Constitutive Parameters for a Nonlinear Cosserat Theory

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- Motivation
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- Constitutive Model
- Numerical Treatment
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- Torsion Test
- Imperfection Algorithm
- Compression Test
- Conclusions and Outlook

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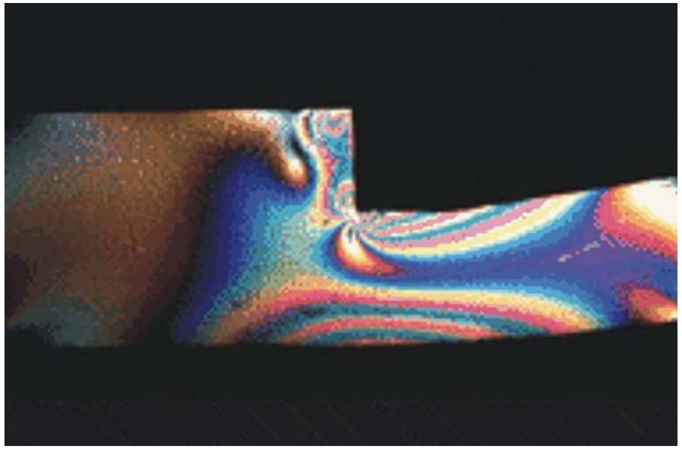
Applications for Cosserat continua:

- stress concentration



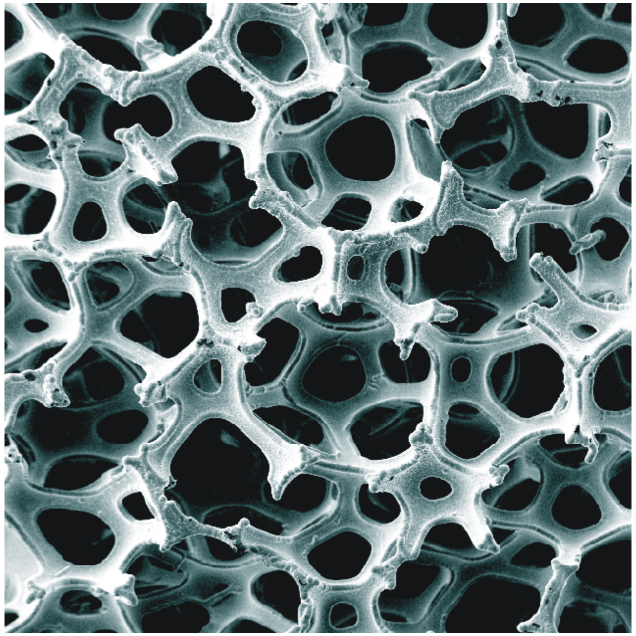
- continua with periodical microstructure

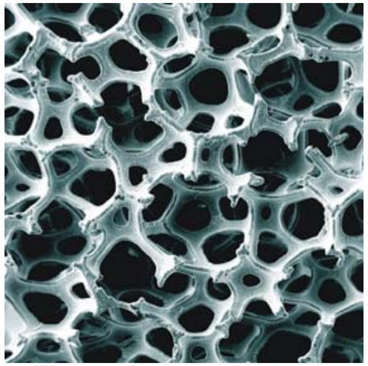
- binary media (suspensions)

- plasticity

- ...

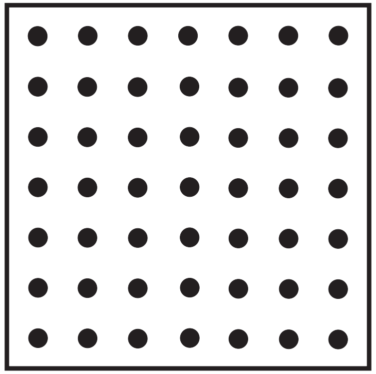
- foams





homogeneous
model without
orientation

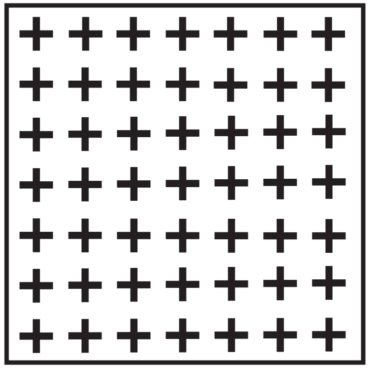
Boltzmann continua



homogeneous
model with
orientation

additional
kinematic

Cosserat continua



→ regularization

→ size effects

→ foam-like behaviour

effects of
(strong) inner
structure

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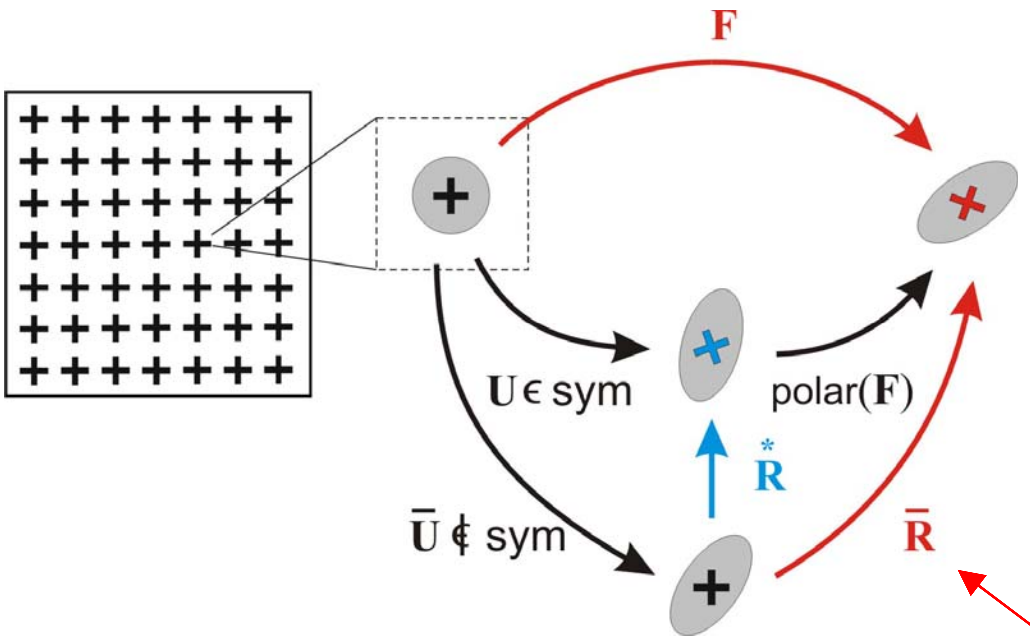
Motivation

Relations for Cosserat theory (e.g. *Ehlers, Bluhm* [1]):



additional
kinematic

Connection between micro- and macrorotation:

$$\bar{\mathbf{R}} = \underbrace{\text{polar}(\mathbf{F})}_{\text{macrorotation}}\ \overset{*}{\mathbf{R}}$$

Nonsymmetric stretch $\bar{\mathbf{U}} = \bar{\mathbf{R}}^T\mathbf{F}$

first Cosserat strain: $\bar{\mathbf{U}} - \mathbf{I} = \bar{\mathbf{R}}^T\mathbf{F} - \mathbf{I}$

second Cosserat strain: $\mathbf{K} = \bar{\mathbf{R}}^T\nabla\bar{\mathbf{R}}$ (curvature)

Kinematic Relations

Relations and linearizations for Cosserat theory (e.g. *Ehlers, Bluhm* [2]):

Euler-Rodrigues formula:
$$\begin{aligned}\bar{\mathbf{R}} &= \exp(\bar{\mathbf{A}}) = \mathbf{I} + \frac{\bar{\mathbf{A}}}{1!} + \frac{\bar{\mathbf{A}}^2}{2!} + \frac{\bar{\mathbf{A}}^3}{3!} + \dots \\ &= \mathbf{e}\otimes\mathbf{e} + (\mathbf{I} - \mathbf{e}\otimes\mathbf{e})\cos\|\gamma\| + (\mathbf{e}\times\mathbf{I})\sin\|\gamma\|\end{aligned}$$

$\mathbf{e}$ ... axis of microrotation vector γ

1) linearization: $\bar{\mathbf{R}}_{\mathsf{lin}} = \mathbf{I} + \bar{\mathbf{A}} = \mathbf{I} - \epsilon\gamma \quad , \quad \gamma = \mathsf{axl}(\bar{\mathbf{A}})$

ϵ ... permutation tensor

first Cosserat strain: $\bar{\mathbf{U}} - \mathbf{I} = \bar{\mathbf{R}}^T\mathbf{F} - \mathbf{I}$

2) linearization: $(\bar{\mathbf{U}} - \mathbf{I})_{\mathsf{lin}} = \nabla\mathbf{u} + \epsilon\gamma = \nabla\mathbf{u} - \bar{\mathbf{A}}$

second Cosserat strain: $\mathbf{K} = \bar{\mathbf{R}}^T\nabla\bar{\mathbf{R}}$ (curvature)

pull back, similar to $\bar{\mathbf{U}} = \mathbf{R}^T\mathbf{F}$

3) linearization: $\mathbf{K}_{\mathsf{lin}} = \nabla\gamma$

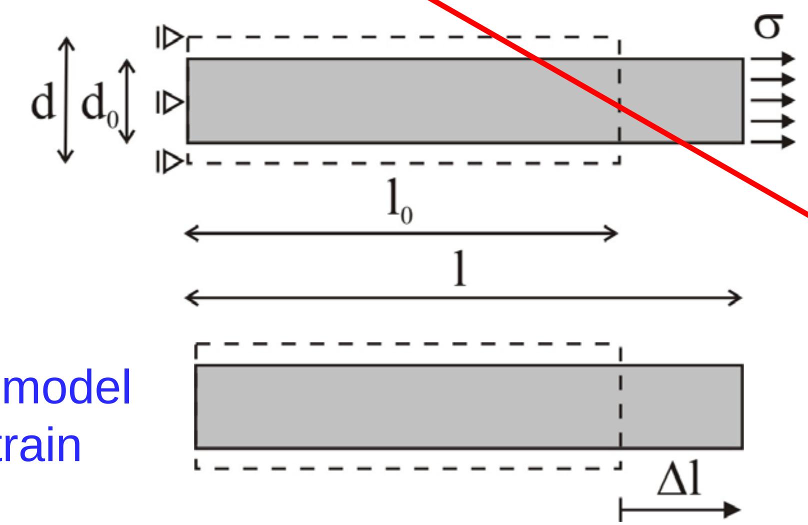
Quadratic ansatz in first Cosserat strain:

$$W_{mp}(\bar{\mathbf{U}}) = \mu \left\| \mathrm{sym}(\bar{\mathbf{U}} - \mathbf{I}) \right\|^2 + \mu_c \left\| \mathrm{skew}(\bar{\mathbf{U}} - \mathbf{I}) \right\|^2 + \frac{\lambda}{2} \left(\mathrm{tr}(\bar{\mathbf{U}} - \mathbf{I}) \right)^2$$

with classical Lamé constants μ and λ

Experiment

Deformation model
uniaxial strain



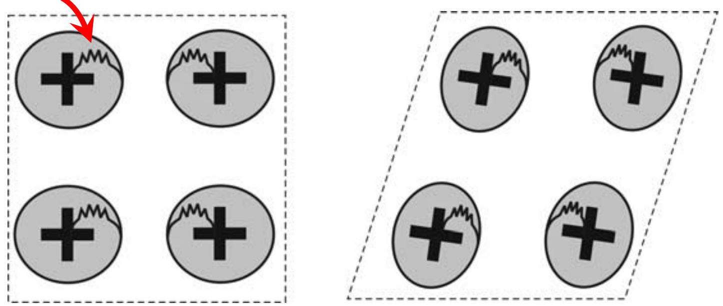
$$\nu = \frac{\left(\frac{d_o - d}{d}\right)}{\left(\frac{l - l_o}{l_o}\right)}$$

$$E = \frac{\sigma_{11}}{\epsilon_{11}} = \frac{\sigma_{11}}{\left(\frac{l - l_o}{l_o}\right)}$$

$$E, \nu \Rightarrow \mu, \lambda, K$$

→ Cosserat couple modulus μ_c penalizes differences of microrotations to macrorotations:

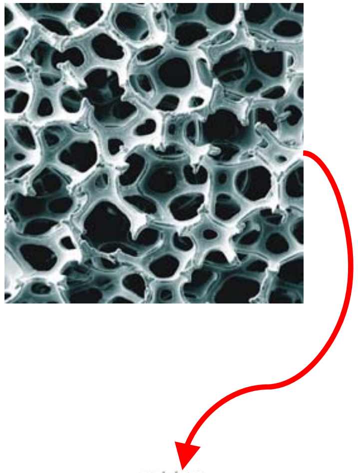
$$\left\| \mathrm{skew}(\bar{\mathbf{U}} - \mathbf{I}) \right\|^2 \to 0 \quad \Rightarrow \quad \bar{\mathbf{R}} \to \mathrm{polar}(\mathbf{F})$$



