

Symmetric Cauchy stresses *do not imply* symmetric
Biot strains in weak formulations of isotropic
hyperelasticity with rotational degrees of freedom.

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Overview

Section

- 1 Introduction
- 2 From Biot to Cosserat
- 3 Symmetry of Cauchy-stresses and Biot-strains
- 4 Conclusions

Notation

Deformation of a Body

- Reference configuration: $\Omega \subset \mathbb{R}^3$
- Deformation $\varphi : \Omega \rightarrow \Omega_{\text{def}} \subset \mathbb{R}^3$
- Deformation gradient: $F := \nabla \varphi \in \text{GL}^+(3)$

Right Polar Decomposition: $F = RU = \text{polar}(F)U$

- Polar rotation: $R, \text{polar}(F) \in \text{SO}(3)$ (exact rotation of $\mathbb{R}^3$)
- Biot-stretch: $U \in \text{PSym}(3)$ (positive definite symmetric)

A Crucial Identity

- $R^T F = U = \sqrt{F^T F} \in \text{PSym}(3)$

Hyperelasticity, Objectivity and Material Isotropy

Hyperelasticity - Minimizing total stored energy

Find minimizers $\varphi_{\min}$ of: $\mathcal{I}(\varphi) = \int_{\Omega} W(\nabla\varphi) \, dV$

- Choice of $W : \mathbb{M}(3, \mathbb{R}) \rightarrow \mathbb{R}$ is a constitutive assumption
- Can apply various boundary conditions ...
- Consider only zero body forces

Isotropy and Objectivity in terms of $F := \nabla\varphi$

- $W(F)$ is **objective** if $\forall Q \in SO(3) : W(QF) = W(F)$
- $W(F)$ is **isotropic** if $\forall Q \in SO(3) : W(FQ) = W(F)$

Objectivity of $W(F)$

$\Rightarrow \exists W^{\sharp} : \text{PSym}(3) \rightarrow \mathbb{R}$ s. th. $W(F) = W^{\sharp}(U(F))$

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The isotropic Biot-model - Standard version

Basic constitutive assumptions

- Objectivity: $W(F) = W^\sharp(U)$, where $U := R^T F = \sqrt{F^T F} \in \text{PSym}$
- Isotropy: $\forall Q \in \text{SO}(3) : W(FQ) = W(F) \iff W^\sharp(Q^T U Q) = W^\sharp(U)$

Example: Most general isotropic quadratic energy in U

$$W^\sharp(U) = \mu \|U - \mathbb{1}\|^2 + \frac{\lambda}{2} \text{tr}[U - \mathbb{1}]^2, \quad U := \sqrt{F^T F}$$

- Linearization equivalent to classical isotropic linear elasticity
- μ, λ the standard Lamé moduli

Biot approach is intrinsically based on a formulation in $U = \sqrt{F^T F}$

- Have to take derivatives of U , i.e., of a matrix square root
- Numerical relaxation $\bar{U} := \bar{R}^T F$ seems to be convenient

Euler-Lagrange equations for the Biot-model

Free variation w.r.t. to φ

$$\begin{aligned}
 0 = \frac{d}{dt} \Big|_{t=0} \mathcal{I}(\varphi + t v) &= \int_{\Omega} \langle D_F W(\nabla \varphi), \nabla v \rangle dV = \int_{\Omega} \langle D_F [W^\sharp(U(F))], \nabla v \rangle dV = \int_{\Omega} \langle D_U W^\sharp(U), D_F U(F) \cdot \nabla v \rangle dV \\
 &= \int_{\Omega} \langle D_U W^\sharp(U), D_F [R(F)^T F] \cdot \nabla v \rangle dV = \dots = \int_{\Omega} \langle R(F) D_U W^\sharp(U), \nabla v \rangle \\
 &= \int_{\Omega} \langle \text{Div}[R(F) D_U W^\sharp(U)], v \rangle dV, \quad \forall v \in C_0^\infty(\Omega, \mathbb{R}^3).
 \end{aligned}$$

Strong form of the equilibrium equation

$$0 = \text{Div}[R(F) D_U W^\sharp(U)], \quad R(F) = \text{polar}(F)$$

A weaker form of the equilibrium equation

$$0 = \text{Div}[\bar{R}(F) D_U W^\sharp(\bar{R}(F)^T F)], \quad \bar{R}(F)^T F \in \text{Sym}$$

