Geometrically exact Cosserat theory for bulk behaviour and thin structures. Modelling and mathematical analysis

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Chapter 1

Preface

This Habilitation-thesis is the outgrow of my investigations into the mathematical structure of models for the mechanical behaviour of materials under large elastic deformations and plasticity.

For infinitesimal displacement, quite some agreement has been reached in the last century on how to formulate the relevant models and the corresponding mathematical analysis of the ensuing systems of linear partial differential equations in the elastic case, or additionally coupled with nonlinear evolution equations in the elasto-plastic case is well understood, with the exception of some degenerated limiting cases e.g. infinitesimal, rate-independent plasticity without hardening.

However, for todays needs in science and technology, it has become clear that infinitesimal models are insufficient to account for important features of material behaviour. Therefore, mechanical models acounting for large deformations and satisfying basic postulates of invariance (observer-invariance) have been proposed in the engineering community in the last part of the century.

Due to the mechanical and mathematical difficulties encountered, no general agreement has been obtained on the structure of the models apart for the case of finite elasticity. And it is only finite elasticity in a variational form which has been treated so far succesfully by the elegant direct methods of variations using Ball's method of combining weak convergence and polyconvex energies.

Only slight progress has been made for certain formulations of finite elasto-plasticity, one has to regard this field still as open.

A main drawback of most of the proposed models is the absence of any internal length scale, which absence may give rise to microstructure and excludes the mathematical treatment of the models along the traditional lines. This lack of an internal length scale is also at the root of convergence problems of corresponding finite element implementations.

In the engineering community one has therefore increasingly considered models, which have a built in length scale, as a means, to either regularize computations or to represent more faithfully the actual material response. Typically, the length scale introduces size effects into the models to the extent that smaller samples of the same material show a relatively stiffer response than a larger sample.

There are many possibilities to introduce length scale effects but we will focus in this investigation on finite Cosserat models, where the length scale is related somehow to the rotational part of the deformation. These models have been introduced by the Cosserat brothers in the beginning of the 20th century into continuum mechanics only to be forgotten soon after. In the second part of the century their general development has been rediscovered with a subsequent focus on the infinitesimal theory.

After some time spent with analytical investigations the infinitesimal Cosserat model was again largely abondened in the late seventies for several reasons: it led to quite elaborate formulas, the influence of specific (additional) boundary conditions on the solution remained unclear as its relation to the classical infinitesimal linear elasticity model and it could never really be validated by experiment or led to inconsistencies. The mathematical theory of the infinitesimal Cosserat model is conceptually simpler than the mathematical treatment of the classical model; existence, uniqueness and continuous dependence on the data has long been established.

We have to differentiate here the Cosserat bulk behaviour from the so-called theory of a Cosserat surface which introduces independent directors to describe thin structures but has no immediate connection to the bulk model. The Cosserat surfaces is well established in continuum-mechanics as a valuable model for thin structures.

Recently, with the advent of the computer, Cosserat bulk models surfaced again in infinitesimal and finite variants, for elasticity and plasticity. Now the Cosserat model is mainly used to stabilize the numerical computations where otherwise shear failure and microstructure would lead to mesh-dependent results. Practically all finite models are simple extensions of infinitesimal models. While the principal conceptual problems remain, the focus has clearly shifted to numerics.

To the authors knowledge, nothing is known as far as mathematical existence for such finite Cosserat models is concerned. This will therefore be our main mathematical concern.

This thesis consists of two parts, treating the bulk behaviour and the behaviour of thin structures separately. Both parts are more or less self-contained and comprise each the corresponding mathematical treatment, which itself is largely self-contained.

In the first part we investigate geometrically exact generalized continua of Cosserat micropolar type. The variational form of these models is introduced and consistently extended to cover finite elasto-plasticity based on the multiplicative decomposition of the deformation gradient only. The decisive stress is the Eshelby energy momentum tensor. It is motivated that the traditional Cosserat couple modulus μ_c can and should be set to zero for macroscopic specimens liable to fracture in shear, still leading to a complete consistent Cosserat theory with independent rotations in the geometrically exact finite case in contrast to the infinitesimal, linearized model.

Depending on material constants different mathematical existence theorems in Sobolev-spaces are given for the resulting nonlinear boundary value problems in the elastic case. These are the first such results known to the author. Various assumptions on the magnitude of deformations and microrotations lead to simplified models which are all analysed mathematically.

Partial focus is set to the possible regularization properties of micropolar models compared to classical continuum models in the macroscopic case of materials failing in shear. The mathematical analysis heavily uses an extended Korn's first inequality (Neff, Proc.Roy.Soc.Edinb.A, 2002) discovered by the author recently. The methods of choice are the direct methods of the calculus of variations.

In the second part we are concerned with the consistent dimensional reduction of the previously introduced finite three-dimensional Cosserat micropolar elasticity model to the two-dimensional situation of thin plates and shells. The resulting membrane energy turns out to be a quadratic, elliptic, first order, non degenerate energy in contrast to classical approaches, the standard bending contribution is augmented with a term representing an additional stiffness of the Cosserat model and the corresponding system of balance equations remains of second order. The model includes size effects, transverse shear resistance, thickness stretch and drilling degrees of freedom. The thin shell limit is non-degenerate due to the additional Cosserat bending stiffness.