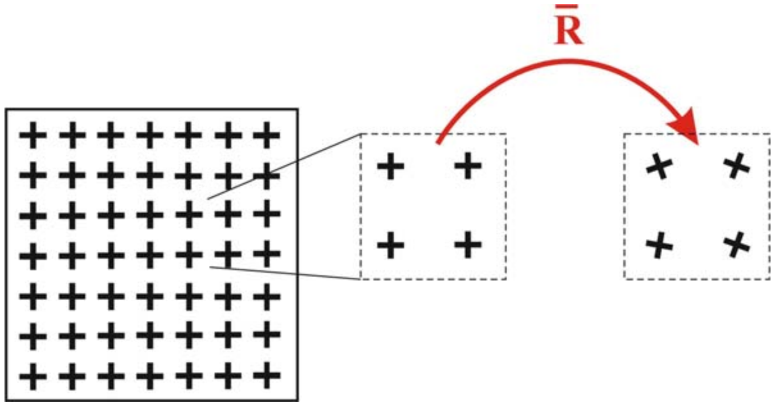
Constitutive Model

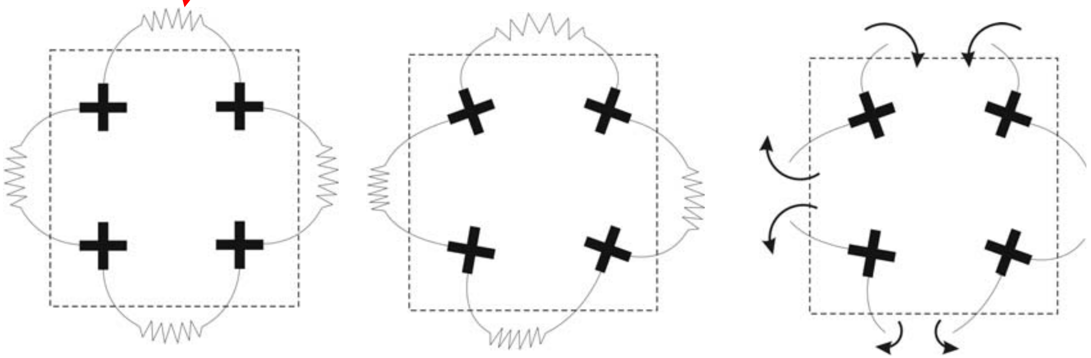
Nonlinear ansatz for curvature energy:

$$W_{curv}(\mathbf{K}) = \frac{\mu}{6}(1 + L_c^2 \|\mathbf{K}\|^2)^3 = \frac{\mu}{6}(1 + L_c^2 \left\|\bar{\mathbf{R}}^T \nabla \bar{\mathbf{R}}\right\|^2)^3$$



phenomenological parameter of inner structure: L_c





$c_M = f(L_c)$

→ acts like torsional spring

→ influences angular momentum

→ L_c penalizes curvature
→ curvature increases for small structures
→ higher stiffness for smaller structures

Constitutive Model

Free Helmholtz energy:

$$W_{\mathsf{mp}}(\bar{\mathbf{U}}) = \mu \left\|\mathsf{sym}(\bar{\mathbf{U}} - \mathbf{I})\right\|^2 + \mu_c \left\|\mathsf{skew}(\bar{\mathbf{U}} - \mathbf{I})\right\|^2 + \frac{\lambda}{2}\left(\mathsf{tr}(\bar{\mathbf{U}} - \mathbf{I})\right)^2$$

$$W_{\mathsf{curv}}(\mathbf{K}) = \frac{\mu}{6}(1 + L_c^2 \,\|\mathbf{K}\|^2)^3 = \frac{\mu}{6}(1 + L_c^2 \left\|\bar{\mathbf{R}}^T \nabla \bar{\mathbf{R}}\right\|^2)^3$$

which turns for the linear Cosserat model into:

$$\begin{aligned} W_{\mathsf{mp}}^{\mathsf{lin}}(\bar{\mathbf{U}}_{\mathsf{lin}}) &= \mu \left\|\mathsf{sym}(\nabla \mathbf{u} - \bar{\mathbf{A}})\right\|^2 + \mu_c \left\|\mathsf{skew}(\nabla \mathbf{u} - \bar{\mathbf{A}})\right\|^2 + \frac{\lambda}{2}\left(\mathsf{tr}(\nabla \mathbf{u} - \bar{\mathbf{A}})\right)^2 \\ &= \mu \,\|\mathsf{sym}(\nabla \mathbf{u})\|^2 + \underbrace{\mu_c \left\|\mathsf{skew}(\nabla \mathbf{u} - \bar{\mathbf{A}})\right\|^2}_{\text{coupling of displacement } \mathbf{u} \text{ and microrotation } \gamma = \mathsf{axl}(\bar{\mathbf{A}})} + \frac{\lambda}{2}(\mathsf{tr}(\nabla \mathbf{u}))^2 \end{aligned}$$

$$W_{\mathsf{curv}}^{\mathsf{lin}}(\mathbf{K}_{\mathsf{lin}}) = \frac{1}{2}\mu\, L_c^2 \,\|\mathbf{K}_{\mathsf{lin}}\|^2 = \frac{1}{2}\mu\, L_c^2 \,\|\nabla \gamma\|^2$$

→ linear isotropic theory decouples for $\mu_c = 0$

- variational formulation and nonlinear 3-d finite element model
- Lagrangean description
- consistent linearization for stiffness matrix and Newtons strategy
- 8 / 27 node brick elements with trilinear / triquadratic shape functions for both fields
- system of algebraic equations:

$$< \delta\mathbf{p}^I, \mathbf{S}(\bar{\mathbf{p}}^I), \Delta\mathbf{p}^I > = - < \delta\mathbf{p}^I, \mathbf{P}(\bar{\mathbf{p}}^I) >$$

$\mathbf{S}\ldots$ stiffness matrix ; $\mathbf{P}\ldots$ residual vector

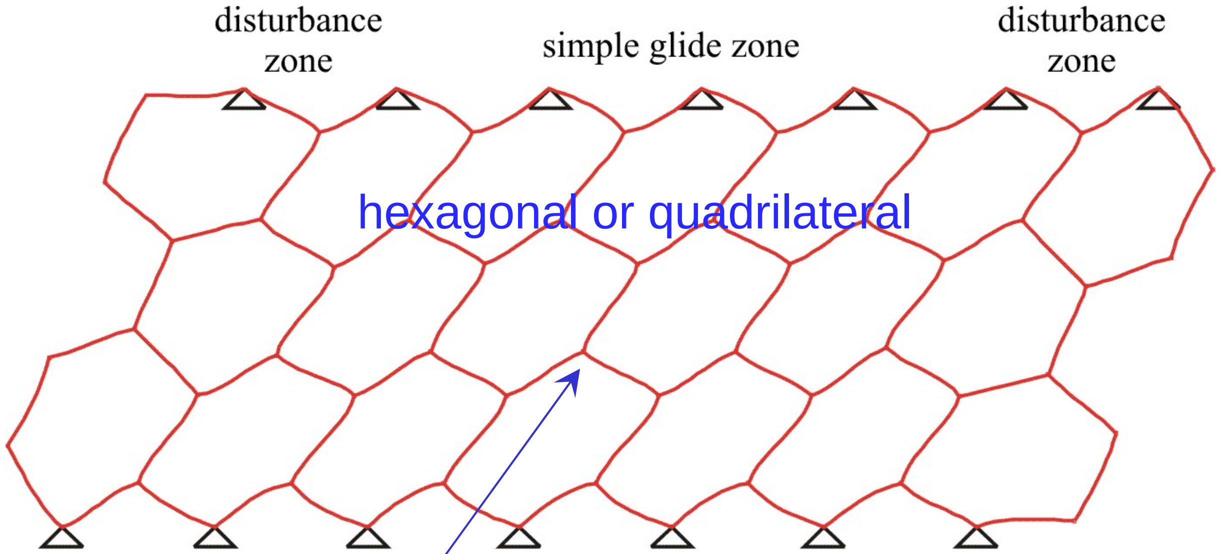
$\bar{\mathbf{p}}^I \ldots$ discrete nodal point vector of displacement and microrotation

$\Delta\mathbf{p}^I \ldots$ discrete increments of displacement and microrotation

- additive update of displacements and infinitesimal microrotations for linear theory
- multiplicative update of microrotations for nonlinear theory (*Sansour,Wagner* [2])

Numerical Treatment

Motivation: from planar shear to simple glide

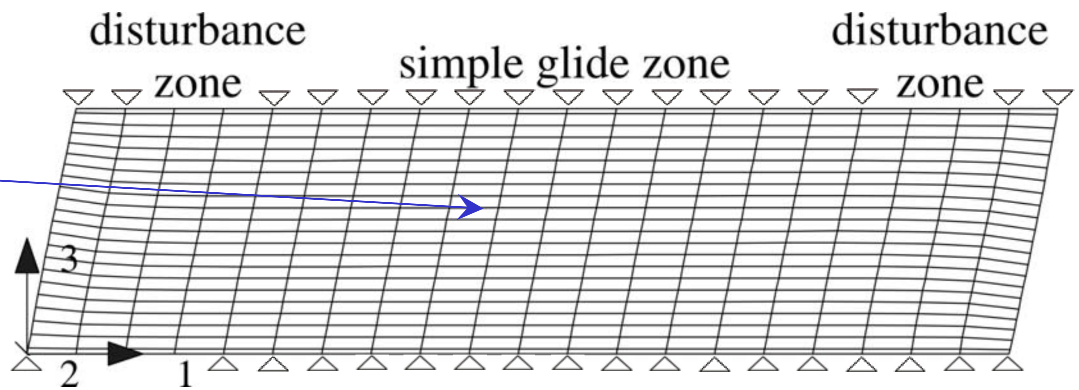


hexagonal or quadrilateral

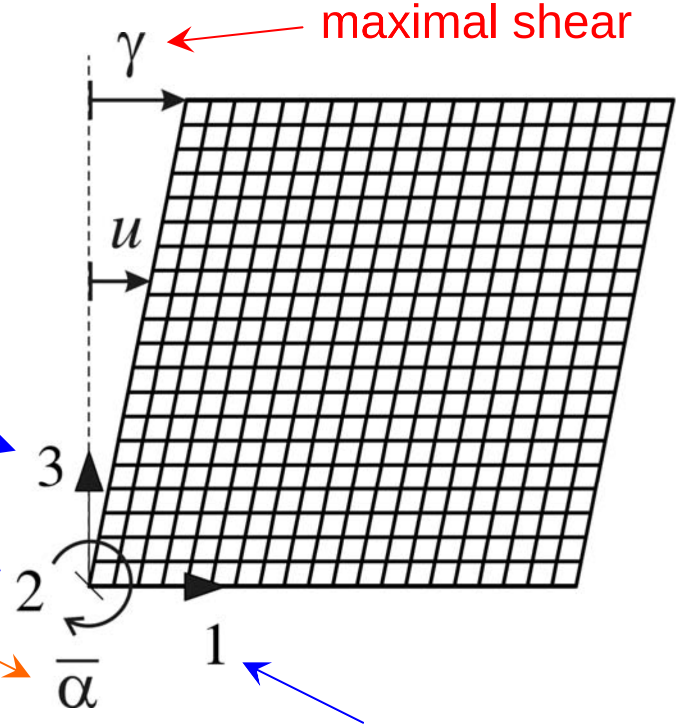
planar: no displacements in 2-direction

different zones of deformation indicate simple glide zone

no displacements in 3-direction
+
no gradients in 1-direction



Analytical investigations
in simple glide:



maximal shear

no displacements in
2- and 3-direction

microrotations $\overline{\alpha}$ (around
2-axis) linked in 1-2-plane

displacements in 1-direction
linked in 1-2-plane → no
gradients in 1-direction

$$\nabla\varphi(x_3) = F = \begin{pmatrix} 1 & 0 & u'(x_3) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overline{R}(x_3) = \begin{pmatrix} \cos\overline{\alpha}(x_3) & 0 & \sin\overline{\alpha}(x_3) \\ 0 & 1 & 0 \\ -\sin\overline{\alpha}(x_3) & 0 & \cos\overline{\alpha}(x_3) \end{pmatrix}$$

Simple glide

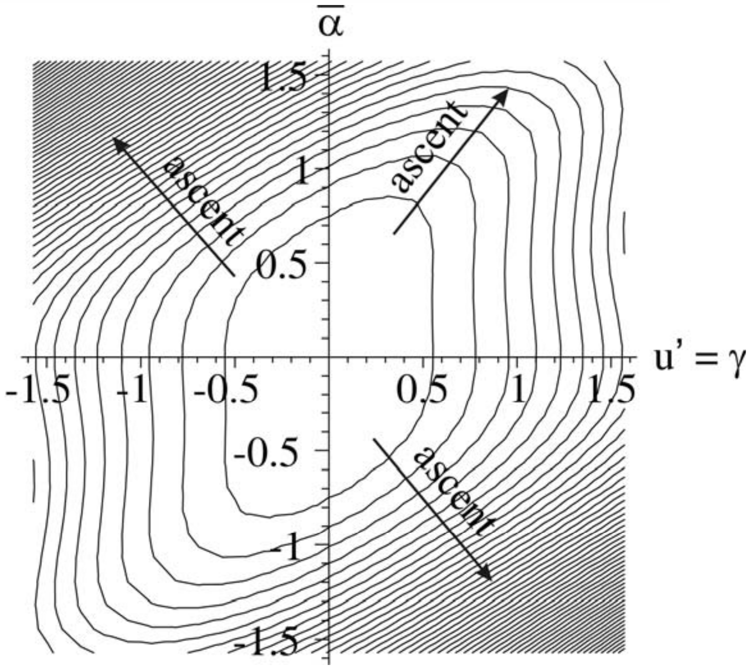
Internal energy expression:

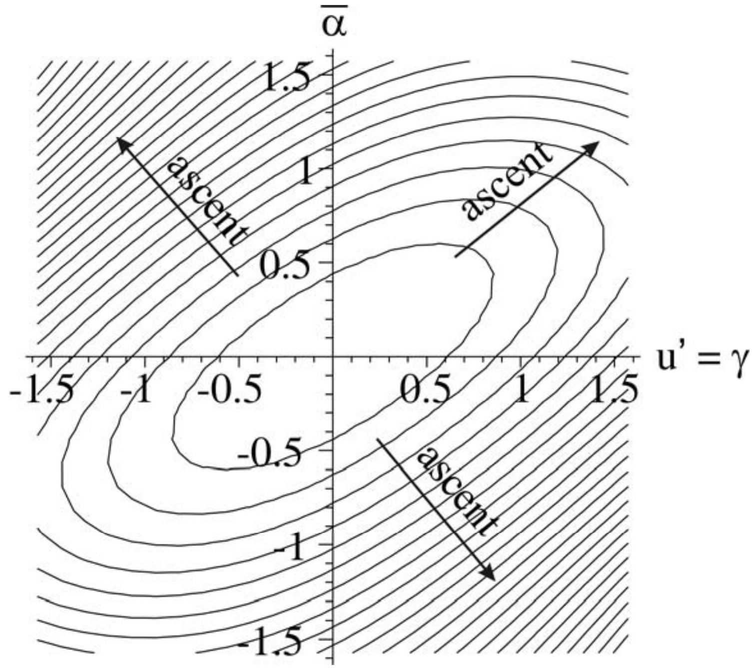
$$W(u',\overline{\alpha},\overline{\alpha}') = \mu\left(2(\cos\overline{\alpha}-1)^2+\frac{u'^2}{2}+\frac{\sin^2\overline{\alpha}\,u'^2}{2}+2(\cos\overline{\alpha}-1)\sin\overline{\alpha}\,u'\right) + \frac{\mu_c}{2}\cos^2\overline{\alpha}\left[2\tan\overline{\alpha}-u'\right]^2+2\mu\,L_c^2|\overline{\alpha}'|^2$$

consider temporary $L_c = 0$, consistent with homogeneous response

$\mu_c = 0 \rightarrow$ non-convex!

