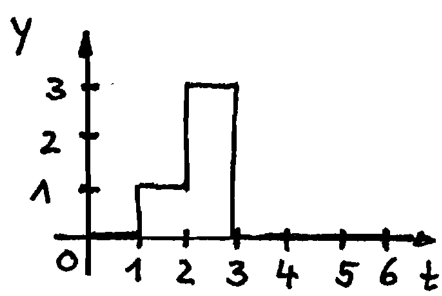
Problem 1

a) A reference value is a variable given from the environment to the controller, which gives the reference for the output as the value to be controlled.

The input variable is acting direct to the system and changes the output of the system. In open loop systems the input variable acts from the environment to the system, in closed loop systems the input is the output of the controller and the input of the system to be controlled.

b)



c) The poles denote those frequencies for which the transfer behavior probably get "∞" (resonance).

The zeros denote those frequencies, which cannot be transferred through the system.

Problem 1

c) For a double pole in the Bode diagram follows an amplitude change of $-40\ ^{dB}/dec.$ and a phase change of $-180°$.

d) $PIDT_1$ - system

ODE

$$T_1\,\dot{y} + y = K \cdot \left(u + \frac{1}{T_I}\int u\,dt + T_D\,\dot{u} \right)$$

transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K \cdot \left(1 + \frac{1}{T_I s} + T_D s\right)}{T_1 s + 1}$$

e) Mathematical definition phase margin Φ_R

$$\Phi_R = 180° + \varphi_o(\omega_s)$$

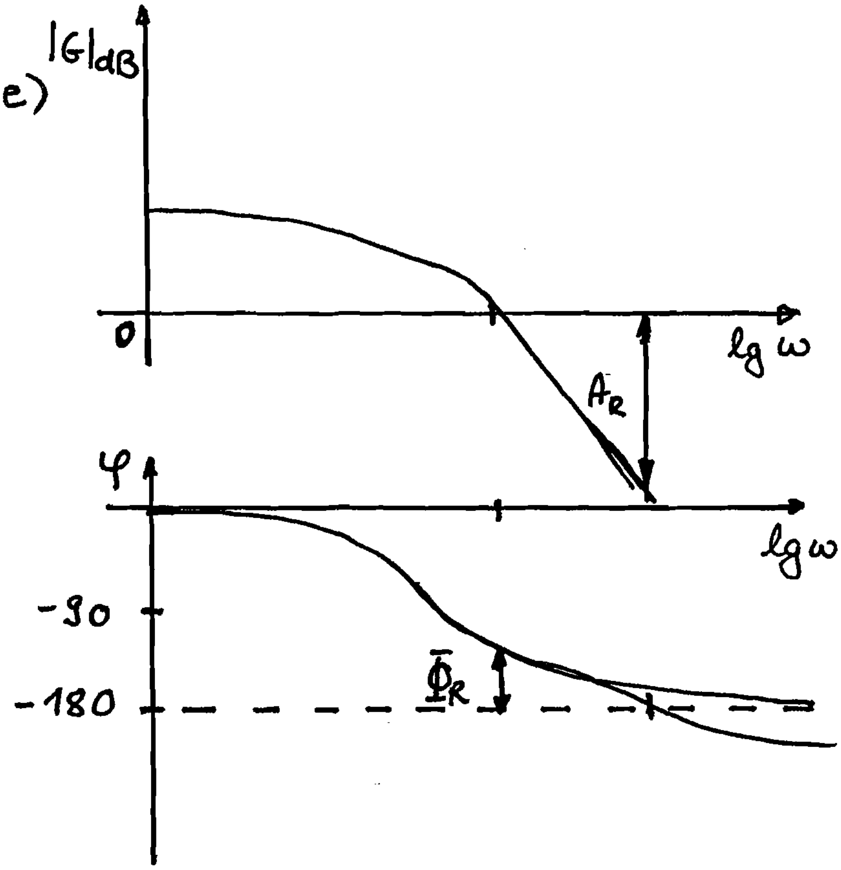
Amplitude margin

$$A_R = \frac{1}{|G_o(j\omega_c)|}$$

The transfer function to be considered is the one of the open loop.

Problem 1

e)



Problem 2

a) PT_2 transfer behavior

$$T_2 \ddot{y} + T_1 \dot{y} + y = K \cdot u$$

$$\mathcal{L}\{\delta(t)\} = 1$$

$$F(s) = G(s) \cdot \mathcal{L}\{\delta(t)\}$$

$$= G(s) \cdot 1$$

~~stationary~~ final value

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s \cdot G(s)$$

$$= \underline{\underline{0}}$$

b) Damping of a pair of eigenvalues

$$d = \cos\left(\operatorname{atan} \frac{\operatorname{Im}\{\lambda_i\}}{\operatorname{Re}\{\lambda_i\}}\right)$$

$$\lambda_1 = -2 \pm 2j \qquad \lambda_2 = -4 \pm 2j$$

$$d_{11} = \cos\left(\operatorname{atan}\left(\frac{2}{-2}\right)\right) = \frac{\sqrt{2}}{2}$$

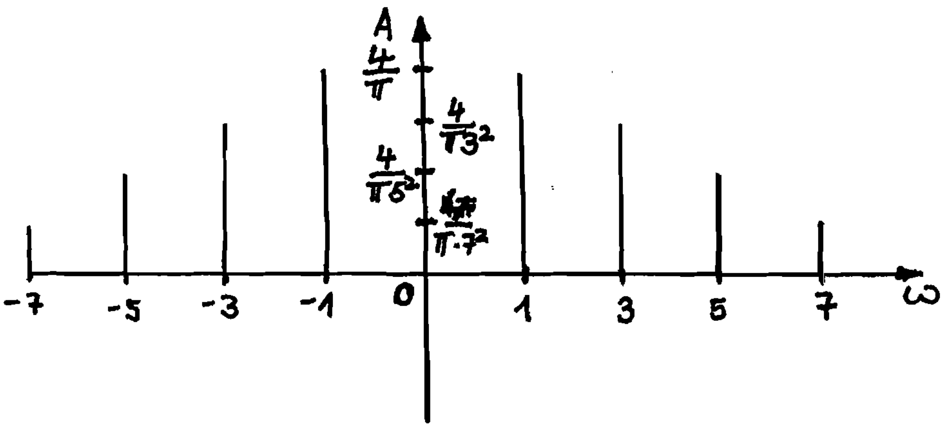
$$d_{21} = \cos\left(\operatorname{atan}\left(\frac{2}{-4}\right)\right) = 0.894$$

$$d_{22} = \cos\left(\operatorname{atan}\left(\frac{-2}{-4}\right)\right) = 0.894$$

$$d_{12} = \cos\left(\operatorname{atan}\left(\frac{-2}{-2}\right)\right) = \frac{\sqrt{2}}{2}$$