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- Weaker means: $\bar{R}^T F \in \text{Sym} \not\Rightarrow \bar{U} = \bar{R}^T F \in \text{PSym}$

Numerical Relaxation

Idea - Generalize a classical model, but preserve its solutions

- $U := R^T F = \text{polar}(F)^T F = \sqrt{F^T F}$ **constrains** rotation

Relaxation of the constraint

- Independent rotations $\bar{R} : \Omega \rightarrow \text{SO}(3)$ (no physics involved!)
- Yields relaxed Biot-stretch $\bar{U} := \bar{R}^T F$ (usually not symmetric)
- Fixing $\bar{R} = \text{polar}(F)$ recovers the non-relaxed Biot-case

Consequences

- Can define relaxed energies $W^\sharp(\bar{U}) = W(F, \bar{R})$
- Relaxation gets rid of U but introduces $\bar{R} \in \text{SO}(3)$
- One obtains a 2-field model!

The relaxed isotropic Biot-model is a Cosserat-model

Hyperelastic Cosserat-model (Neff, 2006)

$$\mathcal{I}(\varphi, \bar{R}) := \int_{\Omega} W_{\text{mp}}(F(x), \bar{R}(x)) + W_{\text{curv}}(\bar{R}(x), D_x \bar{R}(x)) \, dV \mapsto \min. \text{ w.r.t. } (\varphi, \bar{R})$$

Definition of the energetic components: Isotropic case

$$W_{\text{mp}}(\bar{U}) := W_{\text{shear}}(\bar{U}) + W_{\text{vol}}(\det[\bar{U}])$$

$$W_{\text{shear}}(\bar{U}) := \mu \|\text{sym}(\bar{U} - \mathbb{1})\|^2 + \mu_c \|\text{skew}(\bar{U} - \mathbb{1})\|^2$$

$$W_{\text{vol}}(\det[\bar{U}]) := \frac{\lambda}{4} \left((\det[\bar{U}] - 1)^2 + (\det[\bar{U}]^{-1} - 1)^2 \right)$$

$$W_{\text{curv}}(\bar{R}, D_x \bar{R}) := \mu L_c^2 \|\bar{R}^T D_x \bar{R}\|^2$$

Notation

$$\bar{U} := \bar{R}^T F, \quad F := \nabla \varphi, \quad D_x \bar{R} := \left(\nabla(\bar{R} \cdot e_1), \nabla(\bar{R} \cdot e_2), \nabla(\bar{R} \cdot e_3) \right)$$

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The relaxed isotropic Biot-model is a Cosserat-model

Hyperelastic Cosserat-model with $L_c = 0$ and ~~W_{vol}~~

$$\mathcal{I}(\varphi, \bar{R}) := \int_{\Omega} \mu \|\operatorname{sym}(\bar{R}^T F - \mathbb{1})\|^2 + \mu_c \|\operatorname{skew}(\bar{R}^T F - \mathbb{1})\|^2 dV \mapsto \min. \text{ w.r.t. } (\varphi, \bar{R})$$

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The linear setting - $\sigma \in \text{Sym} \iff \text{skew}(\nabla u - \bar{A}) = 0$

Linearizations of the fields

- $F = \mathbb{1} + \nabla u, \nabla u \ll 1$
- $\bar{R} \approx \mathbb{1} + \bar{A}$, where $\bar{A} \in \mathfrak{so}(3)$
- $\bar{U} - \mathbb{1} \approx \nabla u - \bar{A}$
- $\text{polar}(F) \approx \mathbb{1} + \text{skew}(\nabla u)$

Linearized “constraint”

- Finite: $\bar{R} = \text{polar}(F)$
- Linear: $\bar{A} = \text{skew}(\nabla u)$

Definition of $W^\sharp$ - Cosserat without curvature

$$W^\sharp(\bar{U}) := \mu \|\text{sym}(\bar{U} - \mathbb{1})\|^2 + \mu_c \|\text{skew}(\bar{U} - \mathbb{1})\|^2 + \frac{\lambda}{4} \left((\det[\bar{U}] - 1)^2 + \left(\frac{1}{\det[\bar{U}]} - 1 \right)^2 \right)$$

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$W_{\text{quad}}^\sharp$ - Quadratic approximation of $W^\sharp$

$$W_{\text{quad}}^\sharp(\nabla u - \bar{A}) := \mu \|\text{sym}(\nabla u - \bar{A})\|^2 + \mu_c \|\text{skew}(\nabla u - \bar{A})\|^2 + \frac{\lambda}{2} \text{tr} [\nabla u - \bar{A}]^2$$