$\mu_c = \mu \rightarrow$ convex!

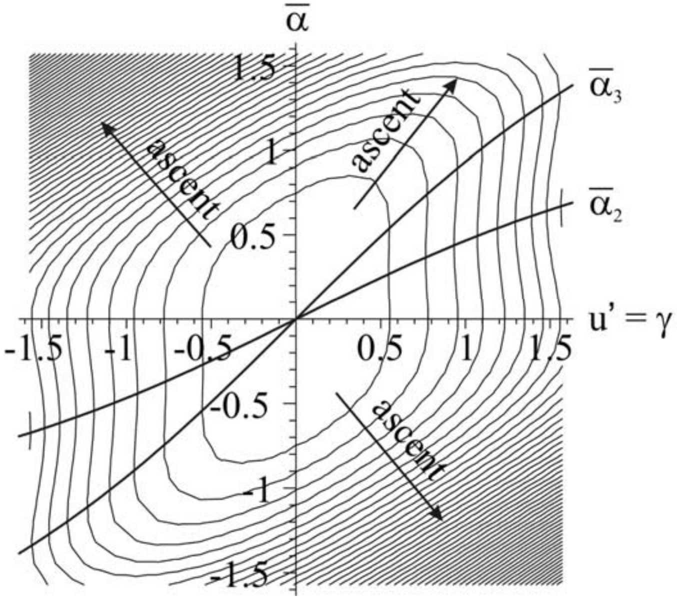




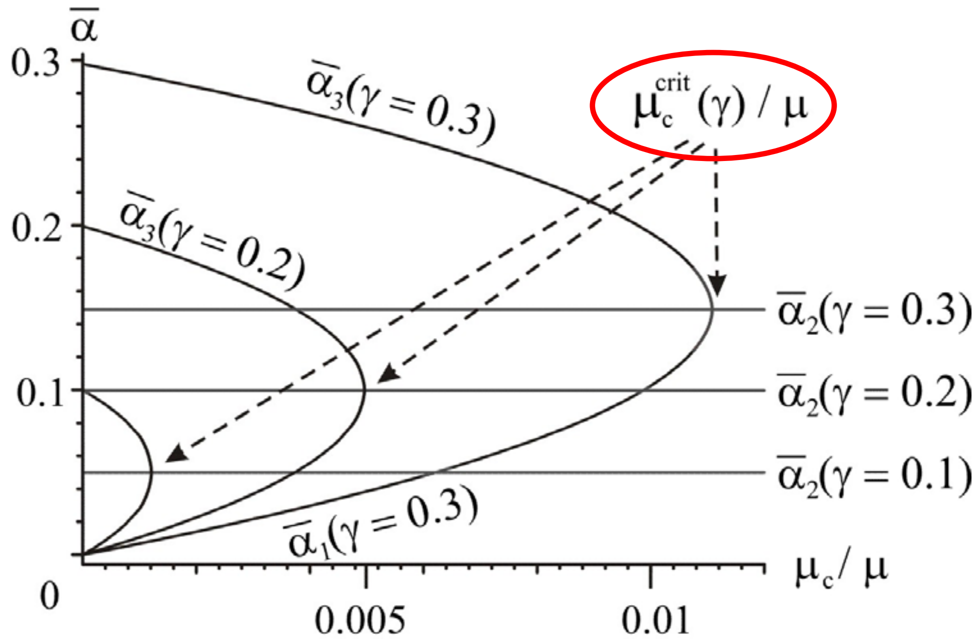
Simple glide

Three possible overall homogeneous solutions for $\mu_c = 0$:

$$u' = \gamma = \text{const.}\,, \quad \overline{\alpha}_1 = 0\,, \quad \overline{\alpha}_2 = \arctan\frac{\gamma}{2}\,, \quad \overline{\alpha}_3 = 2\arctan\frac{\gamma}{2}$$

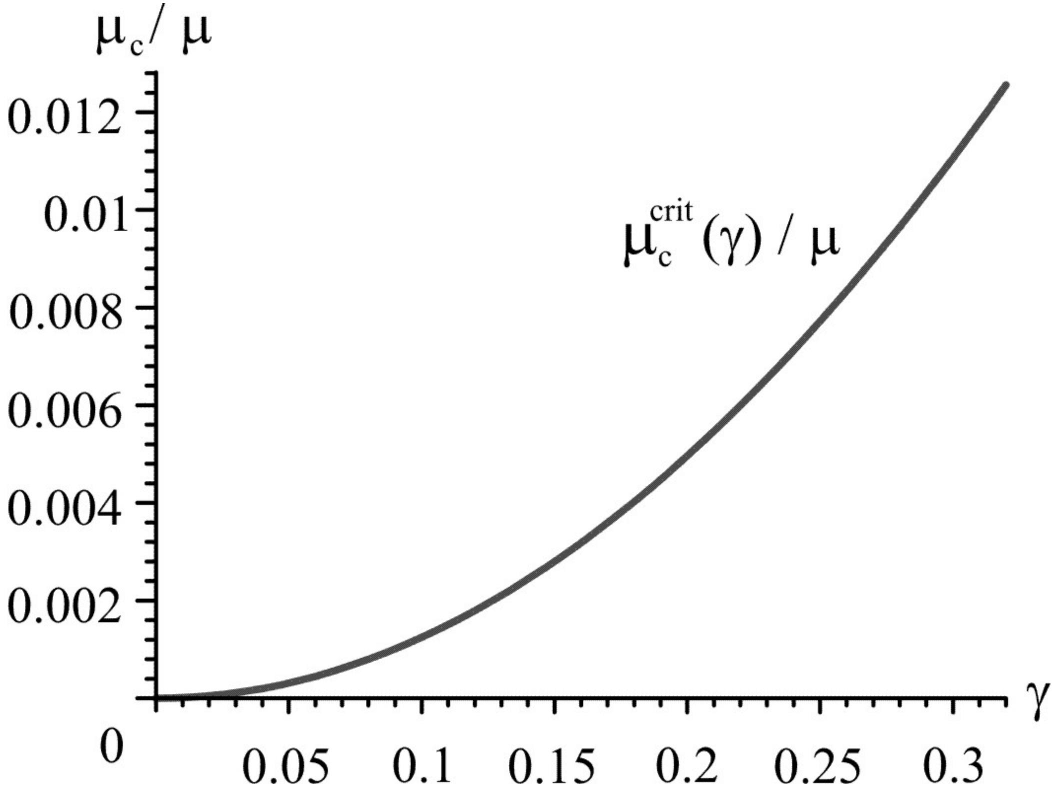


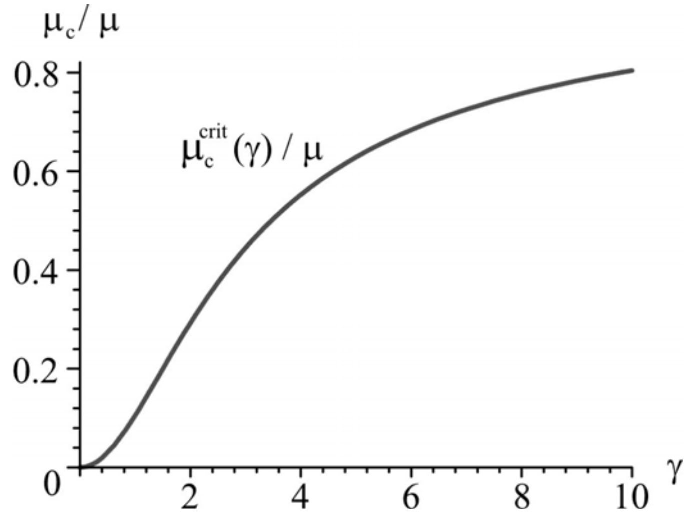
For $\mu_c > 0$ solutions depend on γ :

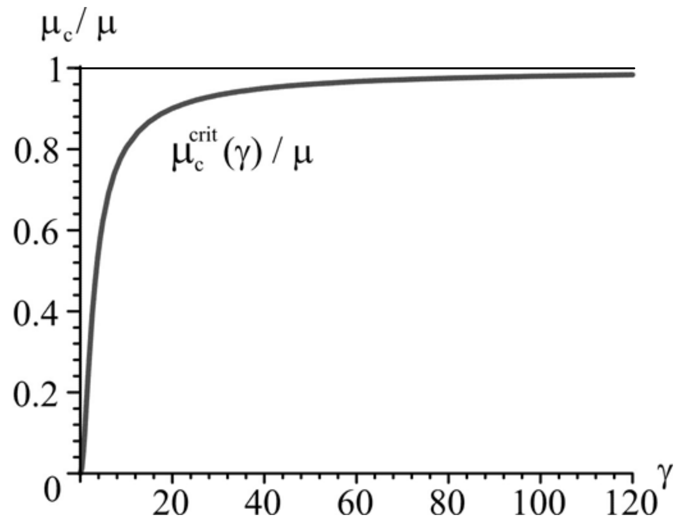


The critical value for μ_c depends on applied displacement γ

$$\mu_c^{\text{crit}}(\gamma) = \mu\,\frac{\sqrt{4+\gamma^2}-2}{\sqrt{4+\gamma^2}}$$







Numerical investigations:

height=width=length=1

$\mu = 1 \cdot 10^5$

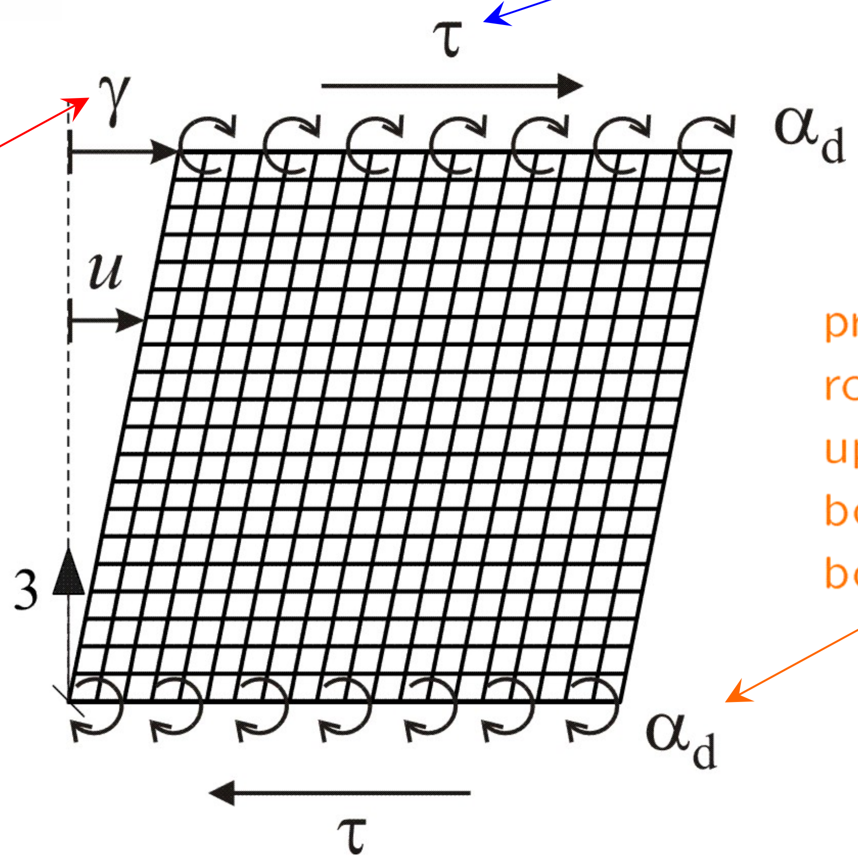
$\lambda = 1.5 \cdot 10^5$

$L_c = 0.05$

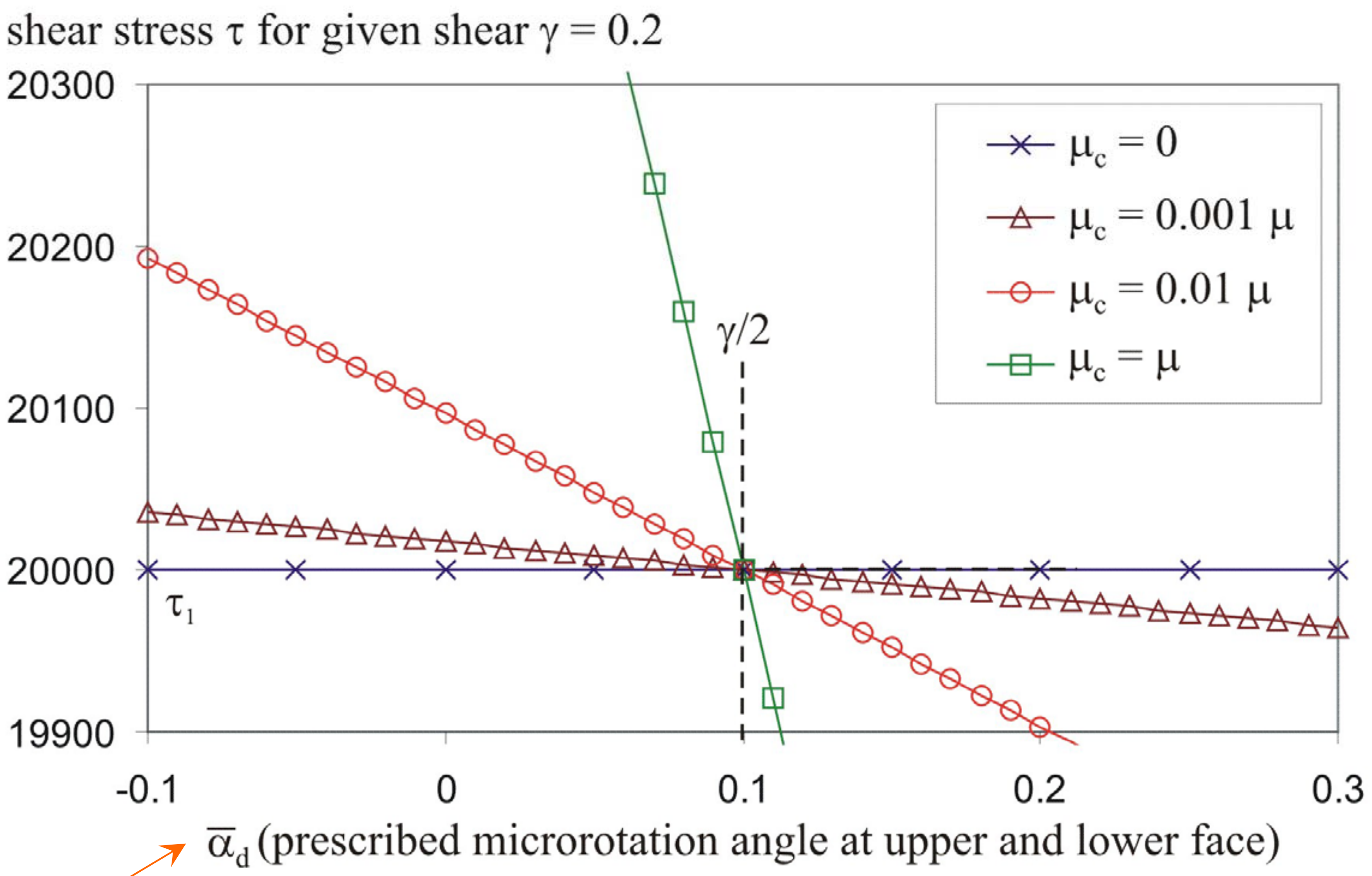
measure 1. P.K. shear stress = reaction force

always $\gamma = 0.2$

prescribe micro-rotation $\overline{\alpha}_{\mathrm{d}}$ on upper and lower border as Dirichlet boundary condition

