Problem 1 (30 Points)

1a) (3 x 5 Points)
Define the properties of MIMO systems with respect to the system description. Which of the following statements is 'True' or 'False'?

NO.	Task/Question/Judgement	True	False
A1	The system can be described in time domain.	⊗	○
A2	The system can be described in frequency domain.	⊗	○
A3	For the MIMO description, using eigenvalues is the suitable approach to describe the dynamical full-state behavior.	⊗	○
A4	The dimensions of weighting matrices (Q and R) are only related to the dimension of the system matrix (A).	○	⊗
A5	The ranks of the matrices B and C are identical and equal to one.	○	⊗

NO.	Task/Question/Judgement	True	False
B1	Eigenvalues are real or conjugate complex.	⊗	○
B2	Applying Hurwitz criterion for exact asymptotic stability check is suitable from a principal point of view.	⊗	○
B3	A system with the zeros $s_{o1,2} = 0.0007 \pm j\omega$, $s_{o3} = -10.000 \pm 0.0007 j\omega$, $s_{o4} = -1 \pm j\omega$, and the poles $s_{p1,2,3,4,5} = -1$ can be denoted as state stable.	○	⊗
B4	State-stability check is related to system stability.	⊗	○
B5	If $r = m = 1$ with Rank $B = m$ and Rank $C = r$, the system is a SISO system.	⊗	○

NO.	Task/Question/Judgement	True	False
C1	The feedback rule $u = -Kx$ is a typical rule for this field.	⊗	○
C2	The controller design defining K can be in general realized using Ackermann formula.	○	⊗
C3	Output control is a possible feedback technique.	⊗	○
C4	Non-controllable eigenvalues are identical to the output decoupling zeros of the system.	○	⊗
C5	Non-observable eigenmodes are identical to the input decoupling zeros of the system.	○	⊗

For the following tasks, a system description is given with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & 0 \\ 1 & 3 \\ 0 & b_2 \end{bmatrix}, \quad C = \begin{bmatrix} c & 0 & 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} d & 0 \end{bmatrix}$$

with $a_{1,2,3} \neq 0$, $b_{1,2} \neq 0$, $c \neq 0$, and $d = 0$.

1b) (2 Points)
State the Rosenbrock matrix of the system depending on the parameters $a_{1,2,3}$, $b_{1,2}$, and c.

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 & 0 & -b_1 & 0 \\ 0 & s & -1 & -1 & -3 \\ -a_1 & -a_2 & s-a_3 & 0 & -b_2 \\ c & 0 & 0 & 0 & 0 \end{bmatrix}$$

1c) (4 Points)
Calculate the characteristic polynomial of the system matrix A depending on the parameters $a_{1,2,3}$, assuming $a_1 = -2$, $a_2 = -3$, and $a_3 = -4$. Is the system asymptotically stable?

$$\det(\lambda I - A) = \lambda^3 + 4\lambda^2 + 3\lambda + 2$$

Hurwitz - Criterion :

① All coefficients have same sign : fulfilled

② Hurwitz determinants : fulfilled

$$H = \begin{vmatrix} 4 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 2 \end{vmatrix}$$

$$H_1 = 4 > 0$$

$$H_2 = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 10 > 0$$

$\Rightarrow$ The system is asymptotically stable.

1d) (1 Point)

Assume $A^* = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ a_1 & a_2 & a_3 \end{bmatrix}$ and $B^* = \begin{bmatrix} 0 & 0 \\ b_1 & 0 \\ 0 & b_2 \end{bmatrix}$ with $a_1 = -5$, $a_2 = 2$, $a_3 = -4$, $b_1 = 1$, and $b_2 = 1$. Give the matrices A^* and B^*.

$$A^* = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$$

$$B^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1e) (4 Points)
Is the eigenvalue $\lambda_1 = -2$ of A^* controllable? State reason (use original Hautus criteria based on eigenvectors).