It is shown that the dimensionally reduced formulation is well-posed along the same line of argument which showed the well posedness of the three-dimensional model [Nef03c]. Decisive use is made of a dimensionally reduced version of an extended Korn's first inequality recently proved by the author [Nef02]. For vanishing Cosserat couple modulus $\mu_c = 0$, the linearized Cosserat plate coincides with the well known Reissner-Mindlin plate formulation.

Let us highlight again the role played by the Cosserat couple modulus μ_c. If $\mu_c = 0$, then there exists no infinitesimal linear Cosserat bulk model and no inconsistencies with experiment occur. Cosserat effects are of second order and intervene only for finite deformations. Consistent with this observation, we impose boundary conditions (which we call consistent coupling) that minimize any non-classical effects. It is shown that the finite Cosserat bulk model is a reasonable physical regularization. It is believed that setting $\mu_c = 0$ is finally doing justice to the initial development of the Cosserats and liberates the theory from the dust piled up by linearization.

Chapter 2

Three-dimensional Cosserat theory

In this chapter we investigate geometrically exact generalized continua of Cosserat micropolar type. The variational form of these models is introduced and consistently extended to cover finite elasto-plasticity based on the multiplicative decomposition of the deformation gradient only. The decisive stress is the Eshelby energy momentum tensor. It is motivated that the traditional Cosserat couple modulus μ_c can and should be set to zero for macroscopic specimens liable to fracture in shear, still leading to a complete consistent Cosserat theory with independent rotations in the geometrically exact finite case in contrast to the infinitesimal, linearized model.

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2.1 Introduction

2.1.1 The development of Cosserat models, motivation and applications

This article addresses the modelling and mathematical analysis of **geometrically exact**[1] generalized continua of **Cosserat micropolar** type in the elastic as well as elasto-plastic case. General continuum models involving **independent rotations** have been introduced by the Cosserat brothers [CC09]. In fact, their original motivation came from the theory of surfaces, where the moving three-frame (Gauss frame) had been used successfully.

Their development has been largely forgotten for decades only to be rediscovered in the beginning of the sixties [Osh55, Gün58, AK61, ES64, Eri68, Tou62, Tou64, GR64, MT62, Sch67, TN65]. At that time theoretical investigations on non-classical continuum theories were the main motivation [Krö68]. The Cosserat concept has been generalized in various directions, for an overview of these so called **microcontinuum** theories we refer to [EK76, Eri99, CG77, Cap89].

Among the first contributions extending the Cosserat framework to (infinitesimal, geometrically linear) elasto-plasticity we have to mention [Saw67, Lip69, Bes74]. More recent (infinitesimal) elastic-plastic formulations have been investigated in [dB92, DSW93, IW98, RV96]. These models directly comprise joint elastic and plastic Cosserat effects. Lately, the models have been extended to a finite elastic-plastic setting, see e.g. [GT01, San98a, San98b, San99, Ste94a, Gra03, FCS97] and references therein. Most of these finite extensions directly comprise of joint elastic and plastic Cosserat effects as well together with an additional split of the curvature into elastic and plastic parts but their physical and mathematical significance is at present much more difficult to asses than models in which Cosserat effects are restricted to the elastic response of the material [FCS97] and references therein. Our own contribution will be of the second type.

It is generally accepted that couple stresses, understood here as the presence of non-symmetric parts of the Cauchy stress, exist in real elastic material [TN65]. Discrepancies between classical linearized elasticity theory and experiments are observed predominantly for high gradients which occur by stress concentrations in the neighbourhood of holes, notches and cracks as predicted by classical elasticity. Indeed, the measured stresses around cracks are smaller than the predicted ones.

[1] Fully frame-indifferent

However, the source, magnitude and significance of couple stresses is still being discussed. A group of researchers [Voi87, HK65, Koi64], supported by experimental evidence [GJ75, Gau82, Sch66, ES67] admits **elastic** couple stresses in elasto statics on a **macroscale** only due to (very small) nonlocal effects such that an infinitesimal elastic Cosserat micropolar theory is meaningless: the infinitesimal continuum rotations must coincide with the infinitesimal microrotations and moreover, couple stresses are altogether neglected since they are supposedly small.[2] Despite this situation, infinitesimal elasto-plastic extensions of the indeterminate couple-stress theory have been investigated in [FMAH94, RV96].

Another group of researchers uses the infinitesimal Cosserat micropolar model, admitting non symmetric infinitesimal constitutive Cauchy stresses as a first order effect due to independent infinitesimal microrotations [Ste94a, IW98, Müh89, dB91, GT01]. Apparently, both views exclude each other. [3]

Notwithstanding, we present a model reconciling both views: the difference of opinion is due to the uncritical use of the infinitesimal, linear Cosserat model but disappears for a geometrically exact description of the Cosserat theory. The **Cosserat couple modulus** μ_c (modulus of local rotational stiffness, Cosserat shear modulus, torsional rigidity cf. (2.2.6)) appearing in both the infinitesimal and geometrically exact description can be **set** to **zero**, still there is a **nonlocal coupling** together with independent finite rotations, while the linearization of this theory has lost all elastic Cosserat effects.

