A Priori Error Estimates for State-Constrained Optimal Control Problems

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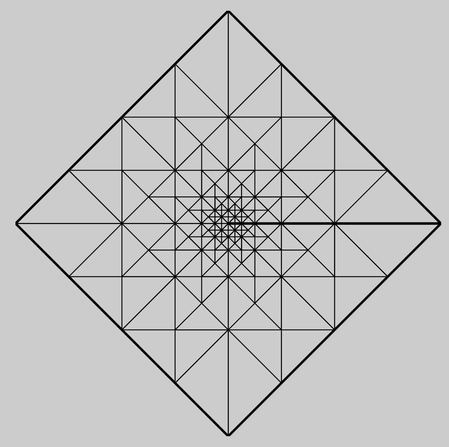


Introduction & Problem Setting

What is **optimal control**?:

- Optimal control means trying to come as close as possible to a desired ***state*** of a system that is controlled by one or more ***control*** variables in an ***optimal***, i.e. cost-effective (e.g. energy minimising) way.
- Applications: e.g. aerodynamics, robotics, vehicle dynamics

The **model optimal control problem**:

$$\left.\begin{aligned} \min_{y,u\in L^2(\Omega)} J(y,u) &:= \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \\ &\text{s.t.} \\ -\Delta y &= u \ \text{ in } \Omega \\ y &= 0 \ \text{ on } \Gamma = \partial\Omega \\ &\text{and} \\ a \le u(x) &\le b, \ \text{ a.e. in } \Omega \\ y(x) &\ge y_c(x) \ \text{ in } \Omega' \subset\subset \Omega, \end{aligned}\right\} \qquad (P)$$

Assumptions:

- $\Omega \subset \mathbb{R}^3$, open, bounded, and of class C^2, $a, b \in \mathbb{R}$, $y_c \in C(\bar\Omega)$

Solvability:

- There exists a **unique solution** $\bar u \in H^1(\Omega)$ and a corresponding optimal state $\bar y = S\bar u \in H^2(\Omega)$, S solution operator to PDE in (P).

However, (P) does not possess a mathematically representable solution. $\Rightarrow$ We now want to **compute** an **approximation** to (P), i.e. **discretise** (P).

Questions:

- How can we discretise (P)?
- How close are the unknown true solution and the computed discrete one: **a priori error estimates**.

Discretisation

Assumptions:

- Triangulate Ω with tetrahedra T, triangulation $\mathcal{T}_h$ is shape regular and uniform, mesh size $h \approx |T|$
- Approximation of control in $U_h := \left\{u \in L^2(\Omega) : u|_T = \text{constant}\right\}$
- Approximation of state in $V_h := \left\{v \in C(\bar\Omega) : v|_T = \text{affine}, \ v|_\Gamma = 0\right\}$

This is the **discretised problem**:

$$\left.\begin{aligned} \min_{y_h\in V_h, u_h\in U_h} J(y_h,u_h) &:= \frac{1}{2}\|y_h-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Omega)} \\ &\text{s.t.} \\ \int_\Omega \nabla y_h(x)\cdot\nabla v_h(x)\,dx &= \int_\Omega u_h(x)v_h(x)\,dx \quad \forall v_h \in V_h \\ &\text{and} \\ a \le u_h(x) &\le b, \ \text{ a.e. in } \Omega \\ y_h(x) &\ge y_c(x) \ \text{ in } \Omega' \subset\subset \Omega, \end{aligned}\right\} \qquad (P_h)$$

Solvability:

- There exists a **unique solution** $\bar u_h \in U_h$ and corresponding state $\bar y_h = S_h\bar u_h$, S_h discretised solution operator to discretised PDE in (P_h).

How good is our approximation?
Typical **a priori error estimates:**

$$\|\bar u - \bar u_h\|_{L^2(\Omega)} + \|\bar y - \bar y_h\|_{L^2(\Omega)} \le ch^\alpha, \quad \alpha > 0.$$

$\Rightarrow$ **How large is α?**

Key tool: optimality conditions for (P) and (P_h):

$$(S^*(S\bar u - y_d), u - \bar u) \ge 0 \quad \forall u \in U^{ad} := \left\{u \in L^2(\Omega) : a \le u \le b, \ Su \ge y_c\right\} \qquad (OC_c)$$

$$(S_h^*(S_h\bar u_h - y_d), u_h - \bar u_h) \ge 0 \quad \forall u_h \in U_h^{ad} := \left\{u_h \in L^2(\Omega) : a \le u_h \le b, \ S_h u \ge y_c\right\} \qquad (OC_d)$$

Derivation of A Priori Estimates

Idea for derivation: Test (OC_c) with functions $u^\delta \in U^{ad}$ close to $\bar u_h$ and (OC_d) $u_h^\sigma \in U_h^{ad}$ close to $\bar u$ & clever rearrangement

Problem: Usual approach $u^\delta = \bar u_h$, $u_h^\sigma = P_h\bar u$ (L^2-projection), however $\bar u_h \notin U^{ad}$, $P_h\bar u \notin U_h^{ad}$

Solution:

- Choose convex linear combinations:

$$\begin{aligned} u_\delta &= (1-\delta)\bar u_h + \delta u_s \\ u_\sigma^h &= (1-\sigma)\bar u + \sigma P_h u_s, \end{aligned}$$

where u_s and $P_h u_s$ are Slater points, i.e.: $Su_s \ge y_c + \tau_c$, $\tau_c > 0$; $S_h P_h u_s \ge y_c + \tau_d$, $\tau_d > 0$.