Yes.

for $\lambda_1 = -2$:

$$\begin{cases} -4\tilde{\tilde{x}}_{11} - 3\tilde{\tilde{x}}_{12} + 5\tilde{\tilde{x}}_{13} = 0 \\ 3\tilde{\tilde{x}}_{11} - 3\tilde{\tilde{x}}_{12} - 2\tilde{\tilde{x}}_{13} = 0 \\ -\tilde{\tilde{x}}_{11} - 3\tilde{\tilde{x}}_{12} + 2\tilde{\tilde{x}}_{13} = 0 \end{cases}$$

$$\Rightarrow \tilde{\tilde{x}}_1^T = [3 \quad 1 \quad 3]$$

$$\Rightarrow \tilde{\tilde{x}}_1^T \cdot B = [1 \quad 3] \neq 0$$

1f) (4 Points)

Which conditions for the system A^* and the elements c_1, c_2 of $C = \begin{bmatrix} 0 & c_1 & 0 \\ 0 & 0 & c_2 \end{bmatrix}$ have to be fulfilled for full observability? Check the observability of the eigenvalues ($\lambda_1 = -2$, $\lambda_2 = 0$ and $\lambda_3 = 1$) using the Hautus criteria calculating first the right eigenvectors.

for $\lambda_1 = -2$:

$$C \cdot \tilde{x}_1 = \begin{bmatrix} 3c_1 \\ -7c_2 \end{bmatrix}$$

$\Rightarrow$ for $c_1 \neq 0$ or $c_2 \neq 0$ λ_1 is observable.

for $\lambda_2 = 0$:

$$C \cdot \tilde{x}_2 = \begin{bmatrix} 3c_1 \\ -11c_2 \end{bmatrix}$$

$\Rightarrow$ for $c_1 \neq 0$ or $c_2 \neq 0$ λ_2 is observable.

for $\lambda_3 = 1$:

$$C \cdot \tilde{x}_3 = \begin{bmatrix} 0 \\ -c_2 \end{bmatrix}$$

$\Rightarrow$ for $c_2 \neq 0$ λ_3 is observable.

$\Rightarrow$ for $c_2 \neq 0$, the system is fully observable.

Problem 2 (35 Points)

Qualify the following statements regarding the analysis of MIMO-systems as well as methods for the design of linear MIMO-systems.

2a) (4 Points)

NO.	Task/Question/Judgement	True	False
1	A system described with eigenvalues $\lambda_{1,2} = 2$, $\lambda_3 = -1 + j$, $\lambda_4 = -1 - j$, $\lambda_5 = \epsilon$ with $\epsilon = 0.001$ is asymptotically stable.	○	⊗
2	A system to be observed ($\hat{=}$ using state observer) has to be full controllable or controllable.	○	⊗
	A system with $A = \begin{bmatrix} 0 & 1 \\ -1 & -a \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ b \end{bmatrix}$ has to be controlled using full state feedback.		
3	In the case $a < 0$, $b \neq 0$ the controlled system can be always stabilized.	⊗	○
4	In the case $a > 0$, $b \neq 0$ the controlled system can never be stabilized.	○	⊗

2b) (4 Points)

From $\det(\lambda_i I - A) = 0$ the left and right eigenvectors $\tilde{\tilde{x}}$ and $\tilde{x}$ can be calculated by $\tilde{\tilde{x}}_i^T(\lambda_i I - A) = 0$ and $(\lambda_i I - A)\tilde{x}_i = 0$ for all λ_i of A. In the detailed case, $\tilde{x}_1$, $\tilde{\tilde{x}}_1$, $\tilde{x}_2$, $\tilde{\tilde{x}}_2$ are calculated as $\tilde{\tilde{x}}_1 = \begin{bmatrix} 8 \\ 1 \\ 4 \\ 2 \\ 9 \end{bmatrix}$, $\tilde{\tilde{x}}_2 = \begin{bmatrix} 8 \\ 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$, $\tilde{x}_1 = \begin{bmatrix} 8 \\ 6 \\ 4 \\ 0 \\ 3 \end{bmatrix}$, and $\tilde{x}_2 = \begin{bmatrix} 8 \\ 0 \\ 9 \\ 2 \\ 9 \end{bmatrix}$ for a system with $B = \begin{bmatrix} 3 \\ 1 \\ -6 \\ 2 \\ -4 \end{bmatrix}$, $C = \begin{bmatrix} 5 & c & 3 & -2 & -7 \end{bmatrix}$.