Another eminent source for couple stresses are granular material [Osh55, MV87, Müh89, MH96, BP91, Bar94, Bar98] where individual grains are supposed to be in contact and to transmit forces by contact couples. Here, effects of couple stresses cannot usually be neglected, however, numerical simulations including a description of the contact mechanics still suggest that they are of second order [Bar98].

Using scaling arguments it is clear, that material length scale effects become the more accentuated the smaller the geometrical dimensions of the specimen are. This suggests the future application of Cosserat models for microscopic specimens or in such fields as thin films and micro actuators.

We remark that it has never really be admitted that Cosserat effects played a role as long as traditional engineering materials in their elastic range on a macroscale were considered. Since a micropolar model is considerably more difficult analytically, the infinitesimal[4] linear elastic Cosserat model was partially abandoned in the early seventies.

Renewed interest in Cosserat models arose with the advent of the computer allowing to circumvent analytical details. Today, apart from the theoretical development, the Cosserat type models are increasingly advocated as a means to **regularize** the pathological **mesh size dependence** of localization computations where shear failure[5] are mechanisms [CH85, MV87, Müh89, BP91, Bar94] play a dominant role, for applications in plasticity see the non-exhaustive list [IW98, DSW93, RV96, dB91, dBS91, dB92]. The occurring mathematical difficulties reflect the physical fact that upon localization of the deformation within narrow bands the validity limit [6] of the classical models is reached. In models without any internal length the deformation should be homogeneous on the scale of a representative volume element of the material [MA91].

Of course, there are many other possibilities available to overcome this deficiency, we mention only higher gradient theories [Aif98] and references therein, nonlocal models [BC84, Eri83] using integral kernels in the constitutive law or incremental variational formulations [Lam02, LMD03, ML03a, ML03b] and viscoplastic regularizations [Nef03a]. While all these models successfully regularize the mesh-dependency, nonlocal and viscoplastic approaches are localization limiters but do not necessarily introduce a specific geometric size effect. The stress-strain diagram in an incremental variational approach is mesh-independent, while the thickness of the localization zone is given by the size of the smallest finite element. Apart from incremental methods all regularizing approaches need additional material parameters. This is a distinctive advantage of incremental methods.

The incorporation of a length scale, which is natural in a Cosserat theory, in principal has the power to remove the mesh sensitivity. The presence of the internal length scale causes the localization zones to have finite width. In [DSW93, IW98] it is explicitly shown in the infinitesimal elasto-plastic context, that mode II failure (shear failure) is ruled out while the formation of holes (mode I failure) is still possible. However, the actual characteristic length scale of a material is difficult to establish experimentally and theoretically [Lak95], and

[2]In [HK65, p.339] we read: "Momentenspannungen sind merklich erst in Bereichen vorhanden, in denen normalerweise nicht nur die Anwendung der linearen, sondern auch der nichtlinearen Elastizitätstheorie nicht mehr sinnvoll ist."

[3]The experimental results of [FMAH94] on the torsion of thin copper wires revealed a strong **geometrical length scale effect** of the **plastic** behaviour: the thinnest wires displayed comparatively the strongest response up into the plastic range. Whether this is due to a genuine Cosserat effect cannot be ascertained. It must be noted that in their experiments, also **grain size effects** interfered, which have nothing to do with geometrical size effects.

[4]By infinitesimal we mean arbitrary small displacements and not just small displacements, certainly far below 1 percent elongation.

[5]In short: shear failure means for us that Legendre-Hadamard ellipticity is violated while the Baker-Ericksen inequalities are satisfied, cf. section (2.8.2).

[6]The occurring high deformation gradients in a shear band suggest that infinitesimal elasticity should not be used for a physically consistent description of shear bands. The same remark applies to calculations of stress concentrations in the vicinity of cracks: certainly, deformations are not any longer infinitesimal small.

remains basically an open question as is the determination of other additionally appearing material constants in the Cosserat framework: the employed non-zero value of the Cosserat couple modulus μ_c remains unmotivated in most of these regularization procedures.

Let us summarize at this stage the gathered circumferential evidence of Cosserat effects for metallic and granular materials:

	infinitesimal elastic	**infinitesimal elastic-plastic**	**finite elastic**	**finite elasto-plastic**
macroscopic specimens, **heterogeneous** on a small scale, rather **homogeneous** on a large scale	**no** significant **Cosserat effects** reported: Ellis, Gauthier, Koiter, Schijve etc., $\Rightarrow \mu_c = 0$	Cosserat models due to: Besdo, de Borst, Fleck, Forest, Iordache, Lippmann, Mühlhaus, Ristinmaa, Steinmann etc., value $\mu_c > 0$ mostly not motivated, inconsistent with infinitesimal elasticity	**significant Cosserat effects** reported for strains > 4 percent in localizations within shear bands: Bardet, Cambou, Mühlhaus etc.	**geometrical size effect** of plastic hardening, grain size effects, Hall-Petch relation, pronounced in **torsion**, small for stretch: Arzt, Fleck, Forest, Stölken, Tsakmakis etc.
microscopic specimen, more or less **homogeneous**	relevant size: atomic bonds, **no** appreciable **size effect** expected: Arzt, Koiter, Kröner, Voigt etc., $\Rightarrow \mu_c = 0$	applied to simulate lattice rotations, dislocations in real single crystals, Forest, $\mu_c \gg 1$, Fleck, $\mu_c = \infty$, inconsistent with infinitesimal elasticity	relevant size: atomic bonds, **no** appreciable **size effect** expected?	microbend tests, plasticity of thin films, single crystals: Forest, Grammenoudis, Tsakmakis etc.
remarks	resulted in **premature** partial **abandoning** of linear infinitesimal indeterminate couple stress theory and linear Cosserat theory altogether	**regularize** pathological **mesh-dependence** in large scale numerical calculations, **no** intrinsic experimental evidence	Cosserat models are used to **regularize shear failure** of granular materials, based on experiments indicating a definite width of the bands	alternative: higher gradient plasticity theory, parabolic flow rule, nonlocal models, incremental variational formulations,

If regularization of widely accepted classical continuum models on the macroscopic scale is our aim, the regularizing effects should be incorporated such that essential salient features of the classical model are still present. Otherwise we face the danger of over-regularization.[7] In the following we will see that such unwanted behaviour is in part linked to the value of certain material constants, notably the Cosserat couple modulus μ_c.

The mathematical analysis of Cosserat micropolar models is at present restricted to the infinitesimal, linear elastic models, see e.g. [Ies71, Duv70, HH69, Ghe74a, Ghe74b]. The major difficulty of the mathematical treatment in the finite strain case is related to the geometrically exact formulation of the theory and the appearance of finite rotations. No general existence theorems are known to the author.

2.1.2 Outline and scope of this contribution

This contribution is organized as follows: first, we review the basic concepts of the geometrically exact elastic Cosserat micropolar theories in a variational context. In contrast to other contributions we keep the third-order tensor character of the curvature tensor $\mathfrak{K}$ and do not use a reduced second order format $\hat{\mathfrak{K}}$ based on the axial representation.

An investigation into the constitutive relations for a geometrically exact Cosserat theory apparently has never been done. We highlight therefore the striking constitutive consequences of the choice for the Cosserat couple modulus, $\mu_c > 0$, $\mu_c = 0$, respectively, in the force balance equation and it is easily seen that $\mu_c > 0$ is **not** a necessary constitutive assumption for the geometrically exact theory. Moreover, it is shown that, contrary to the infinitesimal case, the exact theory does not necessarily reduce to the classical elasticity theory in the limit of a vanishing internal length scale without further provision. To conclude this part, we provide a

[7] If the classical model fails in shear by fracture along a slip line, we impose at least that the onset of fracture is correctly reported.

classification scheme of finite micropolar elasticity and motivate a new boundary condition for microrotations, which we call **consistent coupling** condition.

The elastic formulation is then consistently extended to finite multiplicative plasticity with **non-dissipative** micropolar effects. The decisive stress tensor is nothing else than the elastic **Eshelby energy momentum tensor**. Due to the third order curvature representation $\mathfrak{K}$ we retain also the Eshelby format for the curvature part of the stresses. The obtained general plasticity model is then specialized to a macroscopic case of a polycrystal with grain rotations. It is motivated that for such a model the most natural choice is to set the Cosserat couple modulus $\mu_c = 0$. Various reductions of the geometrically exact model are possible, for conciseness we restrict attention to the infinitesimal micropolar elasto-plastic model, operative however, only for $\mu_c > 0$.

More mathematically inclined readers can safely skip the modelling part and start directly in the analytical section 4. There, the complete problem statement of the geometrically exact elastic Cosserat case in a variational context is repeated.

Existence of minimizers in Sobolev-spaces is established using the direct methods of variations and a novel extended Korn's first inequality. Similar methods allow to treat the various reduced situations as well with stronger results depending on the reductions made. However, only in the completely reduced well known infinitesimal case existence, uniqueness and continuous dependence on the data can be established.

Finally, various alternative forms of the curvature part are investigated and it is argued that Cosserat models can be superior in regularizing shear failure mechanisms than simply taking the quasiconvex hull or numerical approximations of it as a mere computational localization limiter.

In the appendix we provide missing arguments for the Boltzmann axiom and define what we mean by shear failure. In a detailed analysis of simple shear (simple glide) we derive analytical solutions which allow to contrast the different models and underline the merrits of the new approach allowing for sharp interfaces in the limit of vanishing internal length.

2.2 The finite elastic Cosserat micropolar model

Let us now motivate the finite Cosserat approach. The relevant notation will be found in the appendix. For our development we choose a strictly Lagrangean description. First, in the purely elastic case, a Cosserat theory can be obtained by introducing the multiplicative decomposition of the macroscopic deformation gradient F into **independent microrotation** $\overline{R}$ (Cosserat rotation tensor) and **micropolar stretch tensor** $\overline{U}$ (or first Cosserat deformation tensor) with

$$F = \overline{R} \cdot \overline{U} \,, \tag{2.2.1}$$

where $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ and $\overline{U} \in \mathrm{GL}^+(3,\mathbb{R})$ but $\overline{U} \notin \mathrm{PSym}(3)$ such that (2.2.1) is not necessarily the polar decomposition of F. The notion **micro**polar is prone to misunderstandings: $\overline{R}$ must be considered as a macroscopic (average) quantity as the deformation gradient and the resulting model is still phenomenological.