$\Rightarrow$ **Estimate of maximal violation of state constraint**:

- For u^δ:

$$(y_c - Su_h)^+ \le \|(S - S_h)u_h\|_{L^\infty(\Omega)}$$

- For u_h^σ:

$$(y_c - S_h P_h \bar u)^+ \le \|(S - S_h)P_h\bar u\|_{L^\infty(\Omega)} + \|S\|\,\|\bar u - P_h\bar u\|_X$$

Use **FE-approximation estimates** for the $(S - S_h)\cdot$ terms and a cleverly chosen **space X**:

- $S : X \to L^\infty(\Omega)$ continuous.
- $\|\bar u - P_h\bar u\|_X$ small, i.e. 'close to' $\mathcal{O}(h^2)$.

$\Rightarrow$ **Besov space $X = B^{1/2,2,\infty}(\Omega)^*$**

FINAL RESULT depending on regularity of $\bar u \in W^{1,p}(\Omega)$ (at the least: $p = 2$):

$$\|\mathbf{\bar u - \bar u_h}\|_{\mathbf{L^2(\Omega)}} + \|\mathbf{\bar y - \bar y_h}\|_{\mathbf{L^2(\Omega)}} \le \mathbf{ch}^{\min(\mathbf{1}-\varepsilon\,,\,\mathbf{3/2-3/2p})}\|\bar u\|_{W^{1,p}(\Omega)}$$

Numerical Experiments

Problem (P) cannot be solved directly because of poor multiplier regularity for state constraint.

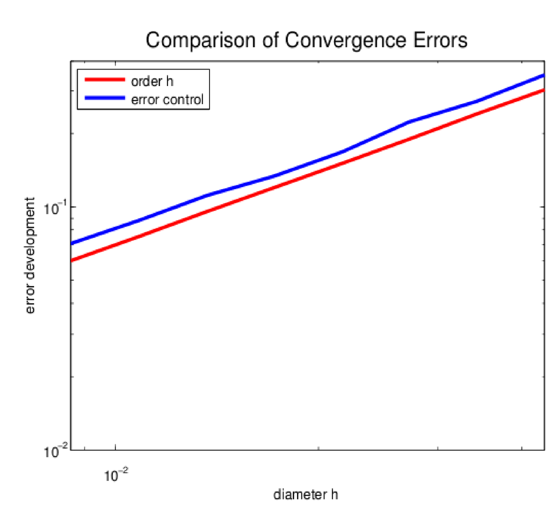
$\Rightarrow$ **Regularisation** needed: i.e. **Virtual Control Approach**. The regularised problem:

$$\left.\begin{aligned} \min_{u,v\in L^2(\Omega)} J(y,u) &:= \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} \\ &\text{s.t.} \\ -\Delta y &= u + v \ \text{ in } \Omega \\ y &= 0 \qquad \text{on } \Gamma \\ &\text{and} \\ a \le u(x) &\le b, \ \text{ a.e. in } \Omega \\ y(x) &\ge y_c(x) - \varepsilon v(x) \ \text{ in } \Omega, \end{aligned}\right\} \qquad (\mathrm{P}_\varepsilon)$$

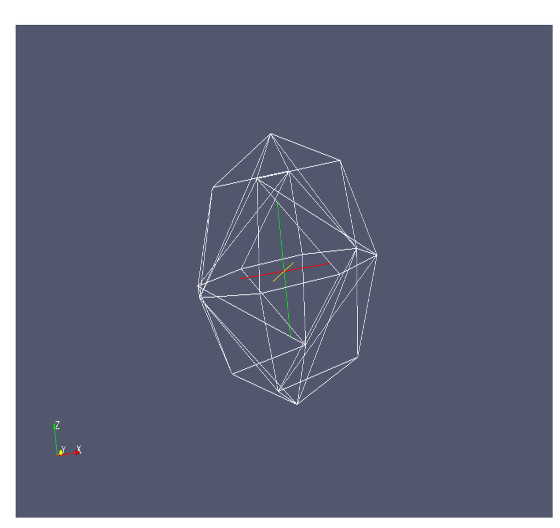
This problem can then be treated efficiently using e.g. the Primal-Dual Active Set Strategy.

The computations were done with the open source FE-package **ALBERTA** (www.alberta-fem.de).

Some results: On the left **rate of convergence** confirming our results, on the right a triangulation of Ω.



(a) L^2-error behaviour for control



(b) A triangulation of Ω

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