What can be concluded from the given facts?

NO.	Task/Question/Judgement	True	False
1	The system has 2 eigenvalues.	○	⊗
2	For $c = 0$ the second mode is not observable.	⊗	○
3	For $c \neq 0$ the second mode is not observable.	⊗	○
4	The first mode is uncontrollable.	○	⊗

2c) (4 Points)

A system with $A = \begin{bmatrix} 0 & 1 \\ -d & -k \end{bmatrix}$, $C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$, $x = \begin{bmatrix} x_2 & x_1 \end{bmatrix}^T$ should be monitored for diagnostic purposes. The damping d can not be modeled, obviously the behavior seems to be perfectly damped, so d is assumed as 10.

NO.	Task/Question/Judgement	True	False
1	The measurements of the system are c_1x_2 and c_2x_1.	○	⊗
2	The measurements are coupled (only one independent measurement exists).	⊗	○
3	The system description shows that $\dot{x}_2 = x_1$, so x_2 can be calculated as $x_2 = \int x_1 dt$ and therefore no measurement for the absolute value of x_2 is necessary.	○	⊗
4	In the case of negative damping $(d < 0)(=$ excitation) the system is unstable. In this case of an unstable system a stable observer can be defined for estimation of x_2 and x_1.	⊗	○

2d) (7 Points)

For the system with

$$\lambda_{1,2} = -3 \pm j$$
$$\lambda_{3,4} = -1 \pm j$$
$$\lambda_{5,6} = 2 \pm 3j$$
$$\lambda_7 = 4$$
$$\lambda_{8,9} = 0 \pm 2j$$

a controller design has to be realized. It has to be stated that the non-observable eigenvalues are $\lambda_{5,6} = 2 \pm 3j$ and $\lambda_{8,9} = 0 \pm 2j$, as well as non-controllable eigenvalues with $\lambda_{1,2} = -3 \pm j$.

NO.	Task/Question/Judgement	True	False
1	The system has 2 output decoupling zeros.	○	⊗
2	The system has at least 6 invariant zeros.	⊗	○
3	The system is stabilizable.	⊗	○
4	The poles are $s_{p1,2} = -1 \pm j$, $s_{p3} = 4$, and $s_{p4,5} = -3 \pm j$.	○	⊗
5	The eigenvalues are $\widetilde{\lambda}_{1,2} = -3 \pm j$, $\widetilde{\lambda}_{3,4} = -1 \pm j$, $\widetilde{\lambda}_{5,6} = 2 \pm 3j$, $\widetilde{\lambda}_7 = 4$, and $\widetilde{\lambda}_{8,9} = 0 \pm 2j$.	⊗	○
6	A full state observer can estimate the mode related to $\widetilde{\lambda}_{8,9} = 0 \pm 2j$.	○	⊗
7	The system is not detectable.	⊗	○

2e) (16 Points)

A system is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ -2 & -3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}, \quad C = \begin{bmatrix} c & 0 & 0 \end{bmatrix}, \quad \text{and} \quad D = [0].$$

i) (3 Points)
The transfer function matrix $G(s)$ is

○ $\frac{(s^2+6s+11)bc}{s^3+6s^2+12s+6}$ ○ $\frac{-(2s+1)bc}{s^3+6s^2+12s+6}$

○ $\frac{sbc}{s^3+6s^2+12s+6}$ ⊗ $\frac{bc}{s^3+6s^2+12s+6}$

☐

ii) (3 Points)
Calculate the eigenvalues of A. The result is:

○ $\lambda_1 = -1,\ \lambda_{2,3} = -2 \pm j$ ○ $\lambda_1 = \sqrt{2},\ \lambda_{2,3} = -\sqrt{2} \pm \sqrt{2}j$

○ $\lambda_1 = -1,\ \lambda_{2,3} = \pm j$ ⊗ None

☐

iii) (2 Points)
Is the system fully controllable?