In the quasistatic case, the Cosserat theory is now derived from a variational principle by postulating the following **'action euclidienne'** [CC09, p.156] for the finite macroscopic deformation $\varphi : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$ and the independent microrotation $\overline{R} : \overline{\Omega} \mapsto \mathrm{SO}(3)$:

$$\begin{aligned} I(\varphi,\overline{R}) &= \int_\Omega W(F,\overline{R},\mathrm{D_x}\overline{R}) - \langle f,\varphi\rangle - \langle M,\overline{R}\rangle \,\mathrm{dV} - \int_{\Gamma_S} \langle N,\varphi\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c,\overline{R}\rangle \,\mathrm{dS} \mapsto \min .\ \text{w.r.t. } (\varphi,\overline{R}), \\ \overline{R}_{|_\Gamma} &= \overline{R}_\mathrm{d} \,, \quad \varphi_{|_\Gamma} = g_\mathrm{d}(t) \,. \end{aligned} \tag{2.2.2}$$

The elastically stored energy W depends on the deformation gradient as usual but in addition on the microrotations together with their space derivatives. Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g, \overline{R}_\mathrm{d}$ for displacements and microrotations, respectively, are prescribed while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions N are applied with $\Gamma \cap \Gamma_S = \emptyset$. The external volume force is f and M takes on the role of external volume couples.[8] In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where surface couples M_c are applied with $\Gamma \cap \Gamma_C = \emptyset$. Variation of the action I with respect to φ yields the equation for balance of linear momentum and variation of I with respect to $\overline{R}$ yields balance of angular momentum.

The standard conclusion from frame-indifference (here: invariance of the free energy under **s**uperposed **r**igid **b**ody **m**otions (**SRBM**) not merely observer invariance of the model [SB97, BS01, Mur03]: $\forall\, Q \in \mathrm{SO}(3,\mathbb{R})$: $W(F,\overline{R},\mathrm{D_x}\overline{R}) = W(QF, Q\overline{R}, \mathrm{D_x}[Q\overline{R}])$ leads to the reduced representation of the energy

$$W(F,\overline{R},\mathrm{D_x}\overline{R}) = W(\overline{R}^T F, 1\!\!1, \overline{R}^T \mathrm{D_x}\overline{R}) = W(\overline{U}, \overline{R}^T \mathrm{D_x}\overline{R}) = W(\overline{U}, \mathfrak{K}) \,, \tag{2.2.3}$$

[8] appearing in a non-mechanical context e.g. as influence of a magnetic field on the polarization of a substructure of the bulk.

where $\mathfrak{K} := \overline{R}^T \mathrm{D_x}\overline{R} = \left(\overline{R}^T\nabla(\overline{R}.e_1), \overline{R}^T\nabla(\overline{R}.e_2), \overline{R}^T\nabla(\overline{R}.e_3)\right)$ is one specific representation[9] of the third order right **micropolar curvature tensor** (or torsion-curvature tensor, wryness tensor, second Cosserat deformation tensor, bending-twist tensor etc.). For a geometrically exact isotropic[10] theory we assume in the following an additive split of the total free energy into local stretch and curvature part according to

$$W = W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{curv}}(\mathfrak{K})\,. \tag{2.2.4}$$

2.2.1 The elastic stretch energy

For a **small elastic strain** theory, which should already cover most cases of physical interest, we require that $W_{\mathrm{mp}}(\overline{U})$ is a non negative isotropic quadratic form [11] normalized to

$$W_{\mathrm{mp}}(\mathbb{1}) = 0, \quad D_{\overline{U}}W_{\mathrm{mp}}(\overline{U})_{|_{\overline{U}=\mathbb{1}}} = 0\,. \tag{2.2.5}$$

The most general form of W_{mp} consistent with (2.2.5) is cf.(2.9.151)

$$W_{\mathrm{mp}}(\overline{U}) = \alpha_1\,\|\,\mathrm{sym}(\overline{U}-\mathbb{1})\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{U}-\mathbb{1})\|^2 + \alpha_3\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}-\mathbb{1})\right]^2, \tag{2.2.6}$$

with material constants $\alpha_1, \mu_c, \alpha_3$ such that $\alpha_1, 3\alpha_3+\alpha_1, \mu_c \geq 0$ from non negativity [Eri99]. By consistency with the classical continuum model without microrotations we can take $\alpha_1 = \mu$, $\alpha_3 = \frac{\lambda}{2}$ with $\mu, \lambda > 0$ the classical Lamé constants and the **Cosserat couple modulus** μ_c remains for the moment unspecified but $\mu_c = 0$ is physically possible since the **micropolar reaction stress** $D_{\overline{U}}W_{\mathrm{mp}}(\overline{U})\cdot\overline{U}^T$ is not symmetric in general, i.e. the problem does not decouple, cf. (2.2.12). For comparison, in [Eri99, p.111] for the infinitesimal case, the elastic moduli are taken to be $\alpha_1 = \mu + \frac{\kappa}{2}$, $\mu_c = \frac{\kappa}{2}$, $\alpha_3 = \frac{\lambda}{2}$ but in this formula, μ cannot be regarded as one of the Lamé constants. [12] In [DSW93, Ste94a, Ste97, FCS97, DFC98, EDV98] the abbreviation μ_c is used while in [Gra03] it is $\mu_c = \alpha$ and $\mu_c = G_c$ in [IW98]. [13]