The least damping is $d_{11}/d_{12} = \frac{\sqrt{2}}{2}$.

c)



d) $$f_a(s) = \frac{1}{T_1 T_2 s^3 + (T_1+T_2)s^2 + s}$$

PT_2 - transfer function

$$G(s) = \frac{1}{T_1 T_2 s^2 + (T_1+T_2)s + 1}$$

$$G(s) = \frac{Y(s)}{U(s)} \Rightarrow Y(s) = G(s) \cdot U(s)$$

$$\Rightarrow \frac{1}{T_1T_2 s^3 + (T_1+T_2)s^2 + s} = \frac{1}{T_1T_2 s^2 + (T_1+T_2)s + 1} \cdot \frac{1}{s}$$

$$U(s) = \frac{1}{s} \quad \Rightarrow \quad \underline{\underline{u(t) = 1(t)}}$$

e) PDT_1 - behavior P - behavior

$$T_1 s y + y = K_1 (u + T_D s u) \qquad y = K_2 \cdot u$$

$$G_1(s) = \frac{K_1 \cdot (1 + T_D s)}{1 + T_1 s}$$

Problem 2

e) Reference transfer function

$$G_W(s) = \frac{Z_R Z_S}{N_R N_S + Z_R Z_S} = \frac{G_0}{1 + G_0}$$

$$= \frac{K_2 K_1 (1 + T_D s)}{(T_1 s + 1) + K_2 K_1 (1 + T_D s)}$$

Disturbance transfer function

$$G_d(s) = \frac{N_R Z_S}{N_R N_S + Z_R Z_S} = \frac{G_S}{1 + G_0}$$

$$= \frac{K_2 \cdot (T_1 s + 1)}{T_1 s + 1 + K_2 K_1 (1 + T_D s)}$$

Characteristic equation

$$T_1 s + 1 + K_1 K_2 + K_2 K_1 T_D s > 0$$

$$\Leftrightarrow \quad s \cdot (T_1 + K_2 K_1 T_D) + 1 + K_1 K_2 > 0$$

$$\Rightarrow \quad T_1 + K_1 K_2 T_D > 0 \quad \wedge \quad 1 + K_1 K_2 > 0$$

$$\Rightarrow \quad K_2 > -\frac{T_1}{K_1 T_D} \quad \wedge \quad \underline{\underline{K_1 K_2 > -1}}$$

$$\Rightarrow \quad \underline{\underline{K_1 K_2 > -\frac{T_1}{T_D}}}$$

Problem 3

a) PT_2 behavior

$$T_1 \ddot{y} + T_2 \dot{y} + y = K u$$

For the given values follows:

$$(s^2 + 10s + 1)\, y = u$$

$$G(s) = \frac{1}{s^2+10s+1}$$

$$\longrightarrow \boxed{\text{Driver}} \longrightarrow \boxed{T_{t_2}} \longrightarrow \boxed{PT_2} \longrightarrow$$

$$G_{Driver_2} = \frac{1}{1+s}\, e^{-(0.6+n)s}$$

analytical equations

phase margin:

1. Define the phase crossover frequency by

$$1 = \left|\frac{1}{1+j\omega_s}\right| \cdot \underbrace{\left|e^{-j\omega_s(-0.6-n)}\right|}_{1} \cdot \left|\frac{1}{(j\omega_s)^2+10j\omega_s+1}\right|$$

$$\Rightarrow 1 = (1+j\omega_s)^2 \cdot \left((j\omega_s)^2 + 10j\omega_s + 1\right)^2$$

$\vdots$

$$\Rightarrow 1 = \left[-\omega_s^6 + 143\,\omega_s^4 - 123\,\omega_s^2 + 1\right] + \left[22\,\omega_s^5 - 224\,\omega_s^3 - 10\,\omega_s^2 + 12\,\omega_s\right] j$$

2. Calculate φ_R by

$$\varphi_R = \pi + \varphi(\omega_s)$$

Problem 3

a) Amplitude margin:

1. $\varphi(\omega_c) = -\pi$

2. $\left| \dfrac{1}{G_0(j\omega_c)} \right| = A_R$

Problem 3

b) state space description

$$10\ddot{y} + 5\dot{y} + k_1 y = g(t) + h(t)$$

$$u(t) = g(t) + h(t)$$

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{10} & -\frac{1}{2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Eigenvalues:

$$\det |\lambda I - A| \overset{!}{=} 0$$

$$\Rightarrow \lambda^2 + \frac{1}{2}\lambda + \frac{k_1}{10} = 0$$

$$\Rightarrow \lambda_{1,2} = -\frac{1}{4} \pm \sqrt{\frac{10-16k_1}{160}}$$

Stodola : $a_0, a_1, a_2 > 0$. means asymptotically stable

$$\overset{a_0}{1 > 0} \;;\; \overset{a_1}{0.5 > 0} \;;\; \overset{a_2}{\frac{k_1}{10}} > 0$$

For $k_1 > 0$ the system is asymptotically stable.

c) BIBO stability

If a system is BIBO stable then the output will be bounded for every bounded input.

Problem 3

d) The Nyquist criteriam

The open loop with $G_o(s)$ leads to an I/O-stable closed-loop, if the polar plot of $G_o(j\omega)$ with $\omega = -\infty : +\infty$ enclosed the critical point $(-1+j0)$ $P_{o,r}$-times counterclockwise.

$$P_{g,r} = P_{o,r} - \hat{U} \quad \| \quad \hat{U} = P_{o,r} - P_{g,r}$$

$P_{g,r}$: poles closed-loop, right s-area
$P_{o,r}$: poles open-loop, right s-area
$\hat{U}$: surrounding counterclockwise

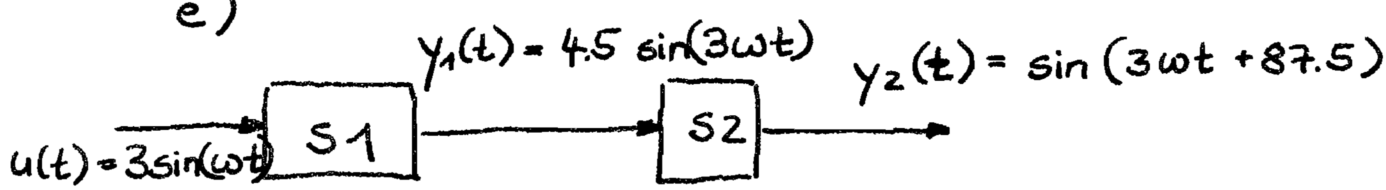
Nyquist criteria of open stable chains ($\Rightarrow P_{o,r} = 0$)