○ Yes. ⊗ Yes, for $b \neq 0$.

○ No. ○ None

☐

iv) (2 Points)
Is the system fully observable?

○ No. ○ Yes.

○ Yes, for $c = 0$. ⊗ Yes, for $c \neq 0$.

☐

v) (3 Points)
Calculate the feedback gains for state control of the given system using pole placement. The desired eigenvalues of the controlled system should be $\lambda_1 = 4$, $\lambda_2 = 2$ and $\lambda_3 = -1$.

- ○ $k_1 = -b/2$, $k_2 = b/3$, $k_3 = b/2$
- ○ $k_1 = -10/b$, $k_2 = 11/b$, $k_3 = 13/b$
- ○ $k_1 = 1/b$, $k_2 = b$, $k_3 = b/3$
- ⊗ $k_1 = 13/b$, $k_2 = 12/b$, $k_3 = -11/b$

☐

vi) (3 Points)
Gilbert approach is considered.

NO.	Task/Question/Judgement	True	False
1	Gilbert approach allows a detailed calculation of parameter dependences between the system A and related output defined by C.	⊗	○
2	Gilbert approach can also be used for stability analysis.	○	⊗
3	Gilbert approach allows mode-wise consideration of observability and controllability.	⊗	○

$\sum$ □

Problem 3 (35 Points)

3a) (4 Points)
An electrical system is modeled using the following differential equation

$$m\ddot{x} + d\dot{x} + kx = 4 \cdot u(t)$$

with scalars $m, d > 0$ and $u(t) = 1(t)$. The variable x is measured.
Taking the vector $z = [z_1,\ z_2]^T = [x,\ \dot{x}]^T$ as state vector, give the matrices A, B, and C.
Calculate the feedback gains for state control of the given system using pole placement. The desired eigenvalues of the controlled system should be

$$\lambda_1 = -\frac{d}{m} + \sqrt{\frac{d^2}{m^2} + \frac{4k}{m}} \text{ and } \lambda_2 = -\frac{d}{m} - \sqrt{\frac{d^2}{m^2} + \frac{4k}{m}}.$$

$$\dot{z} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix}}_{A} \cdot z + \underbrace{\begin{bmatrix} 0 \\ \frac{4}{m} \end{bmatrix}}_{B} \cdot 1(t)$$

$$y = \underbrace{[1 \quad 0]}_{C}\, z$$

$$|\lambda I - (A - Bk)| = \lambda^2 + \left(\frac{d}{m} + \frac{4k_2}{m}\right)\lambda + \frac{k}{m} + \frac{4k_1}{m}$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + \frac{2d}{m} - \frac{4k}{m}$$

$$\Rightarrow k = \left[-\frac{5}{4}k \quad \frac{d}{4}\right]$$

3b) (2 Points)
The system described by the system matrix $A - BK$ has the eigenvalues $\lambda_{1,2} = -2 \pm j\sqrt{5}$, which are not effected by K, while the remaining eigenvalues λ_i, $i = 3...n$ depend on K. Investigate if the system $\dot{x}(t) = Ax(t) + Bu(t)$ is full controllable or not. Also investigate if the mentioned system is stabilizable or not.

The system is not fully controllable, because $\lambda_{1,2}$ are not effected by K.

The system is stabilizable, because $\lambda_{1,2}$ are stable and $\lambda_{3 \cdots n}$ are controllable.

The system $\dot{x}(t) = Ax(t) + Bu(t)$, $y = Cx(t)$ with

$$A = \begin{bmatrix} 0 & 1 \\ -a & -2a \end{bmatrix};\ B = \begin{bmatrix} 0 \\ b \end{bmatrix};\ C = \begin{bmatrix} c & 0 \end{bmatrix}\ (b,\ c \neq 0)$$

has to be investigated.

3c) (3 Points)
For which values of the parameter a an asymptotically stable controlled system using a suitable controller can be achieved? Declare additionally the Rosenbrock matrix of the system.