Linear case $\mu_c > 0$: $\sigma \in \text{Sym} \iff \text{skew}(\nabla u - \bar{A}) = 0$

$$\sigma = D_{\nabla u} W_{\text{quad}}^\sharp = 2\mu \text{sym}(\nabla u) + 2\mu_c \text{skew}(\nabla u - \bar{A}) + \lambda \text{tr} [\nabla u] \mathbb{1}$$

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Linear case $\mu_c = 0$: $\sigma \in \text{Sym}$ is **independent** of $\bar{A}$

$$\sigma = D_{\nabla u} W_{\text{quad}}^\sharp = 2\mu \text{sym}(\nabla u) + \lambda \text{tr} [\nabla u] \cdot \mathbb{1}$$

The linear setting - $\sigma \in \text{Sym} \iff \text{skew}(\nabla u - \bar{A}) = 0$

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Linear case - Equivalence of Symmetry

$$\sigma \in \text{Sym} \iff \text{skew}(\nabla u - \bar{A}) = 0 \iff \text{Linearized } \bar{U} \in \text{Sym}$$

Cauchy-stress and balance of angular momentum in continuum theories with rotational degrees of freedom

Generalized stress tensors in theories with relaxed rotations $\bar{R}$

- 1st Piola-Kirchhoff: $S_1(F, \bar{R}) := D_F W(F, \bar{R}) = \bar{R} D_{\bar{U}} W^\sharp(\bar{U})$
- Cauchy: $\sigma := \frac{1}{\det[F]} S_1(F, \bar{R}) F^T = \frac{1}{\det[F]} \bar{R} D_{\bar{U}} W^\sharp(\bar{U}) F^T$
- $\sigma \in \text{Sym} \iff \bar{R}^T \sigma \bar{R} \in \text{Sym}$

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- $\sigma \in \text{Sym} \iff D_{\bar{U}} W^\sharp(\bar{U}) \bar{U}^T \in \text{Sym} \iff \text{Balance of angular momentum}$

Theorem (Objectivity of $W^\sharp$ implies $\sigma \in \text{Sym}$)

Let $W^\sharp = W(F, \bar{R})$ be differentiable and objective, then the associated Cauchy stress tensor σ is symmetric.

Cauchy-stress and balance of angular momentum in continuum theories with rotational degrees of freedom

Question: Does isotropy of $W_{\text{quad}}^\sharp$ imply $\overline{U}_{\text{lin}} \in \text{Sym}$?

Cauchy-stress and balance of angular momentum in continuum theories with rotational degrees of freedom

Question: Does isotropy of $W_{\text{quad}}^\#$ imply $\bar{U}_{\text{lin}} \in \text{Sym}$?

Answer: Yes (Shown on the previous slide)

Cauchy-stress and balance of angular momentum in continuum theories with rotational degrees of freedom

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Cauchy-stress and balance of angular momentum in continuum theories with rotational degrees of freedom

Question: Does isotropy of $W^\sharp$ imply $\overline{U} \in \text{Sym}$?

Answer: No (Neff, Fischle, Münch - 2007)

Balance of Angular Momentum

Geometrically exact volume terms are irrelevant

- $\det[\bar{U}] = \det[\bar{R}^T F]$ is independent of $\bar{R}$
- Volume term of $W^\sharp$ is irrelevant

Consider $W^\sharp(\bar{U}) := \mu \|\text{sym}(\bar{U} - \mathbb{1})\|^2 + \mu_c \|\text{skew}(\bar{U} - \mathbb{1})\|^2$

$$\begin{aligned}
 \sigma \in \text{Sym} &\iff D_{\bar{U}} W^\sharp(\bar{U}) \bar{U}^T \in \text{Sym} \\
 &\iff [2\mu (\text{sym}(\bar{U} - \mathbb{1})) + 2\mu_c \text{skew } \bar{U}] \bar{U}^T \in \text{Sym} \\
 &\iff [\mu (\bar{U} + \bar{U}^T - 2\mathbb{1}) + \mu_c (\bar{U} - \bar{U}^T)] \bar{U}^T \in \text{Sym} \\
 &\iff (\mu - \mu_c) \bar{U} \bar{U} - 2\mu \bar{U} \in \text{Sym}
 \end{aligned}$$