A stable openloop with the transfer function $G_o(s)$ leads to an input/output stable loop; if the polar plot of $G_o(j\omega)$ with $\omega = -\infty$ to ∞, does not enclose the critical point $(-1+j0)$.

$$\hat{U} = -P_{g,r}$$

assumption :
a) negative feedback
b) gain $K > 0$
c) G_o stable
d) max. two poles in origin
e) order of numerator < order of denominator
f) no pole/zero cancellation

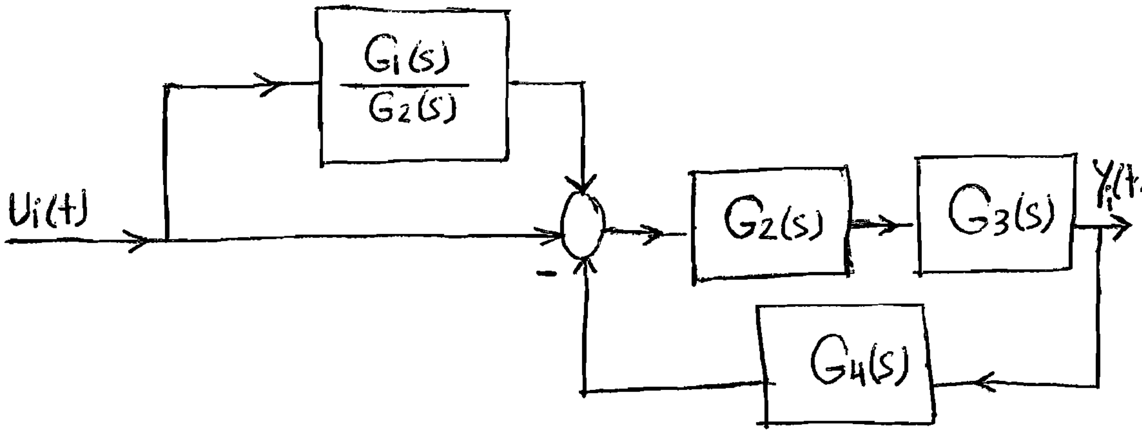
e)

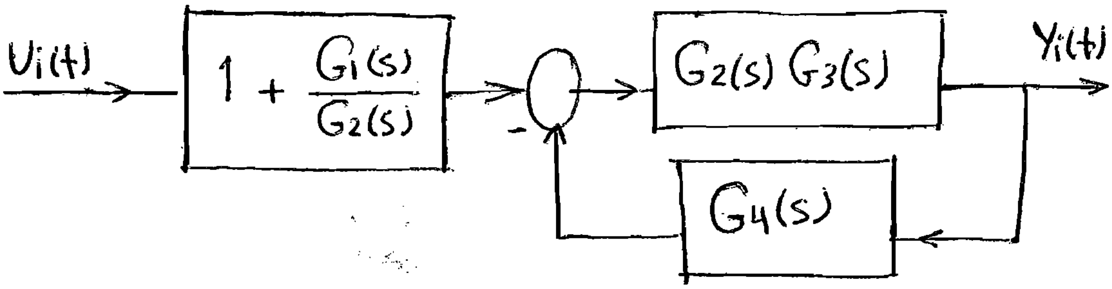


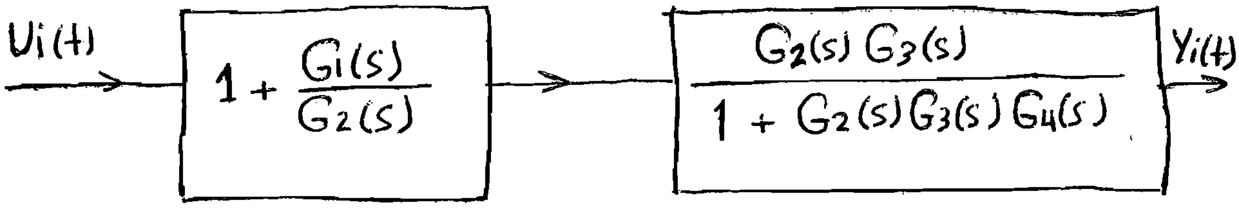
S1 = nonlinear : the frequency is changed
S2 = linear : the changed frequency remains and only the phase is shifted

Problem 3:

a) Transfer function of the marked inner system $G_i(s) = Y_i(s)/U_i(s)$:



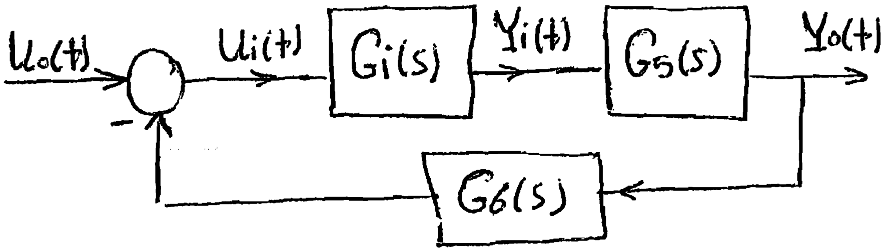


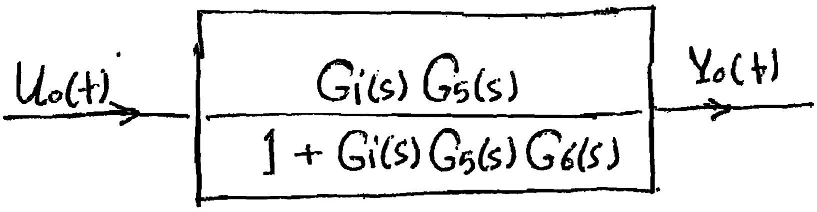


$$G_i(s) = \frac{G_2(s)\,G_3(s) + \dfrac{G_1(s)\,\cancel{G_2(s)}\,G_3(s)}{\cancel{G_2(s)}}}{1 + G_2(s)\,G_3(s)\,G_4(s)}$$

$$G_i(s) = \frac{G_3(s)\left(G_2(s) + G_1(s)\right)}{1 + G_2(s)\,G_3(s)\,G_4(s)} = \frac{Y_i(s)}{U_i(s)}$$

b) Transfer function of the whole system $G_o(s) = \dfrac{Y_o(s)}{U_o(s)}$:





$$G_o(s) = \frac{\left(\dfrac{G_3(s)\left(G_2(s)+G_1(s)\right)}{1+G_2(s)G_3(s)G_4(s)}\right)G_5(s)}{1+\left(\dfrac{G_3(s)\left(G_2(s)+G_1(s)\right)}{1+G_2(s)G_3(s)G_4(s)}\right)G_5(s)G_6(s)} = \frac{Y_o(s)}{U_o(s)}$$

c) The state space model of The system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 6 & 2 \end{bmatrix}, \quad d = 0$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\, u$$

d) The transfer function of the system:

$$2\ddot{y}(t) - 4\dot{u}(t) - 12\int \dot{u}(t)dt = -10\dot{y}(t) - 12y(t) + 2\int u(t)dt + 12u(t)$$

$$\xrightarrow{\frac{d}{dt}}$$

$$2\dddot{y}(t) - 4\ddot{u}(t) - 12\dot{u}(t)dt = -10\ddot{y}(t) - 12\dot{y}(t) + 2u(t) + 12\dot{u}(t)$$

$$\Rightarrow$$

$$\dddot{y}(t) + 5\ddot{y}(t) + 6\dot{y}(t) = 2\ddot{u}(t) + 6\dot{u}(t) + u(t)$$

$$\mathcal{L}[\dddot{y}(t) + 5\ddot{y}(t) + 6\dot{y}(t)] = \mathcal{L}[2\ddot{u}(t) + 6\dot{u}(t) + u(t)]$$

$$[s^3 + 5s^2 + 6s]Y(s) = [2s^2 + 6s + 1]U(s)$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 1}{s^3 + 5s^2 + 6s}}$$

e) $G(s) = \dfrac{2s^2 + 6s + 1}{s^3 + 5s^2 + 6s}$

Characteristic equation:

$$s^3 + 5s^2 + 6s = 0 \Rightarrow s(s^2 + 5s + 6) = 0$$

$$s_1 = 0, \; s_2 = -2, \; s_3 = -3$$

$\Rightarrow$ The system is boundary stable.

Problem 5

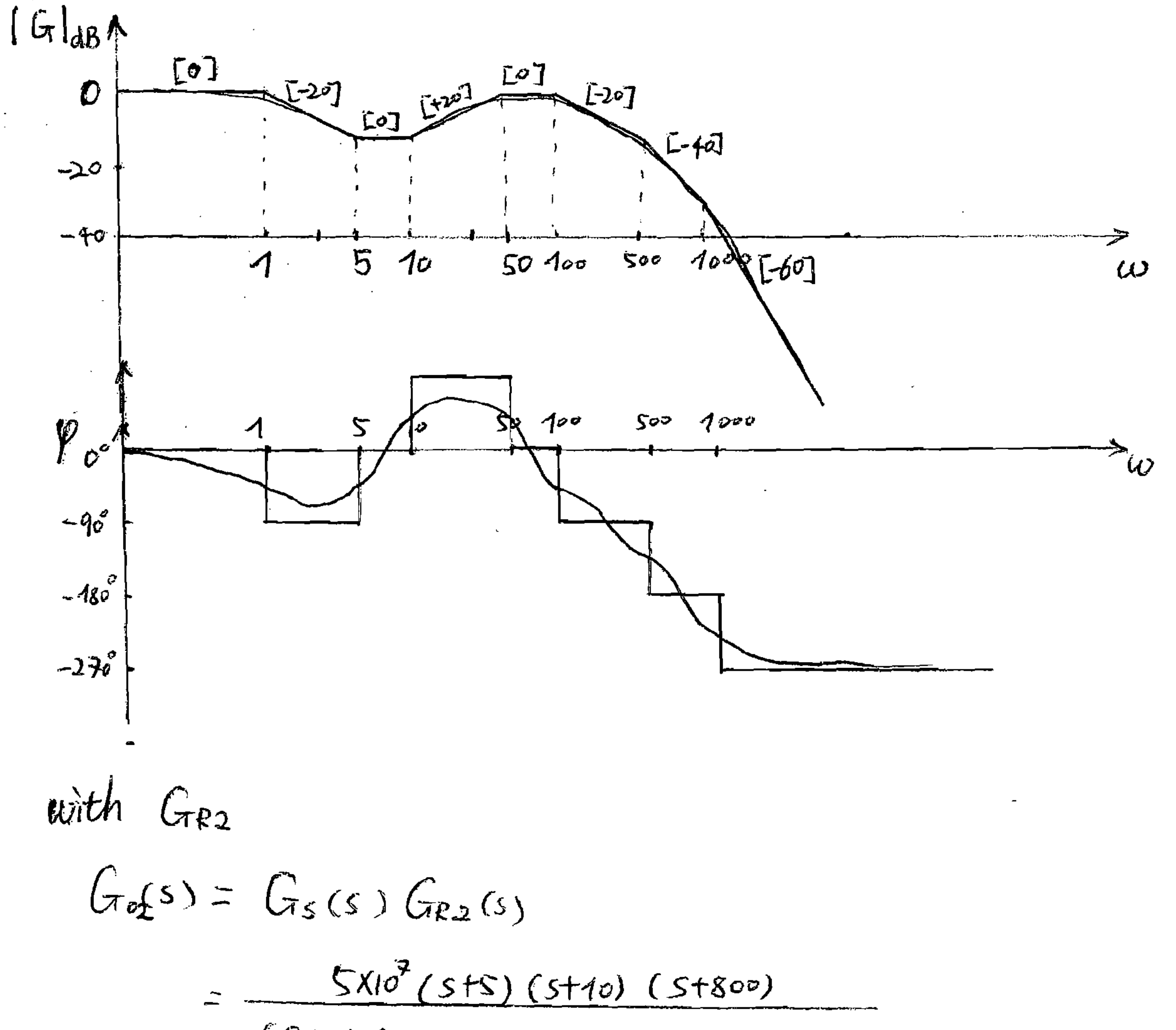
a) $G_{R1} = 1$; $G_{R2} = K(1+T_D s) = 800(1+\frac{1}{800}s) = s+800$

with G_{R1}

$$G_{o1}(s) = G_s(s)\,G_{R1}(s)$$

$$= \frac{5\times10^7 (s+5)(s+10)}{(s+1)(s+50)(s+100)(s+500)(s+1000)}$$

$$= \frac{(\frac{s}{5}+1)(\frac{s}{10}+1)}{(s+1)(\frac{s}{50}+1)(\frac{s}{100}+1)(\frac{s}{500}+1)(\frac{s}{1000}+1)} \qquad (※1)$$