If the system is fully controllable, it is possible to achieve an asymptotically stable controlled system.

$\det(Q_S) = -b^2$.

As $b \neq 0$, for all values of a the requirement is fulfilled.

$$P(s) = \begin{bmatrix} s & -1 & 0 \\ a & s+2a & -b \\ c & 0 & 0 \end{bmatrix}$$

3d) (2 Points)
Which values should the parameters a and b have in order to achieve i) full observability and ii) detectability?

* fully observable independent of a and b

* detectable because of full observability.

3e) (4 Points)
For the design of a linear quadratic optimal state feedback according to the quality criterion

$$J = \frac{1}{2}\int_0^\infty x^T \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix} x + u^2 dt$$

the elements p_{11}, p_{12}, and p_{22} of P have to be assigned. Calculate for $a = 3$ and $b = 2$ the equations to define P.

$$\begin{bmatrix} 0 & -3 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot 1 \cdot [0\ 2]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} -4p_{12}^2 - 6p_{12} + 5 = 0 \\ p_{11} - 6p_{12} - 3p_{22} - 4p_{12}p_{22} = 0 \\ 2p_{12} - 12p_{22} - 4p_{22}^2 + 9 = 0 \end{cases}$$

$\square$

3f) (4 Points)
Give the modal matrix of the system

$$\dot{x}(t) = Ax(t), \text{ with } A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{3} \cdot j}{2}$$

$$v_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} j \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{3}}{2} j \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} j & -\frac{1}{2} - \frac{\sqrt{3}}{2} j \end{bmatrix}$$

3g) (3 Points)

A system is described by A, B, and C. The dimension of the state vector x is 35. The modal matrix of A is V. How can the observable and controllable modes be calculated numerically, using which approach?

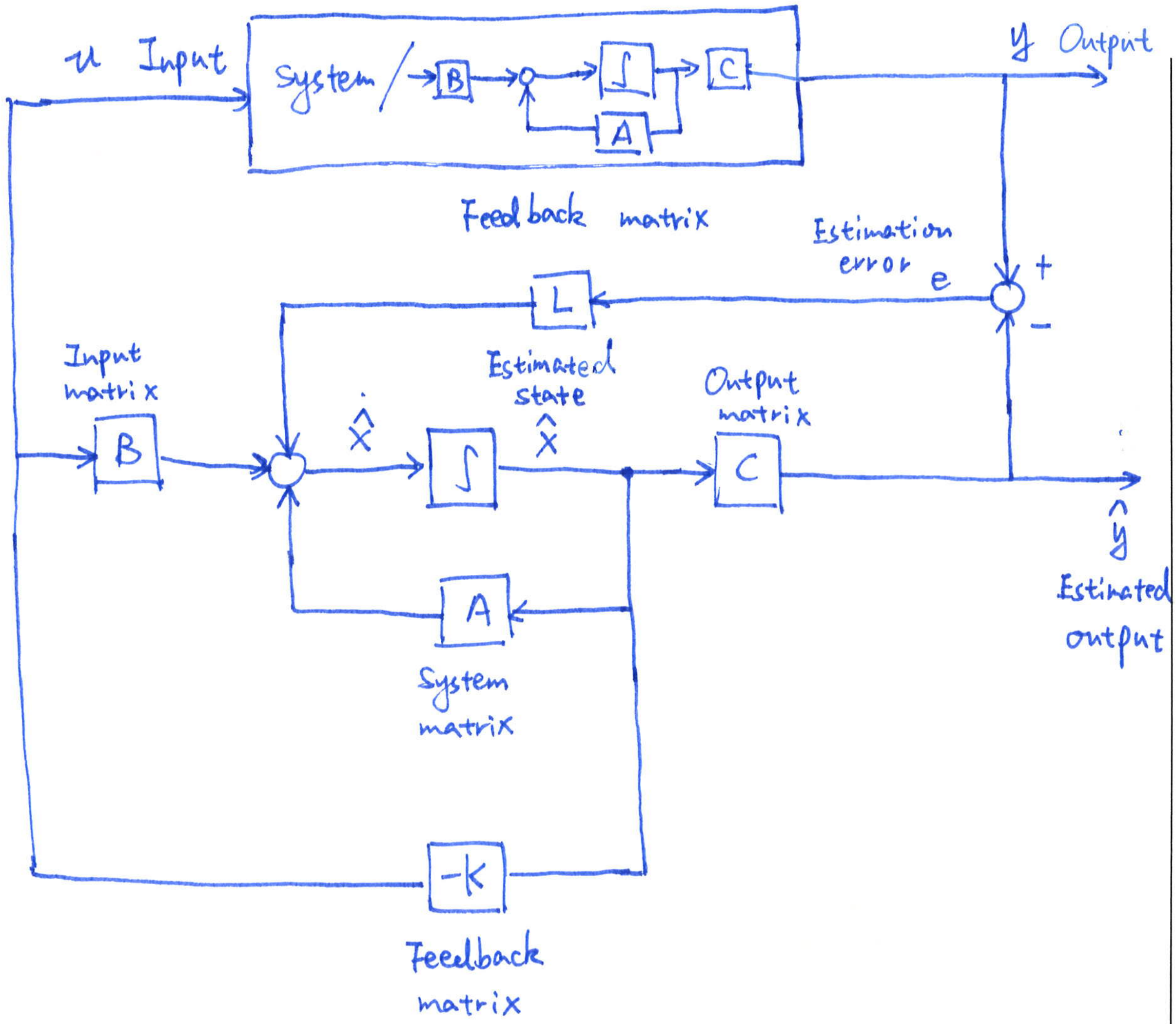
$\tilde{C} = CV$ observable → all colums $\neq 0$ and no dependencies in case of conjugate complex eigenvalues.