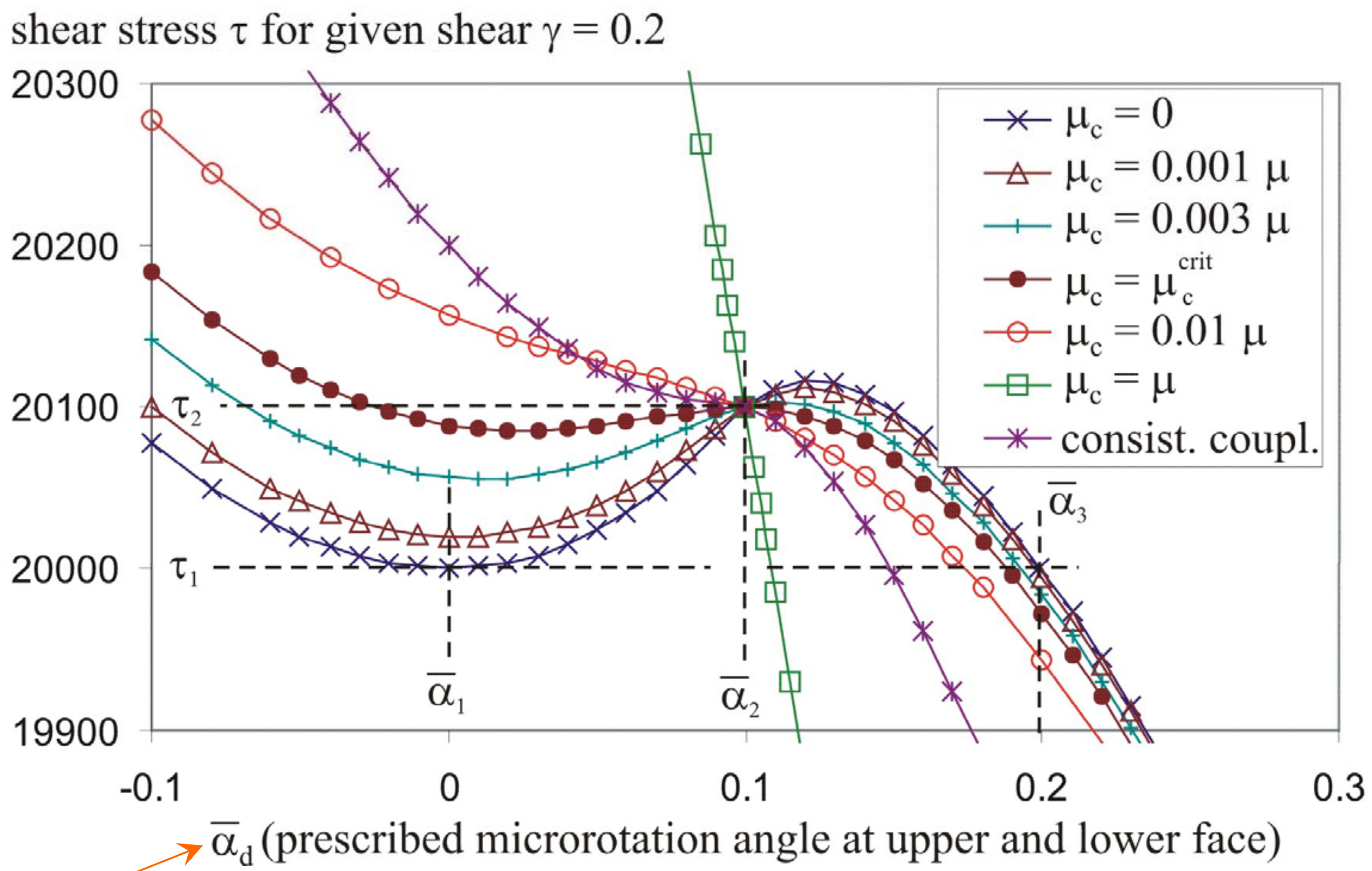


Numerical results for linear Cosserat theory and $L_c = 0.05$



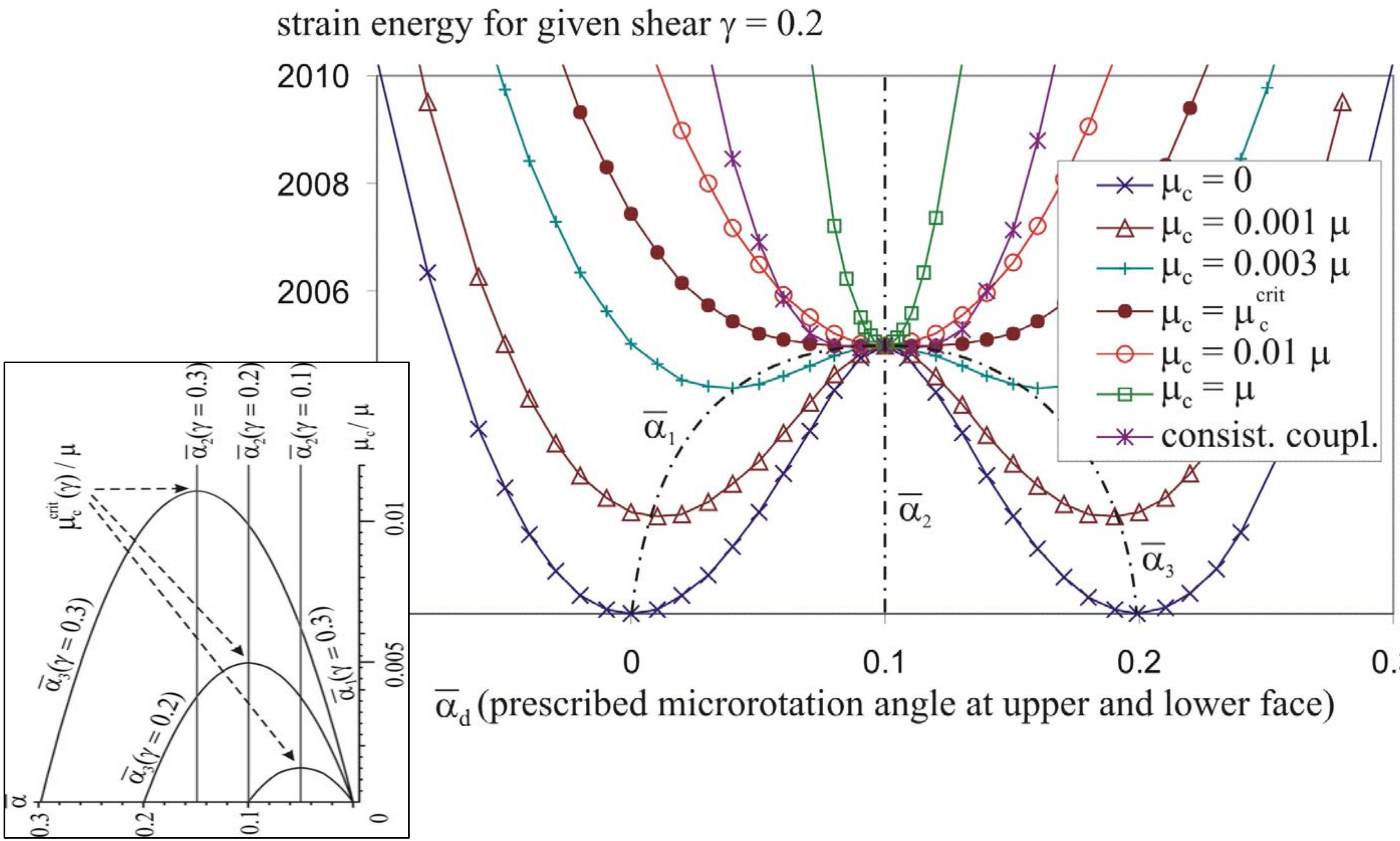
focus on Dirichlet boundary condition for microrotation $\overline{\alpha}$

Numerical results for nonlinear Cosserat theory and $L_c = 0.05$

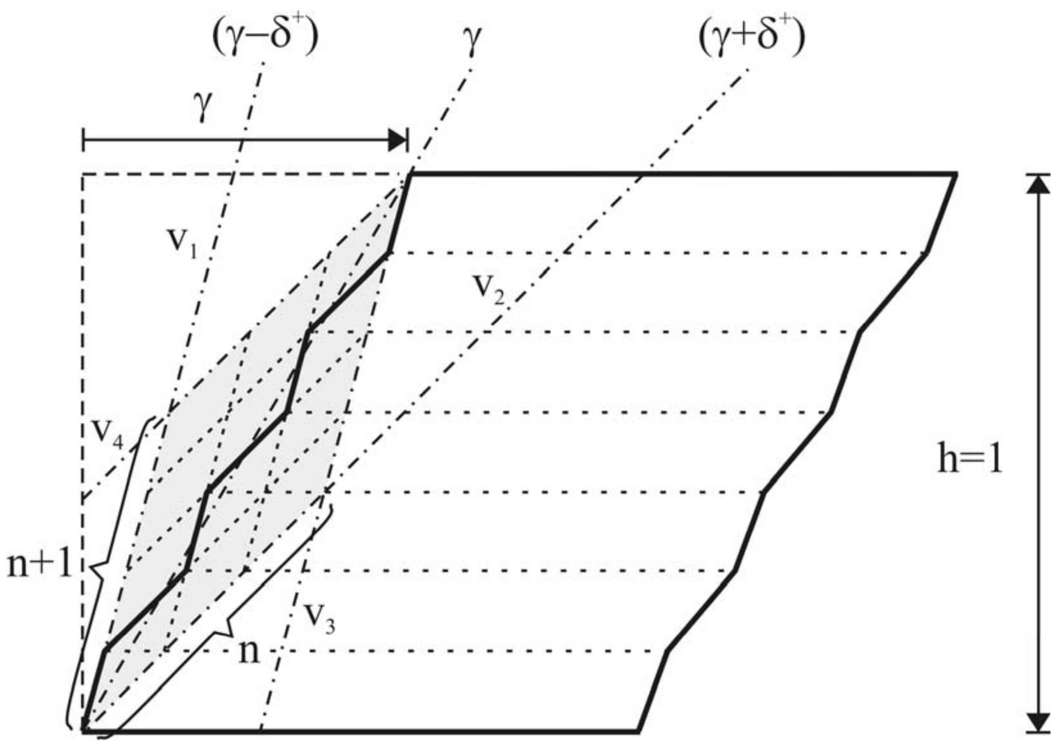


focus on Dirichlet boundary condition for microrotation $\overline{\alpha}$

Numerical results for nonlinear Cosserat theory and $L_c = 0.05$



Lamination microstructure solution for $L_c = 0$ and $\mu_c = 0$



Beats energetically the homogeneous response, construction possible due to non-convexity of $W(u', \overline{\alpha})$

- For $\mu_c < \mu_c^{\text{crit}}$ nonlinear Cosserat theory delivers three possible homogeneous solutions $\rightarrow$ too much freedom $\rightarrow$ need for boundary condition: Dirichlet or consistent coupling

- With consistent coupling boundary condition (micro=macrorotation on Dirichlet boundaries) one homogeneous solution appears $\rightarrow$ clearness