with G_{R2}

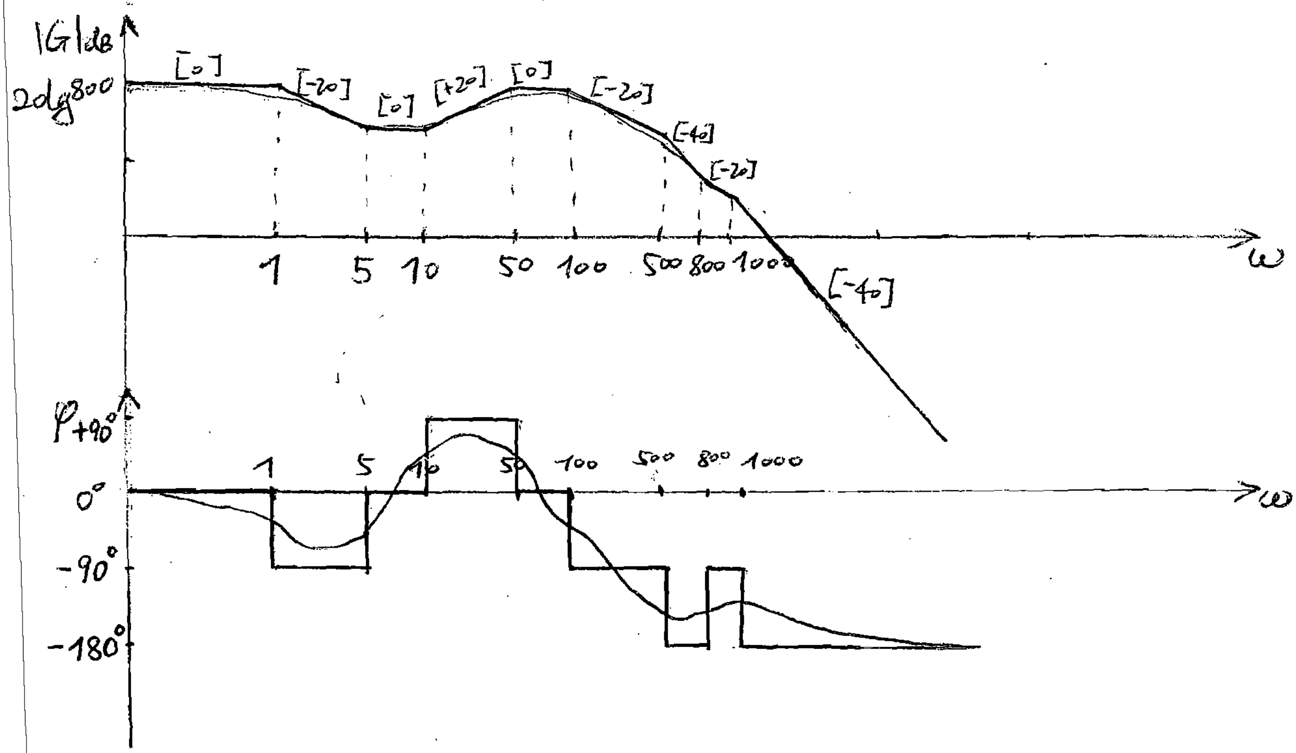
$$G_{o2}(s) = G_s(s)\,G_{R2}(s)$$

$$= \frac{5\times10^7 (s+5)(s+10)(s+800)}{(s+1)(s+50)(s+100)(s+500)(s+1000)}$$

$$= \frac{800(\frac{s}{5}+1)(\frac{s}{10}+1)(\frac{s}{800}+1)}{(s+1)(\frac{s}{50}+1)(\frac{s}{100}+1)(\frac{s}{500}+1)(\frac{s}{1000}+1)} \qquad (※2)$$

Problem 5

a) with G_{R2}



Bode-diagram (see also in the end of the solution).

b) Phase margin (G_{01})

gain of $G_{01}(s)$: $K_{G01} = 1$ (from Eq. (※1))

$|G_{01}(0)| = 0\ dB \Rightarrow \angle G_{01}(0) = 0°$

$\varphi_{R,G01} = 0° - (-180°) = 180°$

Amplitude margin (G_{02})

φ_{G02} doesn't across $-180°$ for $\omega: 0 \to \infty$

$\Rightarrow A_{R,G02} = \infty$

Problem 5

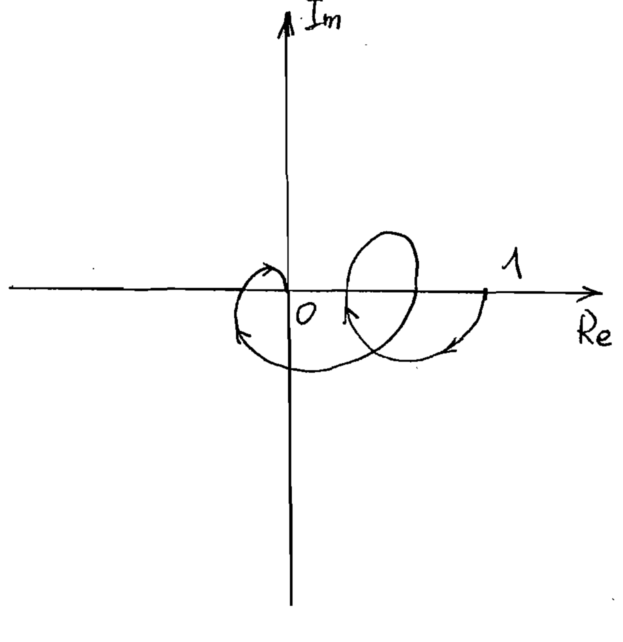
c) Yes.