Equilibrium equation (for angular momentum)

$$(\mu - \mu_c) [\bar{U}^2 - \bar{U}^{T,2}] - 2\mu [\bar{U} - \bar{U}^T] = 0$$

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Example I - Assumptions and Definitions

Assumptions on the structure of F and $\bar{R}$

$$F := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 > 0$$

$$\bar{R} := \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Choose further

- $\rho := \frac{2\mu}{\mu - \mu_c}, \quad 0 \leq \mu_c < \mu$
- $\lambda_1 + \lambda_2 > \rho$
- $\alpha := \arccos\left(\frac{\rho}{\lambda_1 + \lambda_2}\right) \in \left[0, \frac{\pi}{2}\right]$

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$\Rightarrow$ Particular forms for $\bar{R}$ and $\bar{U}$

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⇒ Particular forms for $\bar{R}$ and $\bar{U}$

$\bar{R}$ is given by

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- $\alpha := \arccos\left(\frac{\rho}{\lambda_1 + \lambda_2}\right) \in [0, \frac{\pi}{2}]$

$\bar{U} := \bar{R}^T F$ (relaxed Biot-stretch)

$$\begin{pmatrix} \frac{\rho}{\lambda_1 + \lambda_2} & -\sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & 0 \\ \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & \frac{\rho}{\lambda_1 + \lambda_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\rho \lambda_1}{\lambda_1 + \lambda_2} & \lambda_2 \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & 0 \\ -\lambda_1 \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & \frac{\rho \lambda_2}{\lambda_1 + \lambda_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$\Rightarrow$ Particular forms for $\bar{R}$ and $\bar{U}$

Note that $\bar{U}$ need not be symmetric!

$\bar{R}$ is given by

$\bar{U} := \bar{R}^T F$ (relaxed Biot-stretch)

$$\begin{pmatrix} \frac{\rho}{\lambda_1 + \lambda_2} & -\sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & 0 \\ \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & \frac{\rho}{\lambda_1 + \lambda_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\rho \lambda_1}{\lambda_1 + \lambda_2} & \lambda_2 \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & 0 \\ -\lambda_1 \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & \frac{\rho \lambda_2}{\lambda_1 + \lambda_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example II - Identities

An easy computation

$$\bar{U} - \bar{U}^T = \begin{pmatrix} 0 & (\lambda_1 + \lambda_2) \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & 0 \\ -(\lambda_1 + \lambda_2) \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\bar{U}^2 - \bar{U}^{T,2} = \begin{pmatrix} 0 & \rho (\lambda_1 + \lambda_2) \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & 0 \\ -\rho (\lambda_1 + \lambda_2) \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example II - Identities

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$$\bar{U} - \bar{U}^T = \begin{pmatrix} 0 & (\lambda_1 + \lambda_2) \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & 0 \\ -(\lambda_1 + \lambda_2) \sqrt{1 - \frac{\rho^2}{(\lambda_1 + \lambda_2)^2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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allows us to infer

$$\begin{aligned} \bar{U}^2 - \bar{U}^{T,2} &= \rho (\bar{U} - \bar{U}^T) = \frac{2\mu}{\mu - \mu_c} (\bar{U} - \bar{U}^T) \\ \iff (\mu - \mu_c) (\bar{U}^2 - \bar{U}^{T,2}) - 2\mu (\bar{U} - \bar{U}^T) &= 0 \\ \iff D_{\bar{U}} W^\sharp(\bar{U}) \bar{U}^T &\in \text{Sym} \\ \iff \sigma &\in \text{Sym} \end{aligned}$$

Overview

Section

- 1 Introduction
- 2 From Biot to Cosserat
- 3 Symmetry of Cauchy-stresses and Biot-strains
- 4 Conclusions**

Conclusions and Future Work

The finite Cosserat model and relaxed Biot model

- Absence of Cosserat curvature $\Rightarrow$ “Cosserat = relaxed Biot”
- Isotropy does not imply classical, symmetric Biot-stretches

Influence of μ_c as a penalization parameter

- $\mu_c < \mu$: Asymmetric Biot-stretches with symmetric σ
- $\mu_c \geq \mu$: Exact symmetry of the Biot stretch tensor

Further Research

Characterize a sufficiently large class of isotropic free energies such that moment equilibrium in the relaxed Biot formulation automatically implies the symmetry of the relaxed Biot stretch $\bar{U}$.

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Thank you

- The End -

Thank you for your attention!