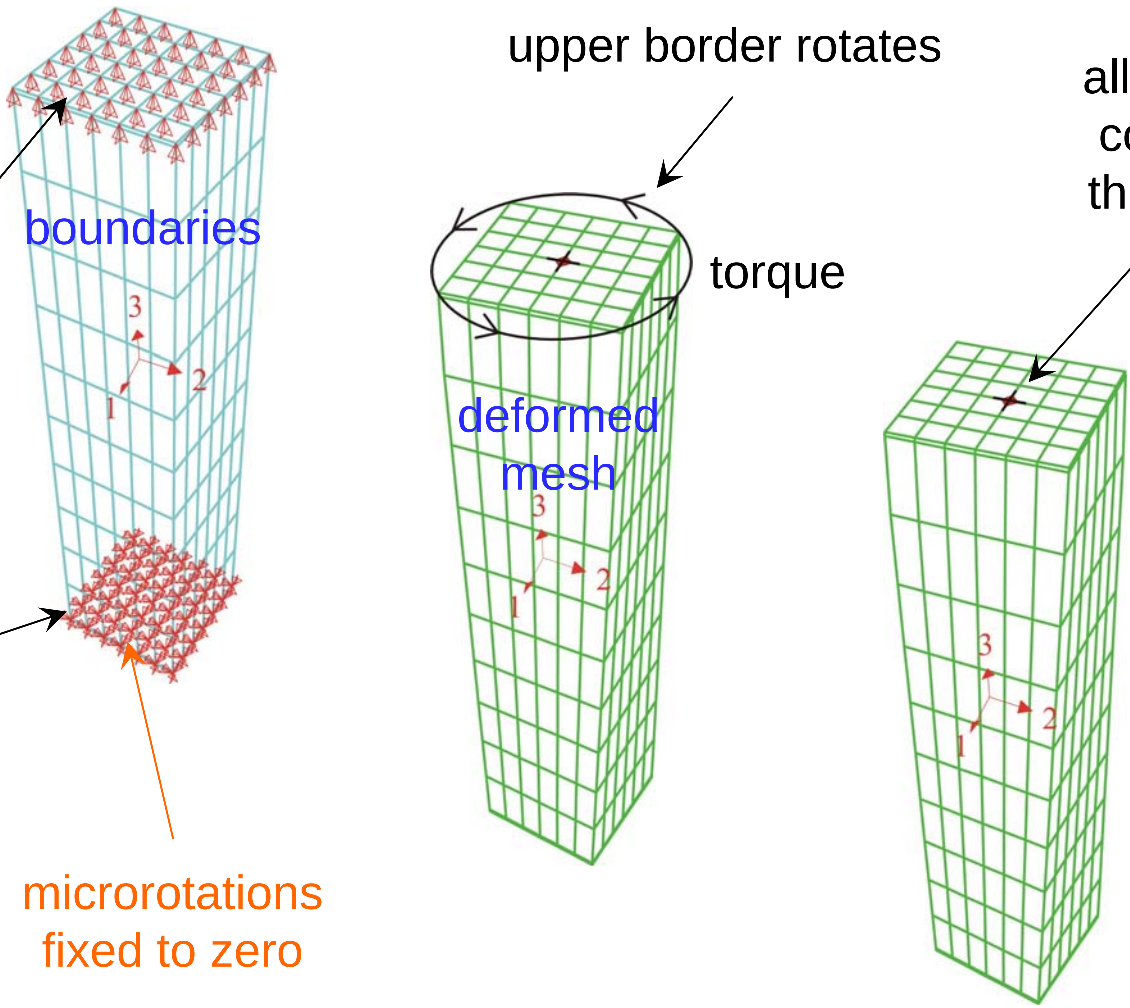
- Boundary conditions on microrotations effect the stiffness behaviour. The greater μ_c, the greater the effect. This interferes with L_c in describing length scale effects.

- $\mu_c = 0$ and $L_c = 0$: lamination microstructure solutions appear

Constant sample values:

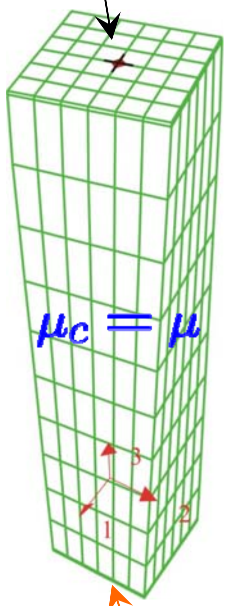
$$height = 2$$
$$width = 0.4$$
$$E = 1 \cdot 10^6$$
$$\nu = 0.3$$

From now on: Nonlinear Cosserat theory



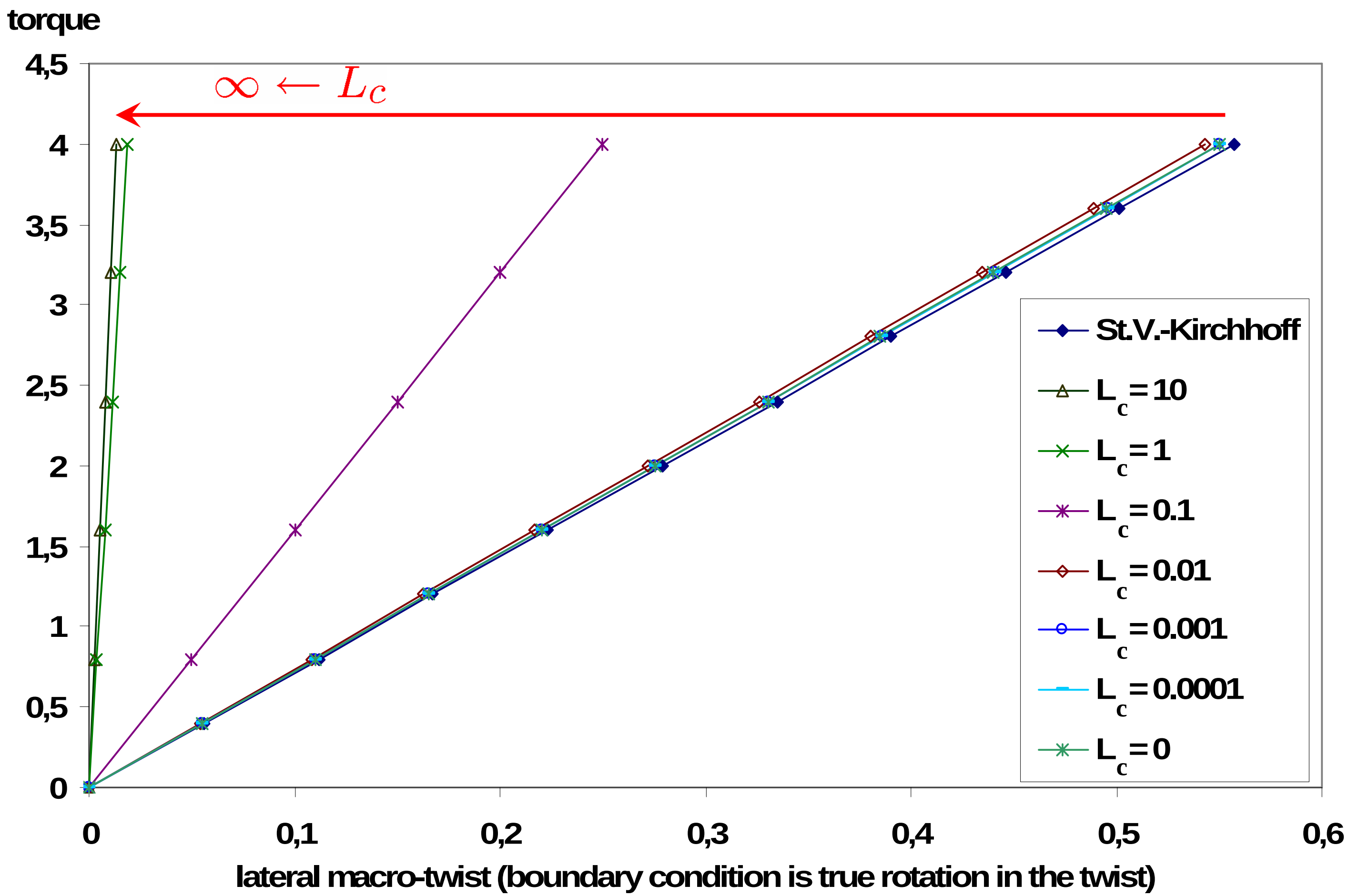
Torsion test

results concern
this point



microrotations
fixed to zero

Macrorotation for various internal length scale L_c



→ solution shows unbounded stiffness for $\mu_c = \mu$ and $L_c \to \infty$

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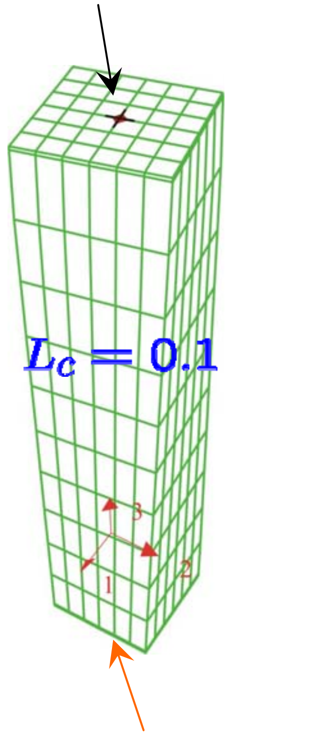
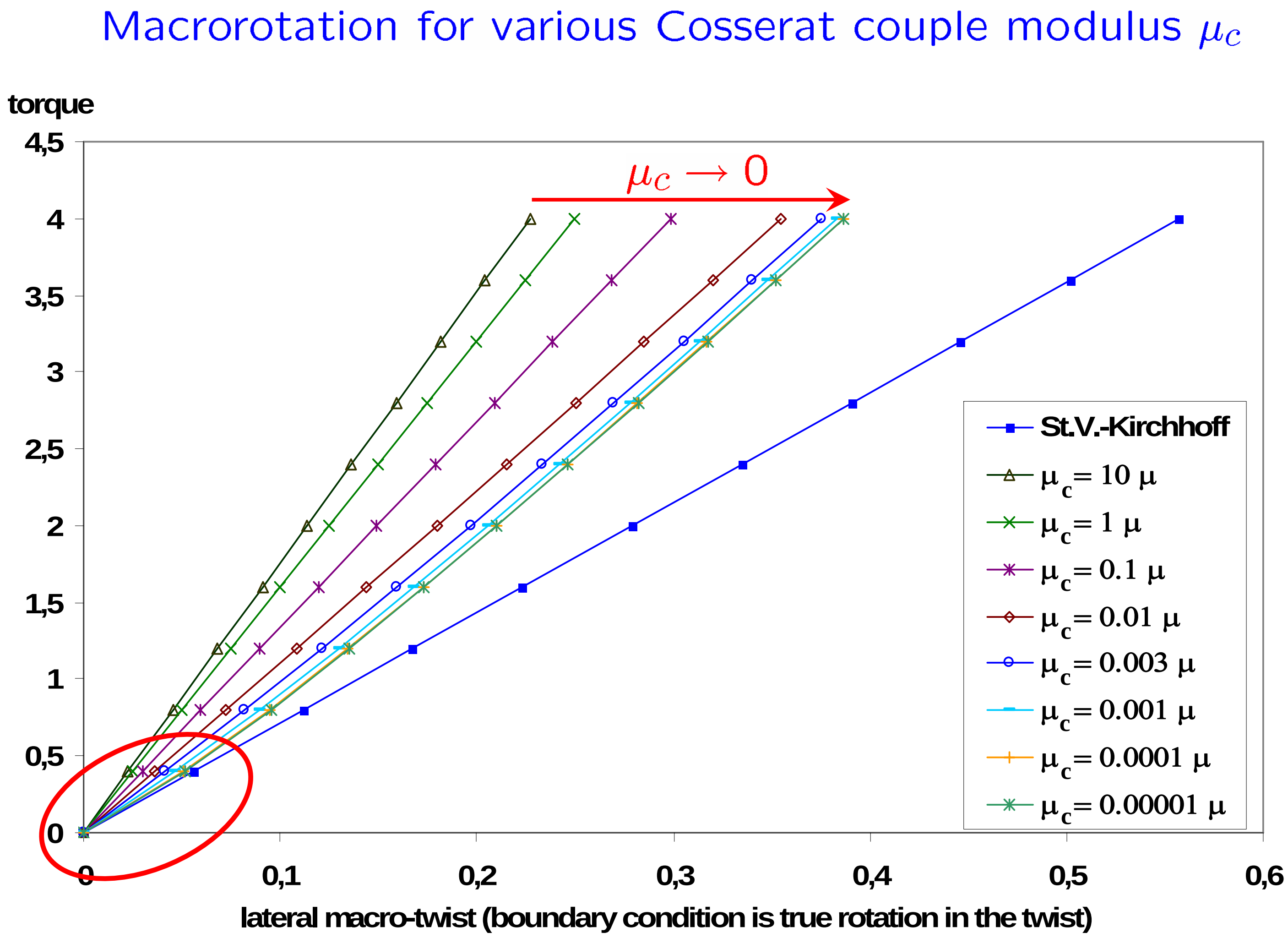
Torsion test