- the open loops are stable, because they have same stable poles $S_1 = -1$ $S_2 = -50$ $S_3 = -100$ $S_4 = -500$ $S_5 = -1000$.
- no poles at the origin.
- negative feedback

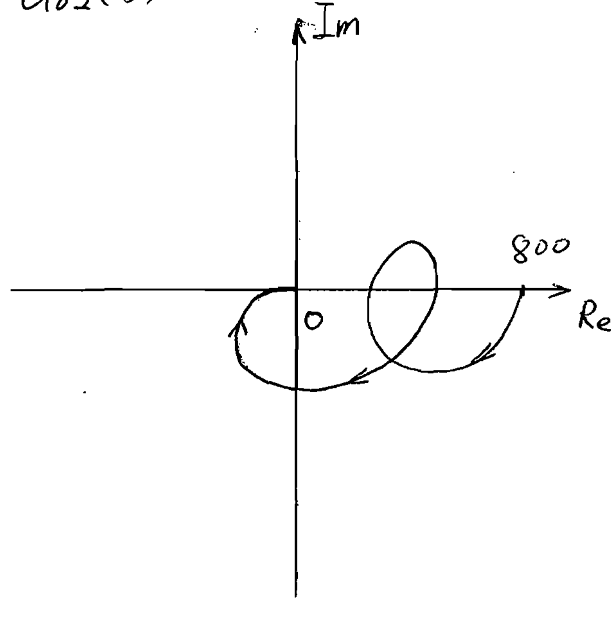
d) The close-loop with $G_p(s)$ and $G_{r1}(s)$ is stable, because the phase margin $\varphi_{R, G_{o1}} > 0°$.
The close-loop with $G_p(s)$ and $G_{r2}(s)$ is stable, because the amplitude margin $A_{R, G_{o2}} > 1$.

e) polar plot

$G_{o1}(s)$



$G_{o2}(s)$



(see next page)

Zeros: poles:

a) $s_{01} = -3$ $s_1 = 0$

$s_{02} = 1$ $s_2 = -0{,}5$

$s_{03} = 2$ $s_3 = -2$

$s_4 = -4$

$$s_{5,6} = -1 \pm \sqrt{-1}$$
$$= -1 \pm i$$

The plant is boundary stable $\rightarrow s_1 = 0$

b) $p_1 = 0$: $D_1 = 1$, $\omega_{0_1} = 0\ s^{-1}$

$p_2 = -0{,}5$: $D_2 = 1$, $\omega_{0_2} = 0{,}5\ s^{-1}$

$p_3 = -2$: $D_3 = 1$, $\omega_{0_3} = 2\ s^{-1}$

$p_4 = -3$: $D_4 = 1$, $\omega_{0_4} = 3\ s^{-1}$

$p_{5,6} = -1 \pm i$: $D_{5,6} = \cos\delta = \dfrac{Re\{\lambda\}}{\sqrt{Re\{\lambda\}^2 + Im\{\lambda\}^2}} = \dfrac{1}{\sqrt{2}}$

$$\omega_{0_{5,6}} = \sqrt{Re\{\lambda\}^2 + Im\{\lambda\}^2} = \sqrt{2}$$

6)

c) $Y(t) = k_P \cdot u(t)$ P

$\frac{1}{2}\ddot{y}(t) - \frac{3}{2}\dot{y}(t) + y(t) = k_D\ \dot{u}(t)$ D-T_2

$G_P(s) = k_P$ => no poles
no zeros

$0{,}5 \cdot Y(s) \cdot s^2 - 1{,}5\ Y(s) \cdot s + Y(s) = k_D \cdot U(s) \cdot s$

$$G_D(s) = \frac{Y(s)}{U(s)} = \frac{2 k_D \cdot s}{s^2 - 3s + 2}$$

$s_{1/2} = 1{,}5 \pm \sqrt{0{,}25}$

$= 1{,}5 \pm 0{,}5$

$s_1 = 1, \quad s_2 = 2$ => > Poles

$s_{01} = 0$ => zero

$$G_D(s) = \frac{2k_D \cdot s}{(s-1)(s-2)}$$

d) $G_{o_1}(s) = G_s(s) \cdot G_p(s)$

$$= \frac{K_p\,(s+3)(s-1)(s-2)}{s(s+0{,}5)(s+2)(s+4)(s^2+2s+2)}$$

$s_{01} = -3$ $\quad s_1 = 0$

$s_{02} = 1$ $\quad s_2 = -0{,}5$

$s_{03} = 2$ $\quad s_3 = -2$

$s_4 = -4$

$s_{5/6} = -1 \pm j$

$G_{o2}(s) = G_s(s) \cdot G_D(s)$

$$= \frac{2 \cdot k_D\,(s+3)}{(s+0{,}5)(s+2)(s+4)(s^2+2s+2)}$$

$s_{01} = -3$ $\quad s_1 = -0{,}5$

$s_2 = -2$

$s_3 = -4$

$s_{4,5} = -1 \pm j$

e) System 1: Number of Poles $n = 6$

Number of zeros $m = 3$

$\rightarrow$ 6 separate branches

$n - m = 3$ branches go to infinity

System 2: Number of poles $n = 5$

Number of zeros $m = 1$

$\rightarrow$ 5 separate branches

$n - m = 4$ branches go to infinity

f) System 1: $$\sigma_{w_1} = \frac{\sum_{i=1}^{n} s_i - \sum_{i=1}^{m} s_{0i}}{n-m}$$

$$= \frac{(0-0{,}5-2-4-1\cancel{+i}-1\cancel{-i})-(-3+1+2)}{3}$$

$$= \underline{\underline{\frac{-8{,}5}{3}}}$$

$$\varphi_{1i} = \frac{180° + i \cdot 360}{n-m} \quad ; \; i = 0, 1 \ldots (n-m-1)$$

$$\Rightarrow \varphi_{11} = 60°$$

$$\varphi_{12} = 180°$$

$$\varphi_{13} = 300°$$

System 2: $$\sigma_{w_2} = \frac{(-0{,}5-2-4-1\cancel{-i}-1\cancel{+i})-(-3)}{}$$