2.2.2 The elastic curvature energy

For the curvature term, to be specific, we assume the general form

$$W_{\mathrm{curv}}(\mathfrak{K}) = \mu\frac{L_c^{1+p}}{12}\,(1+\alpha_4\,L_c^q\,\|\mathfrak{K}\|^q)\,\left(\alpha_5\,\|\,\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\,\|\,\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}, \tag{2.2.7}$$

where L_c is setting an internal length scale with units of length, $\alpha_4 \geq 0, p > 0, q \geq 0$ are additional material constants, the factor $\frac{1}{12}$ only for convenience and $\alpha_5 > 0, \alpha_6, \alpha_7 \geq 0$ as a minimal requirement. We mean $\mathrm{tr}\,[\mathfrak{K}]^2 = \|\mathrm{tr}\,[\mathfrak{K}]\|^2$ by abuse of notation. This choice for W_{curv} does not presuppose any knowledge of the magnitude of curvature [14] in the material and is non-degenerate in the origin $\|\mathfrak{K}\| = 0$, which is not essential however.

In [FCS97, DFC98] the following set of parameters $(\alpha_4, \alpha_5, \alpha_6, \alpha_7, p) = (0, \beta, \gamma, \alpha, 1)$ is used, while in [BGdB98, EDV98] the reduced set $(\alpha_4, \alpha_5, \alpha_6, \alpha_7, p) = (0, \mu, \mu, \alpha, 1)$ is taken and $(\alpha_4, \alpha_5, \alpha_6, \alpha_7, p) = (0, \gamma, \delta, \beta, 1)$ is used in [Gra03]. [15]

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[9] Note that $\mathfrak{K}^i = \overline{R}^T\nabla(\overline{R}.e_i) \notin \mathfrak{so}(3,\mathbb{R})$. Another representation of $\mathfrak{K}$ is given by $\overline{\mathfrak{K}} := \left(\overline{R}^T\partial_x\overline{R}, \overline{R}^T\partial_y\overline{R}, \overline{R}^T\partial_z\overline{R}\right)$. Since $\partial_x(\overline{R}^T\overline{R}) = 0$ for $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ it holds that $\overline{\mathfrak{K}} \in \mathfrak{so}(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R})$. It is therefore possible to base all considerations of curvature on a more compact expression like $\hat{\mathfrak{K}} := \left(\mathrm{axl}(\overline{R}^T\partial_x\overline{R})|\,\mathrm{axl}(\overline{R}^T\partial_y\overline{R})|\,\mathrm{axl}(\overline{R}^T\partial_z\overline{R})\right) \in \mathbb{M}^{3\times3}$. This is the traditional approach, see e.g. [San99, FBC00, Gra03]. We do not use $\hat{\mathfrak{K}}$ since it looses its advantages over $\mathfrak{K}$ if we want to consider **micromorphic** extensions of the theory, e.g. if we would allow for $\overline{R} \in \mathrm{SL}(3,\mathbb{R})$. By extending the theory to multiplicative plasticity it becomes apparent that $\mathfrak{K}$ is a natural representation.

[10] The complete structure for anisotropic infinitesimal formulations has been given in [Kes64].

[11] Hencky-type energies defined on $\ln\overline{U}$ are useless, since $\overline{U} \neq \mathrm{PSym}$ in general. The stretch part could depend in principle on $C = \overline{U}^T\overline{U}$, but would then fail to be altogether quadratic in $\overline{U}$. The same argument excludes a dependence on U. In addition, a possible coupling between $\overline{U}$ and $\mathfrak{K}$ for centrosymmetric bodies can be ruled out [Now86, p.14].

[12] A simple definition of the Lamè constants in micropolar elasticity is that they should coincide with the classical Lamé constants for symmetric situations. Equivalently, they are obtained by the classical formula $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, where E and ν are uniquely determined from uniform traction. Eringens nomenclature unfortunately led to some confusion.

[13] In the remainder, from a modelling point of view, we need to carefully distinguish between micropolar moduli for microscopic specimens and effective moduli for macroscopic specimens, depending on the application aimed at.

[14] The following question merits attention: Is it experimentally possible to determine not only the value of the internal length scale L_c but also to determine the relevant exponents p, q independent of mathematical convenience. Dispersion experiments are the obvious candidate.

[15] Note that this identification is based on the representation of $\mathfrak{K}$ in terms of the axial representation $\hat{\mathfrak{K}}$. All results in the mathematical section hold without modification for $\hat{\mathfrak{K}}$ as well.

