Lecture notes on Modelling of concurrent systems

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Chapter 1

Some properties on bisimilarity

1.1 Closure properties

Definition 1. A symmetric relation $\mathcal{R} \subseteq S \times S$ on the states of a transition system $(S, L, \rightarrow)$ is a *bisimulation* relation if, and only if, the following transfer property is satisfied.

$$\forall_{s,s',t \in S, a \in L}\ (s \xrightarrow{a} s' \wedge s\mathcal{R}t) \implies \exists_{t' \in S}\ t \xrightarrow{a} t' \wedge s'\mathcal{R}t'. \tag{1.1}$$

Two states $s, t \in S$ are bisimilar, denoted $s \leftrightarrows t$, if there exists a bisimulation relation $\mathcal{R}$ such that $s\mathcal{R}t$.

In other words, bisimilarity $\leftrightarrows$ is defined as the union of all bisimulation relations. However, the natural question is: whether bisimilarity is a bisimulation relation? Yes it is, which we prove next.

Lemma 1. *The union of two bisimulation relations is a bisimulation relation.*

Proof. Let $\mathcal{R}, \mathcal{R}'$ be two bisimulation relations on $(S, L, \rightarrow)$. Let $s\mathcal{R} \cup \mathcal{R}'t$ and let $s \xrightarrow{a} s'$. Then we distinguish two cases:

- Let $s\mathcal{R}t$. Then, by (1.1) we find a transition $t \xrightarrow{a} t'$, for some t', such that $s'\mathcal{R}t'$. Clearly, $s'\mathcal{R}\cup\mathcal{R}'t'$.
- Let $s\mathcal{R}'t$. Similar to the previous case. □

Lemma 2. *The relational composition of two bisimulation relations results in a bisimulation relation.*

Theorem 3. *Bisimilarity, i.e.,* $\leftrightarrows$ *is an equivalence relation.*

Proof. Direct from the previous two lemmata. □

1.2 The link with saturated bisimilarity

Definition 2. Let $(S, L, \rightarrow)$ be a transition system. Define a family of symmetric relations $\sim_n \subseteq S \times S$ (for each $n \in \mathbb{N}$) as follows:

- Basis: $\sim_0 = S \times S$.
- Inductive step: $s \sim_{n+1} t \iff \forall_{s' \in S, a \in L}\ (s \xrightarrow{a} s' \implies \exists_{t'}\ t \xrightarrow{a} t' \wedge s' \sim_n t')$.

Then the relation $\sim = \bigcap_{n \in \mathbb{N}} \sim_n$ is called *saturated* bisimilarity.

Lemma 4. $\forall_{m,n \in \mathbb{N}}\ m \leq n \implies \sim_m \supseteq \sim_n$.

Proof. It suffices to show that $\sim_{n+1} \subseteq \sim_n$, for any $n \in \mathbb{N}$. Let $s \sim_{n+1} t$, for some $n \in \mathbb{N}$, and let $s \xrightarrow{a} s'$. Then, $t \xrightarrow{a} t'$ and $s' \sim_n t'$, for some t'. By induction hypothesis $\sim_n \subseteq \sim_{n-1}$. Thus, $s' \sim_{n-1} t'$. Likewise, we can prove the transition originating from t. Hence, $s \sim_n t$. □

Theorem 5. *If the underlying transition system is image-finite, then the bisimilarity and saturated bisimilarity coincides, i.e.,* $\leftrightarrows = \sim$.

Proof. $\boxed{\Rightarrow}$ This direction is obvious and the result follows directly from induction on $\sim_n$.

$\boxed{\Leftarrow}$ Let $s \sim t$ and $s \xrightarrow{a} s'$. Then,

$$\forall_{n>0} \exists_{t_n \in S}\ t \xrightarrow{a} t_n \wedge s' \sim_n t_n.$$

Since the underlying transition system is image finite, we know that the set $\{t' \mid t \xrightarrow{a} t' \wedge \exists_n\ s' \sim_n t'\}$ is finite. I.e., some element in this set that is appearing infinitely often in the sequence $(t_n)_{n\in\mathbb{N}}$. I.e., there is a state t_k (for some $k \in \mathbb{N}$) such that

$$\forall_{m\in\mathbb{N}} \exists_{n\in\mathbb{N}} \quad m \leq n \wedge t_n = t_k. \tag{1.2}$$

