenvalues of this system are $\lambda_1 = \lambda_2 = 1$, and $\lambda_3 = -2$. Which of these eigenvalues are observable, which are controllable? Use suitable criteria to check these properties.

Solution:

$$Q_{\mathrm{S}} = \begin{bmatrix} A & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 6 & -12 \\ 1 & 1 & 1 \end{bmatrix}, \ \det(Q_{\mathrm{S}}) = -26 \neq 0, \ \Rightarrow \ \text{fully controllable}$$

$$Q_{\mathrm{B}} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} \ = \ 0 \ \Rightarrow \ \text{not fully observable}$$

$$\lambda_1 = 1: \ \operatorname{rank} \begin{pmatrix} \lambda_1 I - A \\ C \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 0 & 0 & -1 \\ -2 & 3 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 3 \Rightarrow \ \text{observable}$$

$$\lambda_3 = -2: \ \operatorname{rank} \begin{pmatrix} \lambda_3 I - A \\ C \end{pmatrix} = \operatorname{rank} \begin{pmatrix} -3 & 0 & -1 \\ -2 & 0 & -2 \\ 0 & 0 & -3 \\ 1 & 0 & 1 \end{pmatrix} = 2 \Rightarrow \ \text{not observable}$$

4f) (8 Points)

For a similiar system the eigenvalues $\lambda_1 = -4$, $\lambda_2 = -1$, and $\lambda_3 = -2$ and the corresponding eigenvectors

$$q_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \; q_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T, \text{ and } q_3 = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^T$$

are given.
The B and C matrices are known as

$$B = \begin{bmatrix} 4f_1 \\ f_1 - f_2 \\ 1 - f_1 \end{bmatrix} \text{ and } C = \begin{bmatrix} f_1 & f_2 & 1 \end{bmatrix}.$$

Check the range of the parameters $f_i, i = 1, 2$ for the system to be fully controllable. Therefor use the GILBERT criterion. For which parameters is the system fully observable?

Solution:

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \; V^{-1} = \begin{bmatrix} 1 & 0.5 & -0.5 \\ 0 & 1 & 0 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

$$\tilde{B} = V^{-1}B = \begin{bmatrix} 5f_1 - 0.5f_2 - 0.5 \\ f_1 - f_2 \\ -f_1 + 0.5f_2 + 0.5 \end{bmatrix}$$

$\Rightarrow$ fully controllable, if $5f_1 - 0.5f_2 - 0.5 \neq 0 \;\wedge\; f_1 \neq f_2 \;\wedge\; -f_1 + 0.5f_2 + 0.5 \neq 0$

$$\tilde{C} = CV = \begin{bmatrix} f_1 & f_2 + 1 & f_1 + 2 \end{bmatrix}$$

$\Rightarrow$ fully observable, if $f_1 \neq 0 \;\wedge\; f_2 \neq -1 \;\wedge\; f_1 \neq -2$