The form (2.2.7) is motivated by consistency with an expansion for a corresponding shell theory. But care has to be exerted in the finite regime: W_{curv} should preferably be **coercive** in the sense that we impose pointwise

$$\exists c^+ > 0 \,\exists r > 1 : \;\forall \mathfrak{K} \in \mathfrak{T}(3) : \quad W_{\text{curv}}(\mathfrak{K}) \geq c^+ \,\|\mathfrak{K}\|^r , \tag{2.2.8}$$

or less demanding

$$\exists r > 1 : \quad \frac{W_{\text{curv}}(\mathfrak{K})}{\|\mathfrak{K}\|^r} \to \infty \quad \text{as } \|\mathfrak{K}\| \to \infty \,, \tag{2.2.9}$$

which implies necessarily $\alpha_6 > 0$ in (2.2.8). This is at variance with the infinitesimal case (2.4.60), where $\alpha_6 = 0$ is still acceptable. A major step forward in the subsequent mathematical treatment would be to show that giving up (2.2.9), i.e. $\alpha_6 = 0$, still yields a well posed geometrically exact finite micropolar theory.

If it is known in advance that the curvature remains small, i.e. $L_c \cdot \|\mathfrak{K}\| \ll 1$, then $\alpha_4 = 0$, $p = 1$ may be a reasonable choice [GT01, BGdB98, EDV98, FCS97, Ste94a] but we will see that this leads to a loss of control (in the critical case $\mu_c = 0$) in the corresponding finite boundary value problem that can be overcome by taking $\mu_c > 0$, which will be seen below. Consistent with this observation (but not based on), in [Sch62, Bes74, San99] the parameter μ_c has been set to $\mu_c = \mu$ such that

$$W_{\text{mp}}(\overline{U}) = \mu \,\|\overline{U} - 1\!\!1\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\overline{U} - 1\!\!1\right]^2 , \tag{2.2.10}$$

superficially coinciding with the functional form of the classical free energy defined on the positive definite right stretch tensor U. In [Ste94a, Ste94b] the strain energy

$$W_{\text{mp}}(\overline{U}) = \mu \,\langle \frac{\overline{U}}{\det[\overline{U}]^{1/3}} - 1\!\!1, 1\!\!1\rangle + h(\det[\overline{U}]) \tag{2.2.11}$$

is proposed. Upon linearization, this corresponds as well to the choice $\mu = \mu_c$.

2.2.3 The balance equations

For the choices we have made we note the resulting material form of the field equations on the reference configuration (with $\alpha_4 = 0$, $p = 1$) which can be obtained after some algebraic manipulations.

$$\begin{aligned} 0 &= \operatorname{Div}\left(S_1(F, \overline{R}) + 2\,\mu_c\, \overline{R}\operatorname{skew}(\overline{R}^T F)\right) + f_{\mathbb{R}^3}, \qquad \textbf{force balance} \\ 0 &= \operatorname{skew}(D_{\overline{U}} W_{\text{mp}}(\overline{U})\,\overline{U}^T) + \operatorname{skew}\left(\overline{R}^T \operatorname{Div}\left[\overline{R}\, D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K})\right]\right) + \operatorname{skew}\left(D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K}) \mathfrak{K}^T\right) \\ &\quad + \operatorname{skew}(\overline{R}^T M)_{\mathbb{M}^{3\times 3}}, \qquad \textbf{angular momentum balance}\,, \end{aligned} \tag{2.2.12}$$

where S_1 is the first Piola-Kirchhoff stress (for $\mu_c = 0$) with the functional form

$$S_1(F, \overline{R}) = \overline{R}\left[\mu(F^T\overline{R} + \overline{R}^T F - 2 1\!\!1) + \lambda \operatorname{tr}\left[F^T\overline{R} - 1\!\!1\right] 1\!\!1\right] , \tag{2.2.13}$$

as in [Nef03a, (P3)] and $D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K})$ is the material **micropolar moment tensor** (or **couple-stress tensor**). A similar form of the unconventional[16] balance of angular momentum equation has been given in [Cap89, p.63]. In our variationally based development, the balance equations will not play a prominent role.

2.2.4 Constitutive consequences of the value for the Cosserat couple modulus

Looking at (2.2.6) with $\mu_c > 0$ we see that the implication of this choice at a first glance is an inoccuous rise in the macroscopic elastic strain energy $W_{\text{mp}}(\overline{U})$ if $\overline{R} \neq \operatorname{polar}(F)$, but $\overline{R}$ is generically assumed to be independent of $\operatorname{polar}(F)$. The choice $\mu_c > 0$ acts like a local 'elastic spring' between both continuum rotations and microrotations.