Next, we claim that $\forall_{m\in\mathbb{N}}\ s' \sim_m t_k$. Let $m \in \mathbb{N}$. Clearly, from (1.2) we have some $n \in \mathbb{N}$ such that $m \leq n$ and $t_n = t_k$. And using Lemma 4 we conclude that $s' \sim_m t_k$. Hence, $s' \sim t_k$. ☐

1.3 The link with Hennessy-Milner logic

Recall the Hennessy-Milner logical formulas are generated from the following grammar:

$$\Phi_{\text{HML}} ::= \quad \top \quad | \quad \langle a \rangle \varphi \quad | \quad \neg\varphi \quad | \quad \varphi \wedge \varphi'.$$

Define a modal depth δ as a function of type $\Phi_{\text{HML}} \longrightarrow \mathbb{N}$:

$$\delta(\top) = 0 \qquad\qquad \delta(\neg\varphi) = \delta(\varphi)$$
$$\delta(\langle a \rangle\varphi) = \delta(\varphi) + 1 \qquad\qquad \delta(\varphi \wedge \varphi') = \max(\delta(\varphi), \delta(\varphi')).$$

Theorem 6. *Let $\Phi_{HML}(n) = \{\varphi \in \Phi_{HML} \mid \delta(\varphi) \leq n\}$ be the set of logical formulas of modal depth n and let $\Phi_{HML}(s, n) = \{\varphi \in \Phi_{HML}(n) \mid s \models \varphi\}$. Then, two states are saturated bisimilar up to depth n if, and only if, they satisfy the same set of modal formulas of depth n. In symbols,*

$$\forall_{n\in\mathbb{N}}\ s \sim_n t \iff \Phi_{HML}(s, n) = \Phi_{HML}(t, n).$$

Proof. $\boxed{\Leftarrow}$ Consider the above theorem statement as the definition of $\sim_n$. Clearly, $\sim_0 = S \times S$ because all states satisfy $\top$ and $\Phi(s, 0) = \{\top\}$. Furthermore, observe that

$$\forall_{n\in\mathbb{N}}\ \Phi_{\text{HML}}(s, n) = \Phi_{\text{HML}}(t, n) \implies \forall_{a\in L}\ s \xrightarrow{a} \iff t \xrightarrow{a}. \tag{1.3}$$

Now for the inductive case, assume $\Phi_{\text{HML}}(s, n) = \Phi_{\text{HML}}(t, n)$ to prove the contrapositive statement, i.e,

$$s \not\sim_{n+1} t \implies \Phi_{\text{HML}}(n + 1, s) \neq \Phi_{\text{HML}}(n + 1, t).$$

Suppose $s \not\sim_{n+1} t$ and, without loss of generality, let $s \xrightarrow{a} s'$. Furthermore, we fix $\text{Moves}(a, t) = \{t' \mid t \xrightarrow{a} t'\}$. Note that $\text{Moves}(a, t) \neq \emptyset$ due to (1.3). Since the underlying transition system is image finite, we can enumerate the set $\text{Moves}(a, t)$ by a finite nonempty index set $I = [0, n]$ for some $n \in \mathbb{N}$. Then, we find that $\forall_{i\in I}\ s' \not\sim_n t'_i$. By induction hypothesis we get $\Phi_{\text{HML}}(s', n) \neq \Phi_{\text{HML}}(t'_i, n)$ for all $i \in I$. I.e., there are formulae $\varphi_i \in \Phi_{\text{HML}}(s', n)$ such that $\varphi_i \notin \Phi_{\text{HML}}(t'_i, n)$. So consider the formula $\varphi = \langle a \rangle \bigwedge_{i\in I} \varphi_i$. Clearly, $\delta(\varphi) = n + 1$ and, more importantly, we have $s \models \varphi$ but $t \not\models \varphi$. Thus, $\Phi_{\text{HML}}(s, n + 1) \neq \Phi_{\text{HML}}(t, n + 1)$.

$\boxed{\Rightarrow}$ Left as an exercise. ☐

Corollary 7. *For an image-finite transition system, logical equivalence coincides with bisimilarity.*

Proof. Direct from Theorem 5 and Theorem 6. ☐