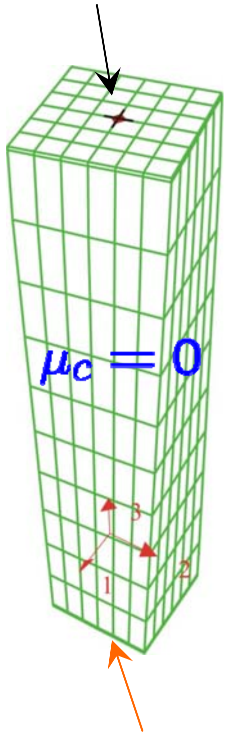




→ nonlinear Cosserat theory with $\mu_c \to 0$ starts like classical elasticity

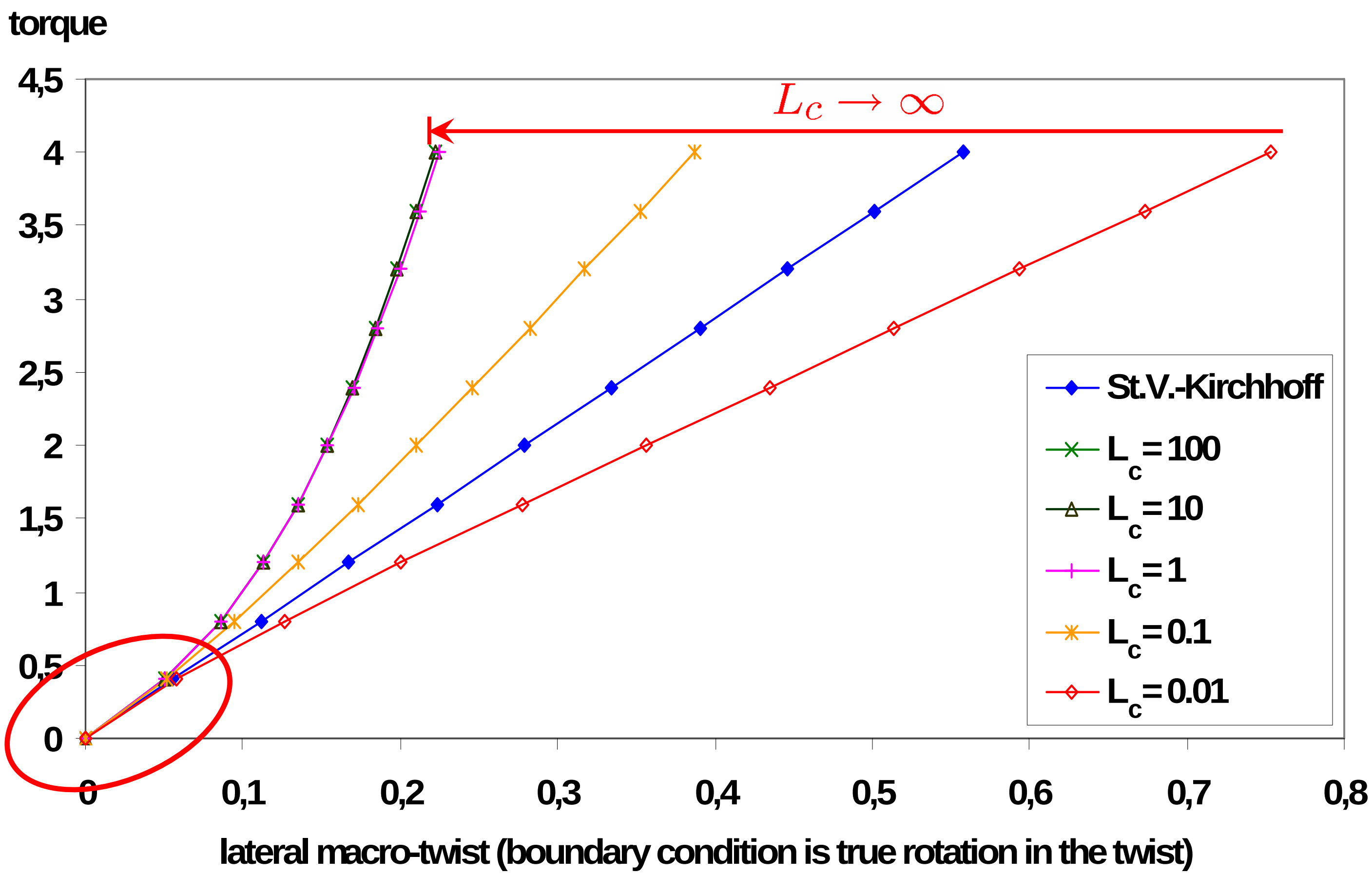
Torsion test

results concern
this point



microrotations
fixed to zero

Macrorotation for various internal length scale L_c



→ nonlinear Cosserat theory with $\mu_c = 0$ starts deformation similar to classical elasticity for any L_c

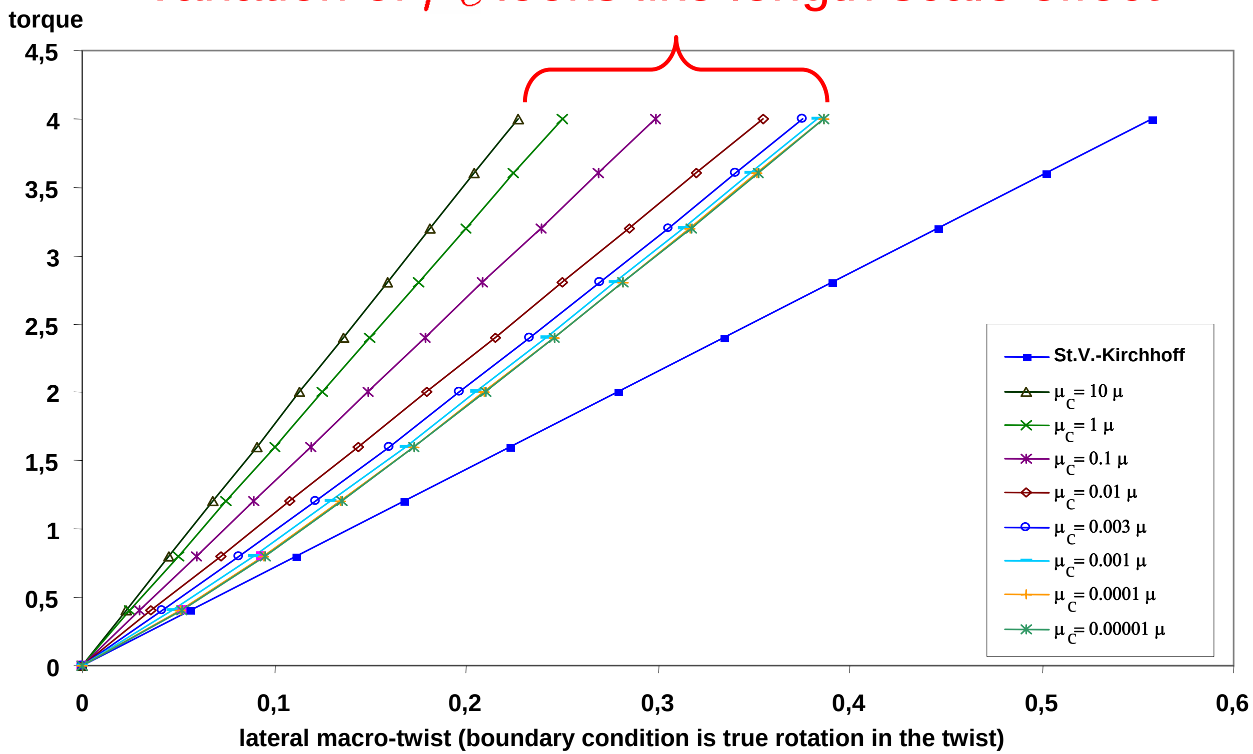
→ solution shows bounded stiffness for $\mu_c = 0$ and $L_c \to \infty$

Torsion test

→ For $L_c > 0$ nonlinear Cosserat theory starts deformation similar to classical theory only if $\mu_c = 0$

→ curvature energy should be responsible for length scale effects!

variation of μ_c looks like length scale effect



→ $\mu_c > 0$ interferes with L_c in describing length scale effects

→ results in general difficulty to determine $\mu_c > 0$ independent of L_c

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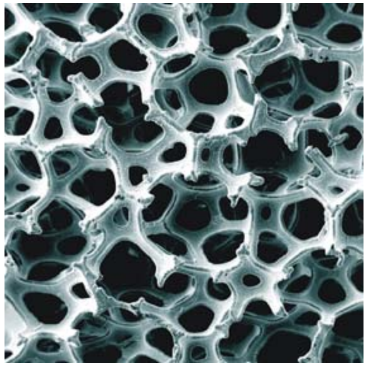
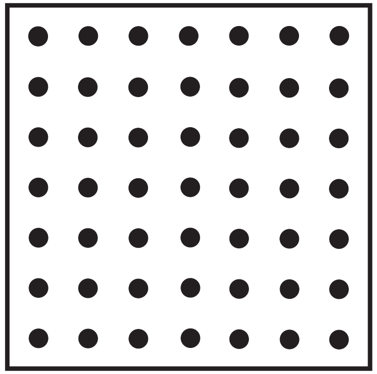
Conclusions in Torsion Test



homogeneous model without orientation

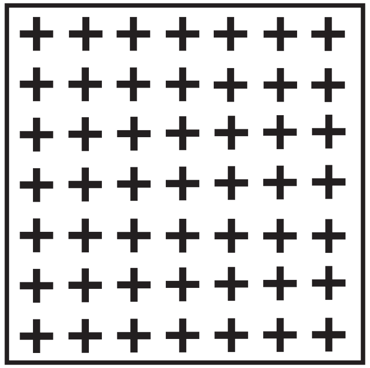
Boltzmann continua



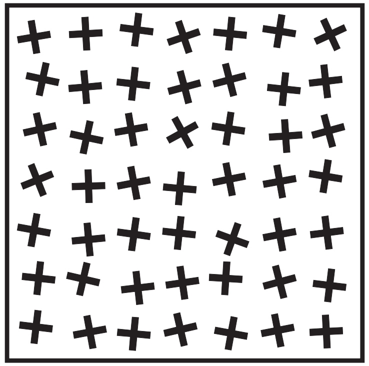
homogeneous model with orientation

For perfect samples and perfect boundary conditions the full spectrum of possible solutions can often not be reached numerically → disturb perfect situations

Cosserat continua



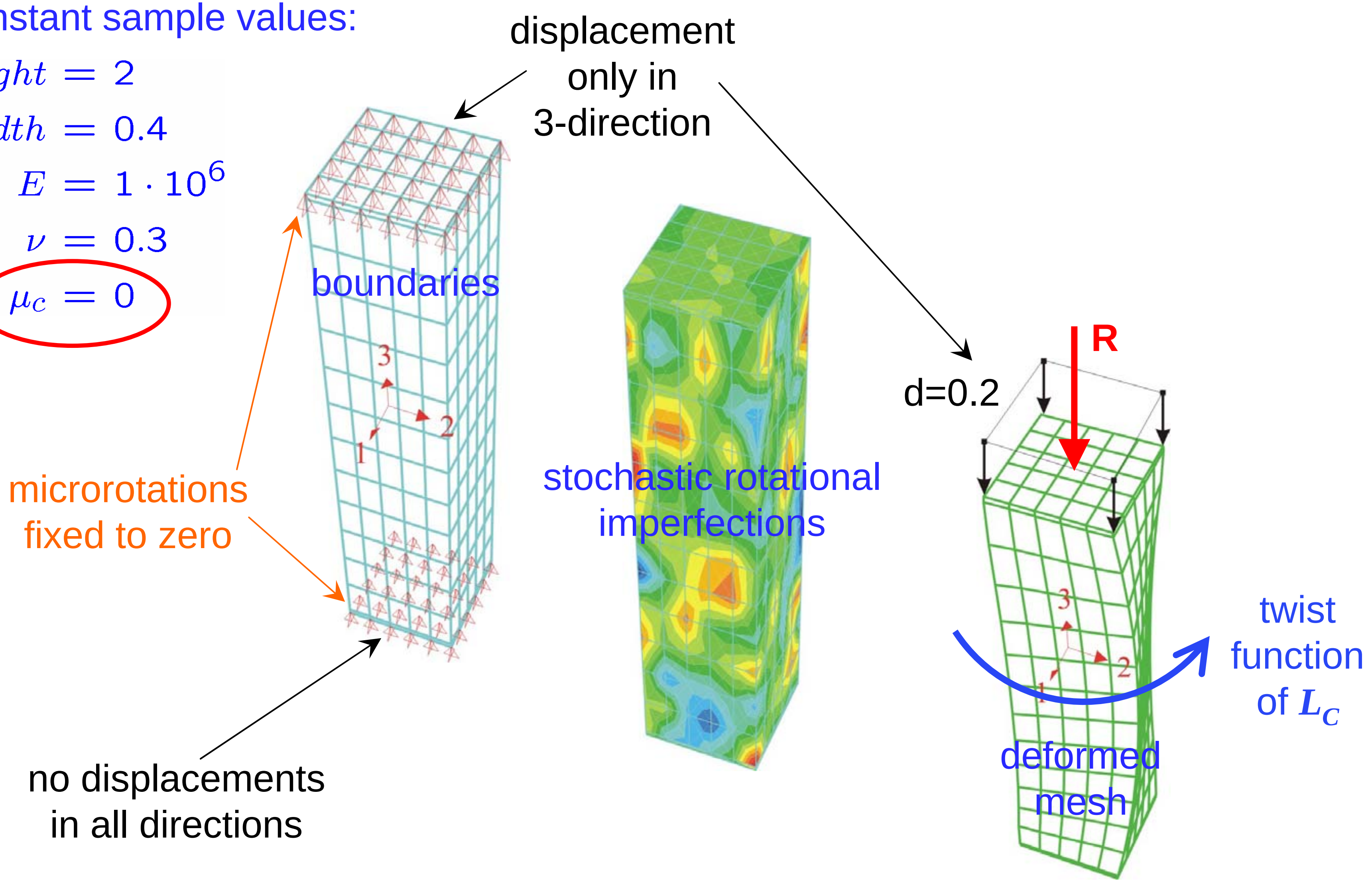
stochastic rotational imperfections



Imperfection Algorithm

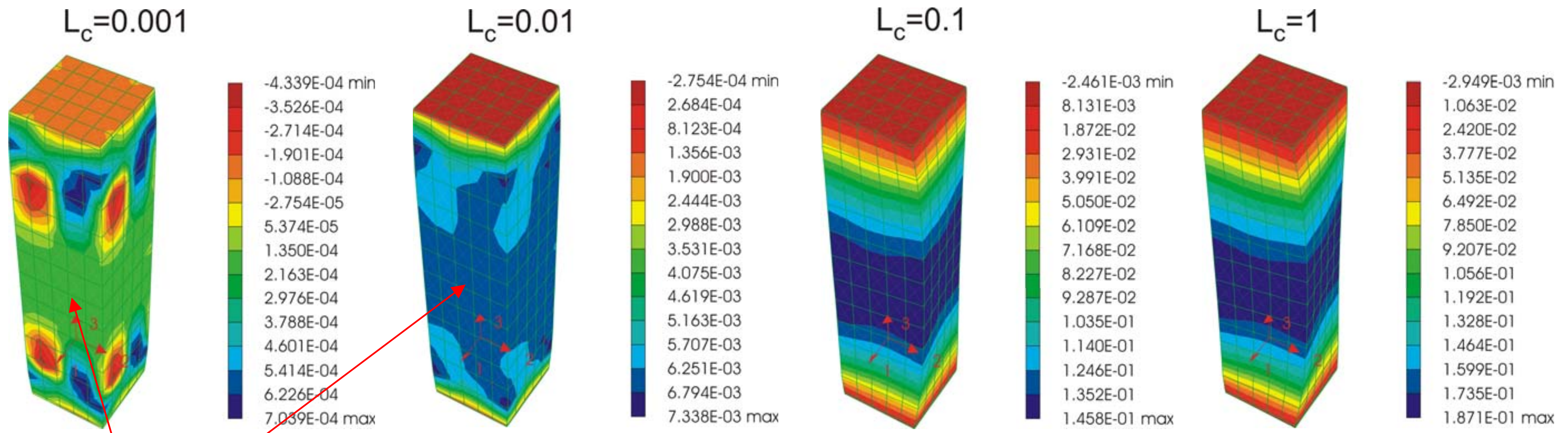
Constant sample values:

$$height = 2$$
$$width = 0.4$$
$$E = 1 \cdot 10^6$$
$$\nu = 0.3$$
$$\mu_c = 0$$

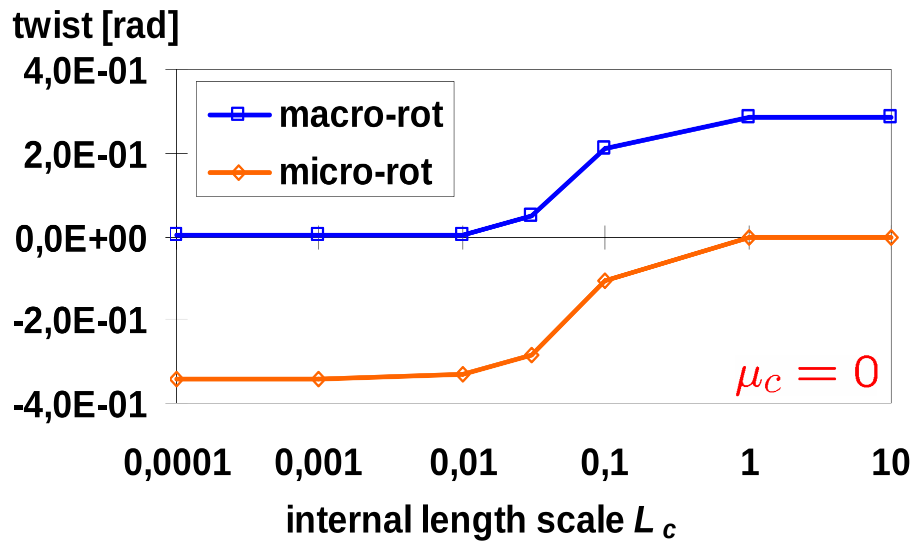


Compression test

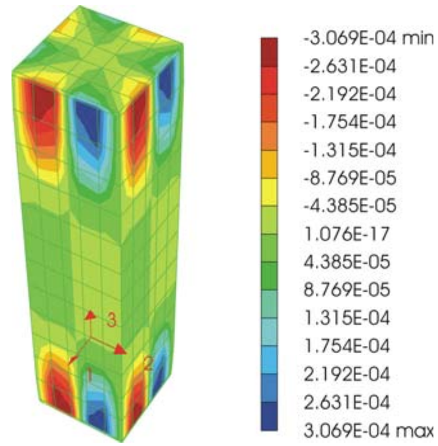
Macrorotation arround vertical axis for various internal length scale factors L_c



Rotations measured at half height



Boltzmann continua



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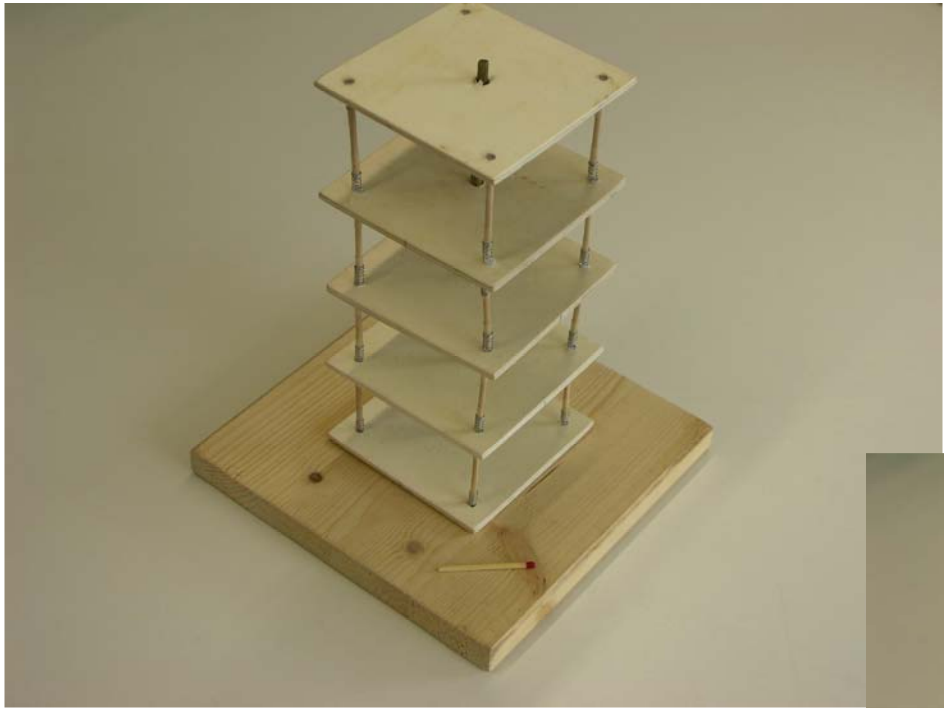
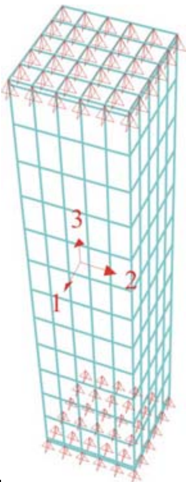
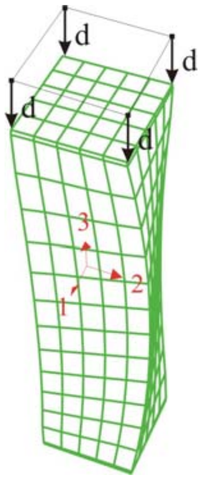
Compression test

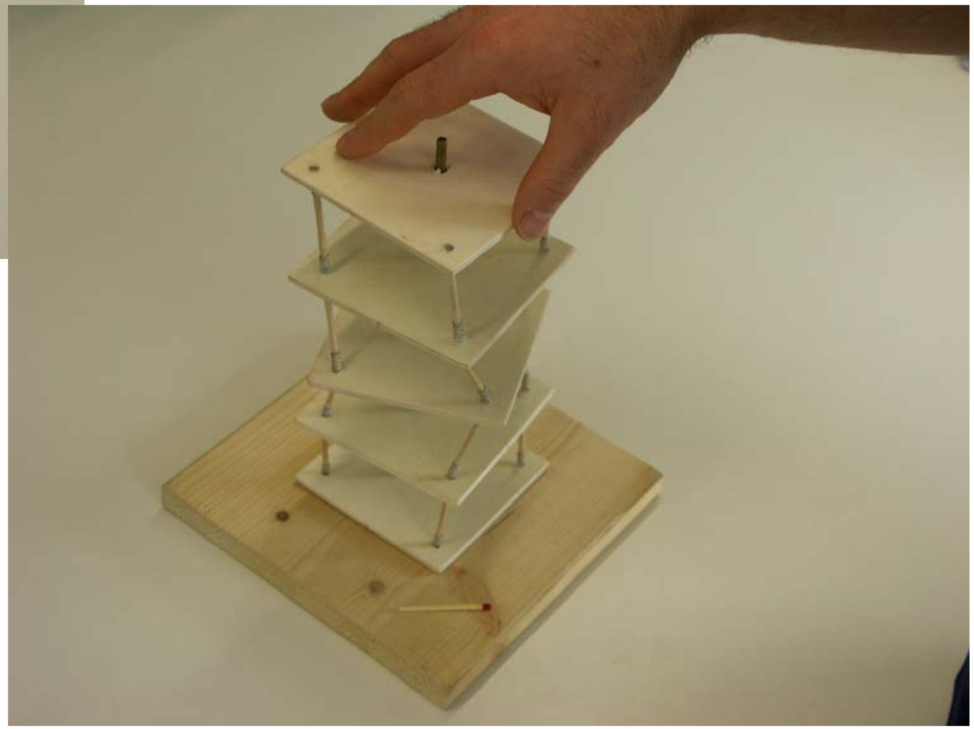


Simple model to motivate shortening through twist



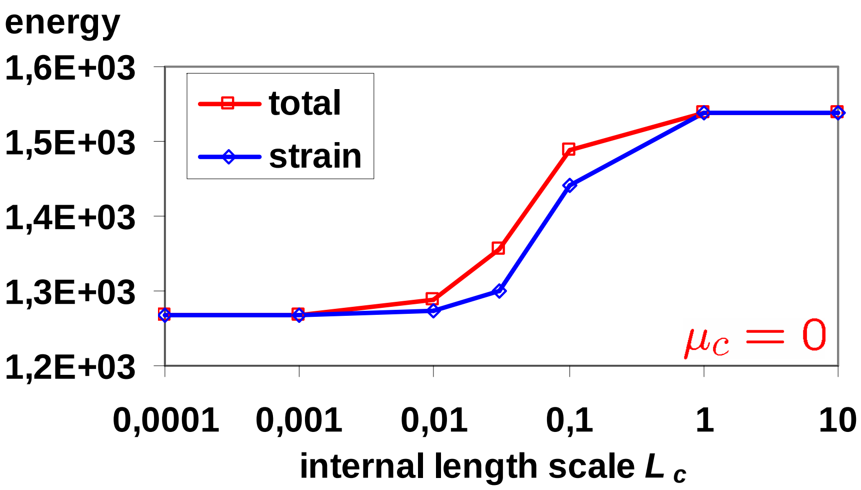
only with
$\mu_c = 0,$
$L_c > 0.01$
$\longrightarrow$
and
stochastic
imperfections

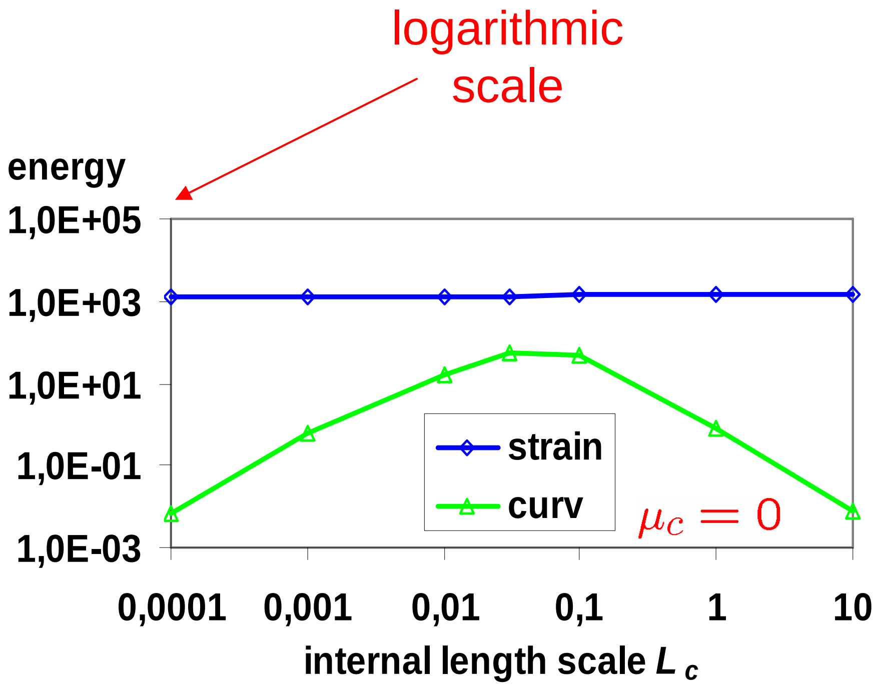




Compression test

Inner energies for various internal length scale factors L_c



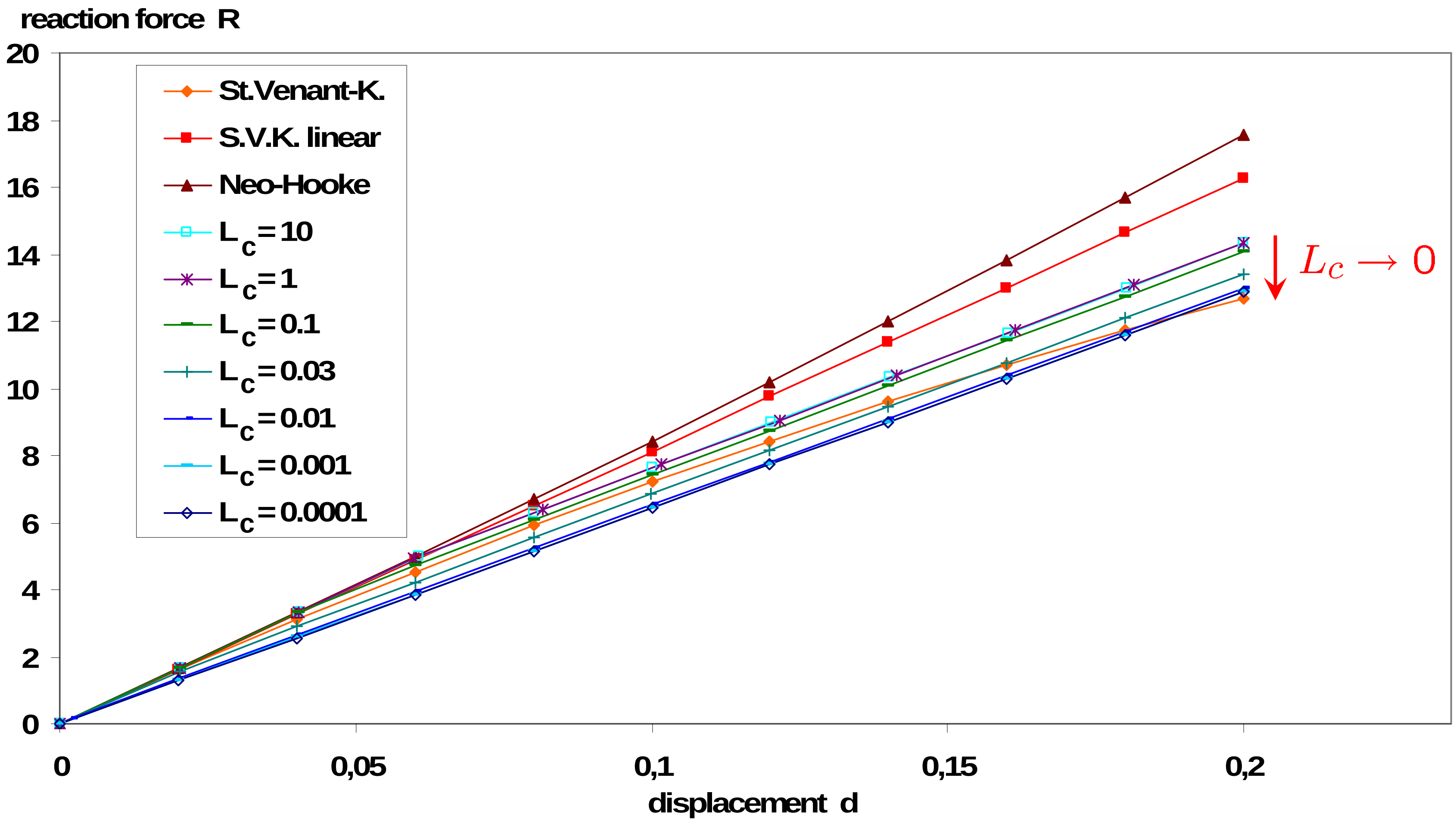


- total energy increases for increasing internal length scale factor – but not arbitrary