$$= \frac{-8{,}5+3}{4}$$

$$\Rightarrow \varphi_{2_1} = 45°$$

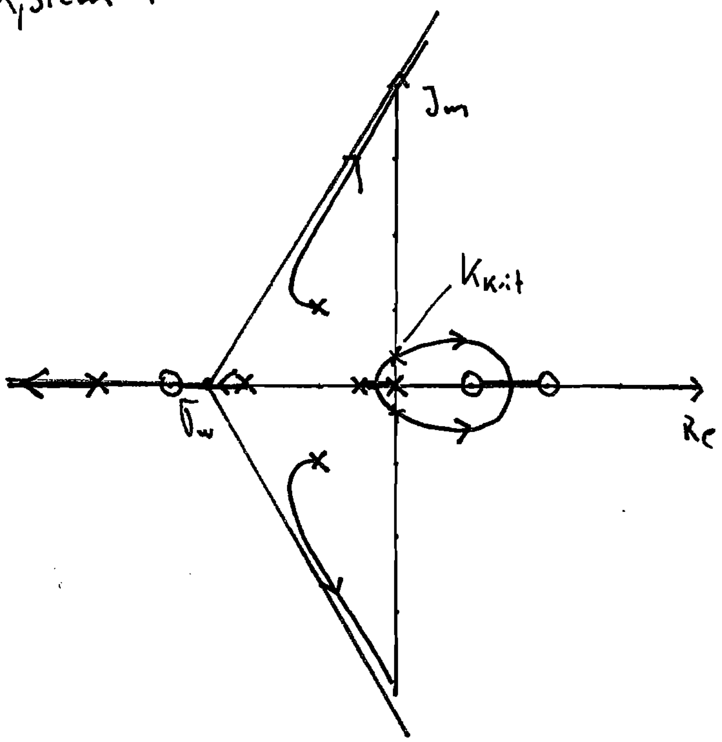
$$\varphi_{2_2} = 135°$$

$$\varphi_{2_3} = 225°$$

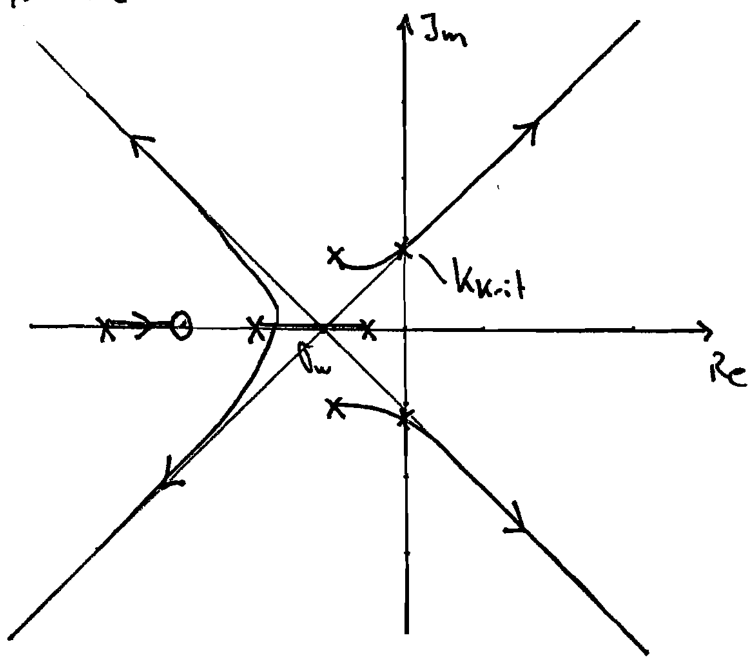
$$\varphi_{2_4} = 315°$$

g)

System 1:



System 2:

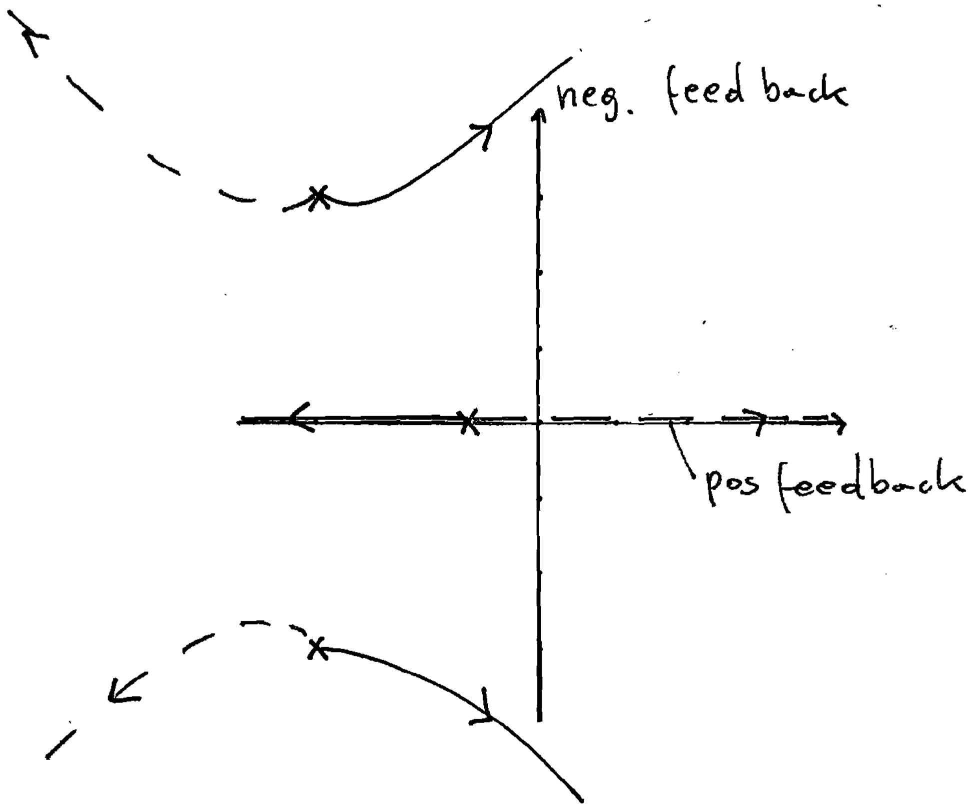


h) $G_s(s) = \dfrac{1}{(s+1)(s^2+6s+18)} \Rightarrow k_s = \dfrac{1}{18}$

$s_1 = -1$

$s_{2/3} = -3 \pm \sqrt{9-18}$
$= -3 \pm 3i$

no zeros



i) $G_o = k_p \cdot \dfrac{1}{(s+1)(s^2+6s+18)} = -1$

$\Leftrightarrow k + s^3 + 6s^2 + 18s + s^2 + 6s + 18 = 0$

$\Leftrightarrow k + s^3 + 7s^2 + 24s + 18 = 0$

$$H = \begin{pmatrix} 7 & 18+k & 0 \\ 1 & 24 & 0 \\ 0 & 7 & 18+k \end{pmatrix}$$

$a_i > 0 \Rightarrow k > -18$

$|H_1| = 7 > 0$ ✓

$|H_2| = \underbrace{7 \cdot 24 - 18}_{150} - k > 0$

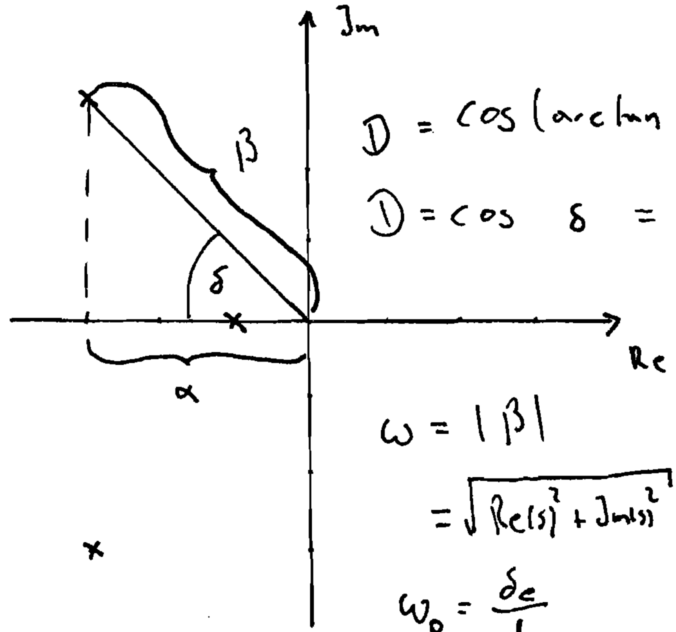
$150 > k$

$|H_3| = |H_2| \cdot a_0 = 150 \cdot (18+k) > 0$

$\Rightarrow k > -18$

$\Rightarrow -18 < k < 150$

j)