$\tilde{B} = V^{-1}B$ controllable → all rows $\neq 0$ and no dependencies in case of conjugate complex eigenvalues.

Using Gilbert

3h) (3 Points)

Draw the scheme of a model-based controller using Luenberger observer, denote the matrices, the signal flows, and the inputs and outputs.



3i) (3 Points)
How can the observer be used for diagnostic purposes? Explain additionally the term residuum.

compare online the estimation with measurement over the time and use the estimation as an auxiliary information to detect possible faults in the system.

Residuum: $r =$ measurement − estimation
$= y(t) - \hat{y}(t)$

if $|r(t)| < \varepsilon$: no fault
else : fault exists.

3j) (4 Points)
State recommendations about the relation between the eigenvalues of the closed-loop system and the eigenvalues of the observer. Which condition is required for the sampling rate of the calculation unit?

* The eigenvalues λ_{Bi} of the observer should definitely lie left to the eigenvalues of the closed-loop system on the left half space of the complex plane.

$$|Re\{\lambda_{Bi}\}| \overset{!}{=} 2 \sim 6 \, |Re\{\lambda_{s-left}\}|$$

where λ_{s-left} is the leftmost eigenvalue of the system.

* According to the shannon theorem, the sampling rate of the calculation unit ω_T should be greater than $2\omega_{max}$, typically $(5 \sim 10)$, where ω_{max} is the maximum frequency of the eigenvalues of the closed-loop system and the observer.

3k) (3 Points)

The system matrix A of a linear dynamical system is defined as

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}.$$

Calculate the solution matrix P using Ljapunov approach (with the weighting matrix $Q = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$) and determine the stability of the system.

if for all matrices $Q = Q^T$ positive definite $(Q > 0)$ the Ljapunov equation $A^T P + PA = -Q$ has a unique solution $P = P^T > 0$, the system is asymptotically stable.

$$\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\begin{cases} 6P_1 + 2P_2 = -2 \\ 3P_2 + P_3 + P_1 = 0 \\ 2P_2 = -4 \end{cases}$$

$$P = \begin{bmatrix} \frac{1}{3} & -2 \\ -2 & \frac{17}{3} \end{bmatrix}$$

$\lambda_1 > 0 \Rightarrow P$ is not positive definite.

$\lambda_2 < 0 \Rightarrow$ the system is not asymptotically stable.

Σ