- pronounced length scale effects for $L_c \in [0.001; 1]$

- curvature energy only of second order (maximal 4% of strain energy)

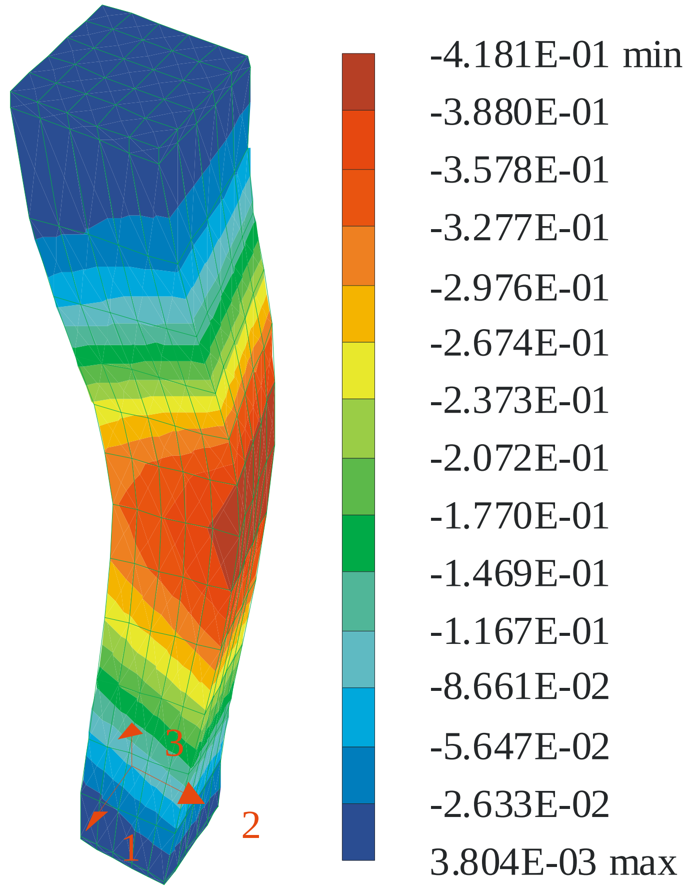
Load-displacement curves for $\mu_c = 0$ and various L_c



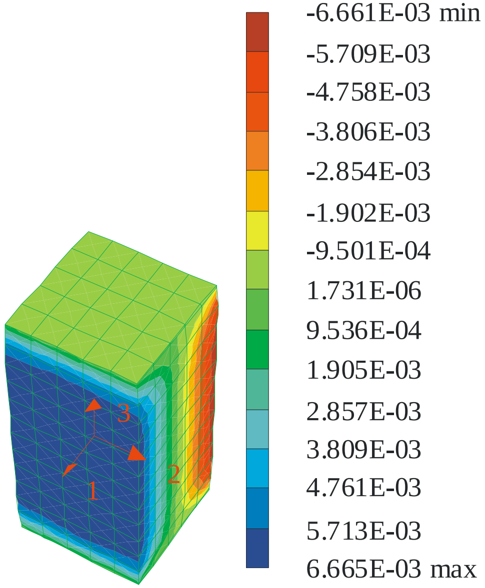
→ Size effects not as significant as in torsion test → as expected !!!

Compression test with various slenderness ratios (different aspect ratio), constant internal length scale factor and constant maximal strain d / h = 10 %

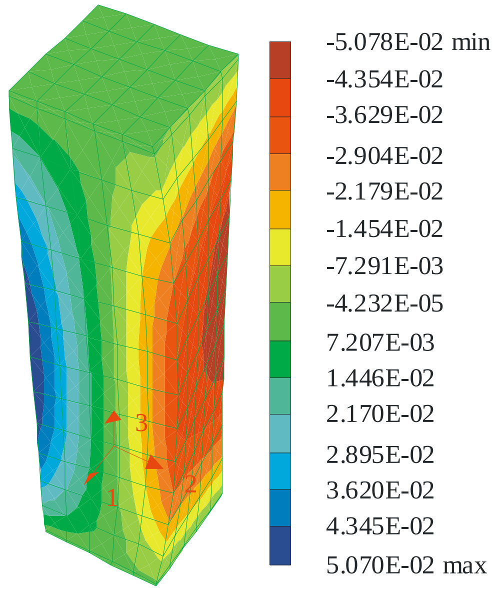
h=4: deformation with twist and buckling



h=1.0: symmetric deformation



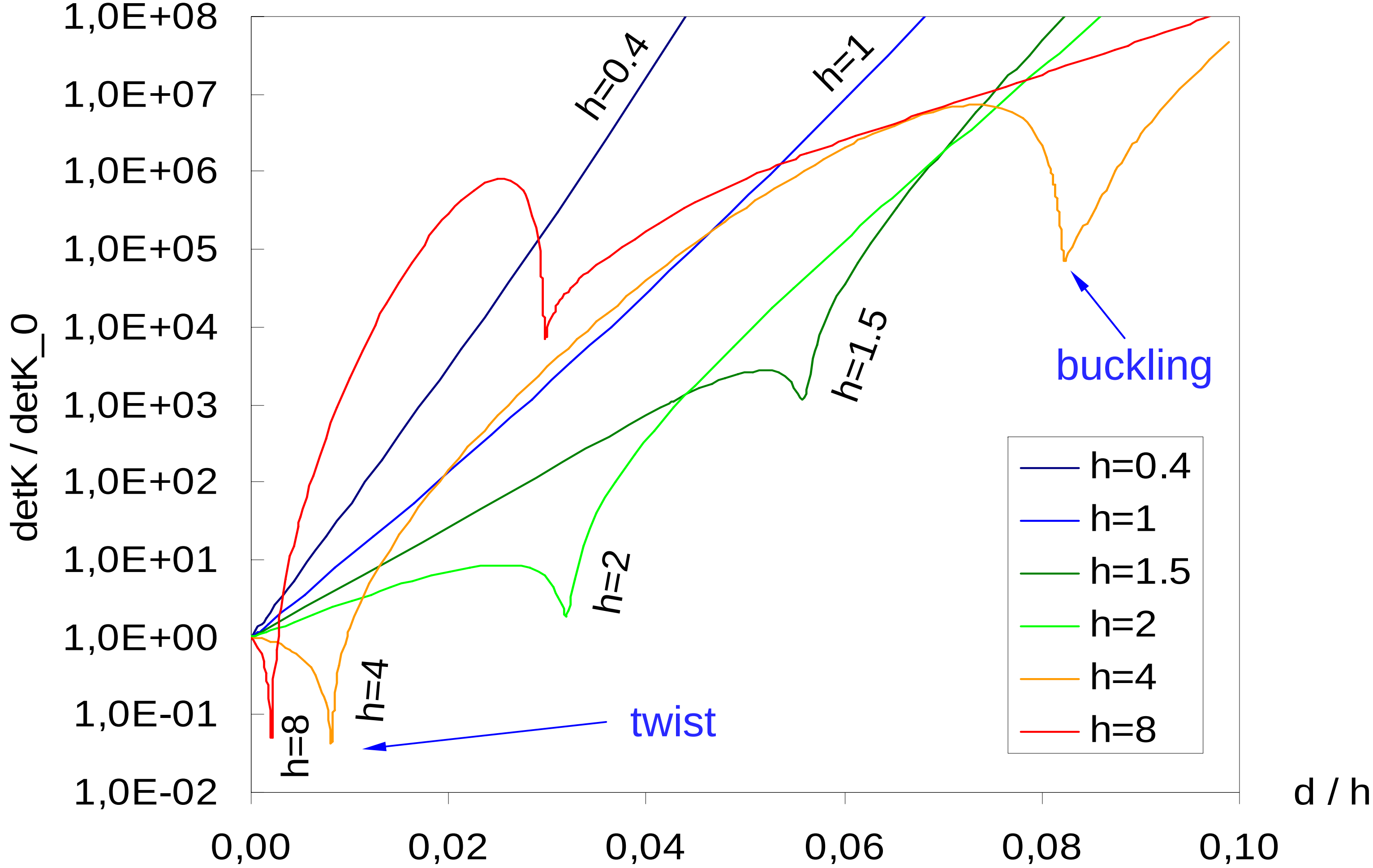
h=2.0: deformation with twist



deformed mesh and coloured displacement in 1-direction

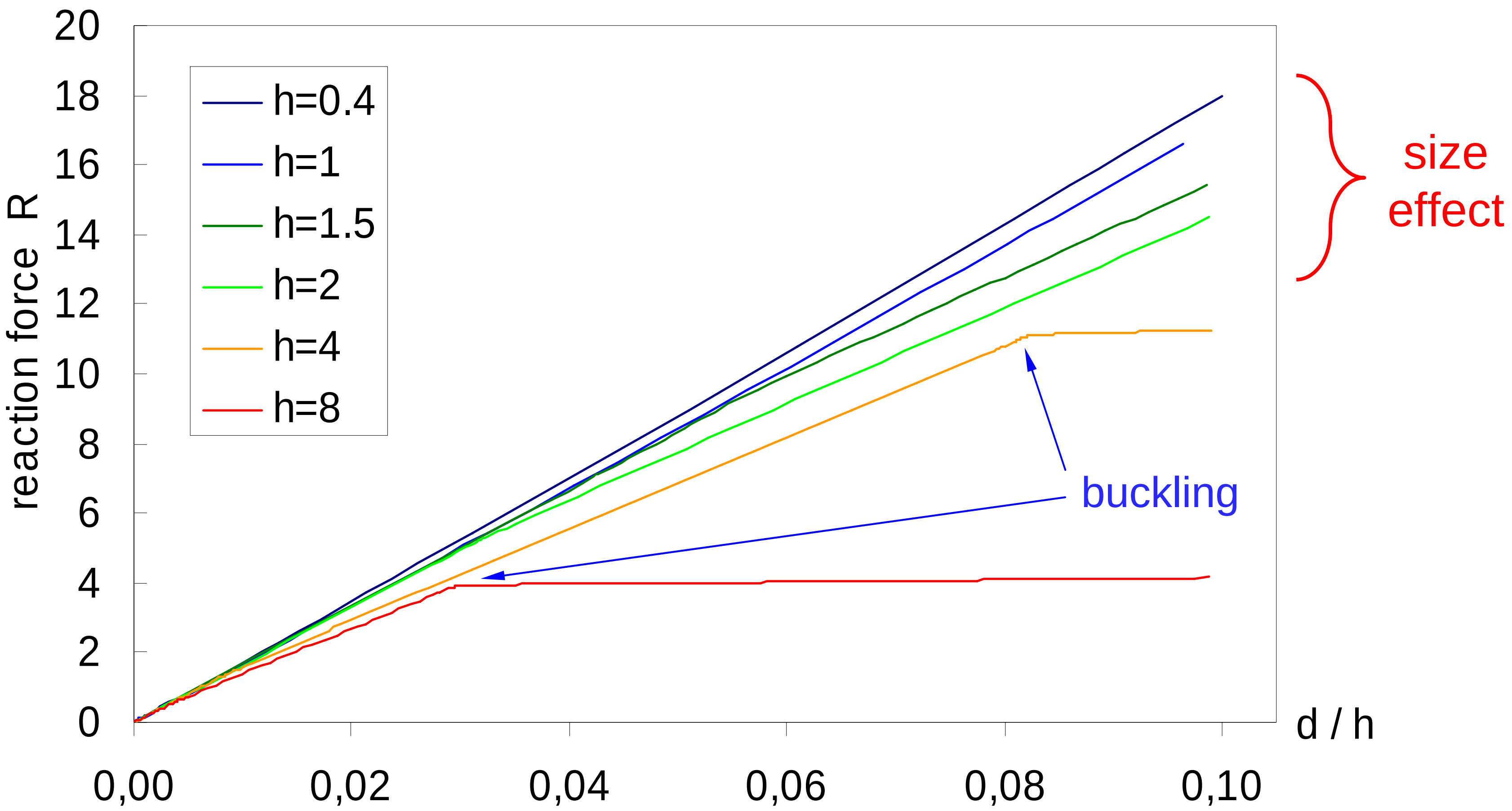
Compression test

Determinant of global stiffness matrix; various heights of structure



Compression test

Reaction force for various heights of structure



→ size effect (caused by twist) is much less significant than buckling

→ size effect hardly measureable in practical tests

→ $\mu_c = 0$ necessitates nonlinear framework and boundary conditions for microrotations (e.g. consistent coupling)

→ $\mu_c = 0$ is consistent with linear elasticity for small strains and any internal length scale L_c

→ bounded stiffness for any internal length scale L_c in torsion test if $\mu_c = 0$

→ non-classical deformation modes (twist) appear for $\mu_c = 0$

Outlook: Extension to micromorphic theory (simulation of foams)

Open question: Right choice of boundary condition for microrotations (is there a physical interpretation of consistent coupling?)

END

References:

[1] *W. Ehlers, J. Bluhm*: Porous Media – Theory, Experiments and Numerical Applications, Springer 2002

[2] *C. Sansour, W. Wagner*: Multiplicative updating of the rotation tensor in the finite element analysis – a path independent approach, Comp. Mech. 31, Springer 2003