$$D = \cos\left(\arctan\left(\frac{\omega_e}{\delta_e}\right)\right) = \cos \phi_d$$

$$D = \cos \delta = \frac{\alpha}{\omega}$$

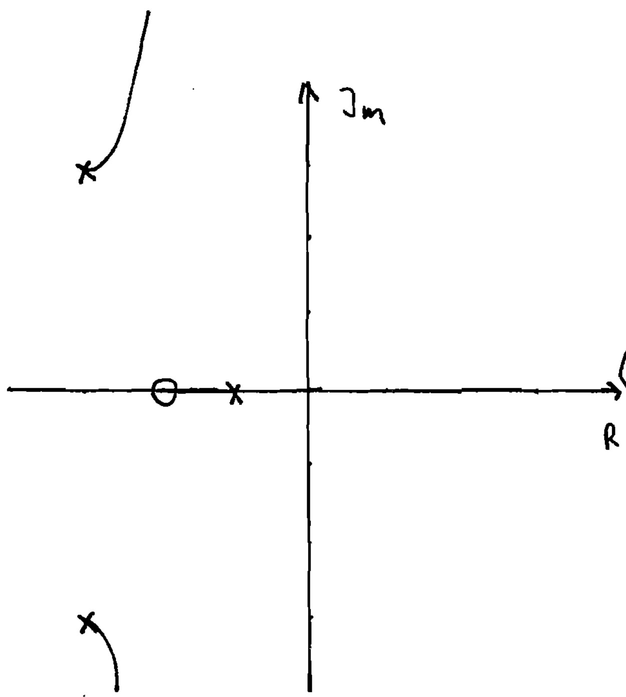
$$\omega = |\beta| = \sqrt{Re(s)^2 + Im(s)^2}$$

$$\omega_0 = \frac{\delta_e}{d_e}$$

k)

$$k = |(-4+1)((-4)^2 + 6(-4) + 18)|$$
$$= |(-3)(16 - 24 + 18)|$$
$$= |-30| = \underline{\underline{30}}$$

l)



PT_1 : one additional Pol

I : one additional Pol in the origin

PD : one additional zero

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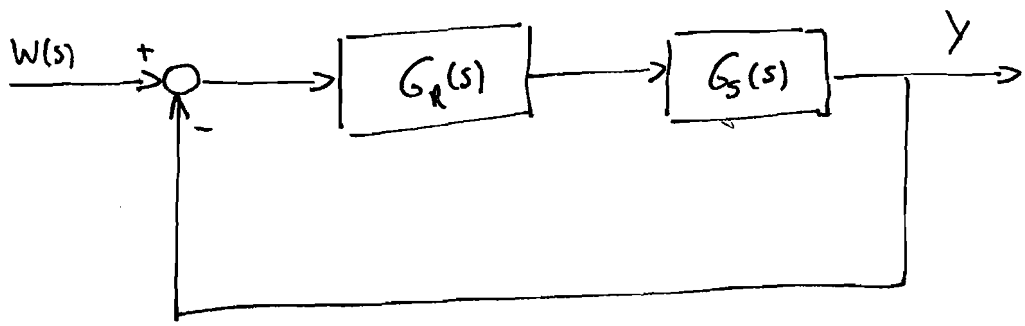
m)

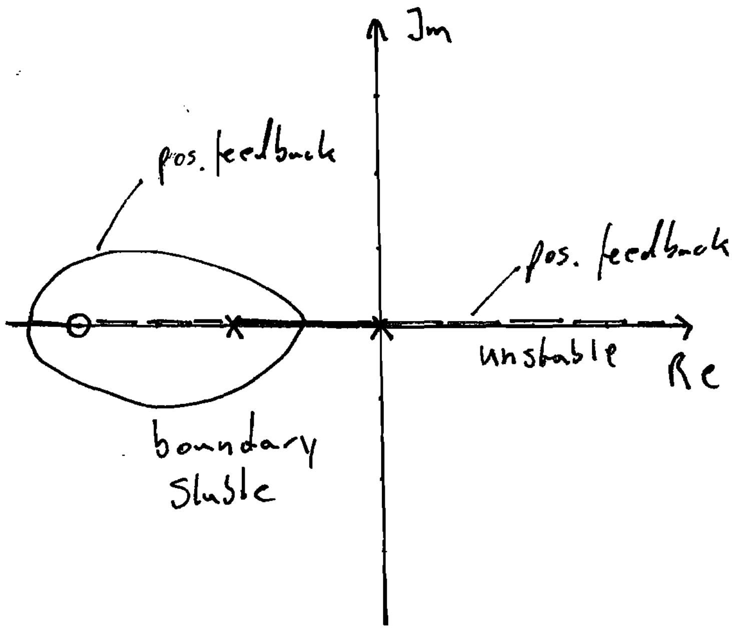
$$G_S(s) = \frac{1}{T_I s(1+sT_1)} = \frac{2}{s(1+s\cdot\frac{1}{3})}$$

$$G_R(s) = k(1+sT_d) = 3(1+s\cdot\frac{1}{4})$$

$$k_0 = k_I \cdot k = 2\cdot 3 = 6$$



o)



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$$\frac{1}{s} + \frac{1}{s+3} = \frac{1}{1+4}$$

$$<=> (s+4)(s+3) + s(s+4) = s(s+3)$$

$$s^2 + 7s + 12 + s^2 + 4s = s^2 + 3s$$

$$s^2 + 8s + 12 = 0$$

$$s_{1,2} \pm = -4 \pm \sqrt{16-12}$$

$$s_1 = -2 \quad \wedge \quad s_2 = -6$$

p)

No, the system has no complex poles