[16] because we have not transformed the tensor equation into a related vector format, which is usually preferred. Following [Cap89] we can identify an external volume couple b_c in the equilibrium vector-format with $\operatorname{axl}(\operatorname{skew}(\overline{R}^T M))$. Then b_c is a volume couple which is not a dead load. We note that a term $\operatorname{skew}\left(D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K})\, \mathfrak{K}^T\right)$ does not directly appear in derivations based on $\hat{\mathfrak{K}}$ since e.g. $\hat{\mathfrak{K}}^1 = \operatorname{axl}(\overline{R}^T \partial_x \overline{R})$ and variation along the one-parameter group introduced in (2.9.154) yields

$$\delta\hat{\mathfrak{K}}^1 = \operatorname{axl}((A\overline{R})^T \partial_x \overline{R} + \overline{R}^T \partial_x[A\overline{R}]) = \operatorname{axl}(-\overline{R}^T A \partial_x \overline{R} + \overline{R}^T(\partial_x A)\overline{R} + \overline{R}^T A\partial_x \overline{R}) = \operatorname{axl}(\overline{R}^T(\partial_x A)\overline{R})\,.$$

This is not at variance with $(2.2.12)_2$ since differentiation is carried out differently. Observe that $\operatorname{skew}\left(D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K})\, \mathfrak{K}^T\right) = 0$ if $\alpha_5 = \alpha_6$, $\alpha_7 = 0$, i.e. if couple stresses are proportional to the curvature tensor.

Let us consider the mathematical implications of $\mu_c = 0$ and $0 < \mu_c \le \mu$, respectively, in more detail. It is readily verified that for the elasticity tensors (differentiating the stretch energy $W_{\mathrm{mp}}(\overline{R}^T F)$ at fixed $\overline{R}$ w.r.t. F)

$$\begin{aligned} \mu_{\mathbf{c}} > 0 \Rightarrow \quad & \forall\, H \in \mathbb{M}^{3\times 3}: \quad D_F^2 W_{\mathrm{mp}}(\overline{R}^T F).(H,H) \ge 2\,\mu_c\,\|H\|^2 \\ \mu_{\mathbf{c}} = 0 \Rightarrow \quad & \forall\, H \in \mathbb{M}^{3\times 3}: \quad D_F^2 W_{\mathrm{mp}}(\overline{R}^T F).(H,H) \ge 2\,\mu \|\frac{1}{2}(\overline{R}^T H + H^T \overline{R})\|^2 . \end{aligned} \tag{2.2.14}$$

Hence the choice $\mu_c > 0$ leads to **uniform convexity** of $W_{\mathrm{mp}}(\overline{R}^T F)$ w.r.t. F and **unconditional elastic stability** on the macroscopic level: regardless of what distribution of microrotations $\overline{R}(x)$ is given, the macroscopic equation of balance of linear momentum is uniquely solvable and this equation is insensible to any deterioration of the spatial features of the microstructure. Uniform convexity is difficult to accept from a constitutive point of view since it is impossible for a geometrically exact description in the framework of a classical macroscopic continuum but clear from the above discussion: the additional elastic spring between micro- and continuum rotation extremely rigidifies the material and completely changes the type of the mathematical boundary value problem compared with the classical finite theory.[17]

Fortunately, such a far reaching unsatisfactory conclusion does not hold for $\mu_c = 0$, in which case we have

$$D_F^2 W_{\mathrm{mp}}(\overline{R}^T F).(\xi\otimes\eta, \xi\otimes\eta) = \mu\left(\|\overline{R}^T \xi\otimes\eta\|^2 + \langle \overline{R}^T \xi\otimes\eta, \eta\otimes\overline{R}^T\xi\rangle\right) = \mu\left(\|\overline{R}^T \xi\otimes\eta\|^2 + \langle \overline{R}^T \xi, \eta\rangle^2\right),$$

which shows the physically much more appealing inequality

$$D_F^2 W_{\mathrm{mp}}(\overline{R}^T F).(\xi\otimes\eta, \xi\otimes\eta) \ge \mu\,\|\xi\|^2\cdot\|\eta\|^2\,, \tag{2.2.15}$$

expressing nothing but uniform **Legendre-Hadamard ellipticity** of the acoustic-tensor with ellipticity constant μ **independent** of $\overline{R}$. The Legendre-Hadamard condition has the most convincing physical basis [Ant95, p.461] in that it implies the reality of wave speeds and the Baker-Ericksen inequalities (stress increases with strain, [MH83, p.19]). The choice $\mu_c = 0$ leads to the strain energy density proposed in [Nef03a, (P3)] and [NW03, M1] if the appearing independent viscoelastic rotations there are identified with the independent elastic Cosserat microrotations here. [18]

2.2.5 The Boltzmann axiom

In the absence of volume couples and curvature terms, i.e. without internal length scale, $L_c = 0$, the second equation in (2.2.12) reduces to the classical symmetry condition [SFH92, (6)], the so called **Boltzmann axiom**,

$$D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T \in \mathrm{Sym} \Leftrightarrow \mathrm{skew}\left(D_F W(\overline{R}^T F)\,F^T\right) = 0 \Leftrightarrow S_2 := F^{-1} S_1(F,\overline{R}) \in \mathrm{Sym}, \tag{2.2.16}$$

postulating the symmetry of the second Piola-Kirchhoff stress S_2 and we note that trivially $\left[\overline{U}\in\mathrm{Sym} \Leftrightarrow \overline{R} = \mathrm{polar}(F)\right] \Rightarrow D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$. However, for the converse we state a first result:

Lemma 2.2.1 (Limit rotations with zero internal length scale)
Let W_{mp} be defined as in (2.2.6). If $\alpha_1 = \mu_c$ and $\mathrm{tr}\left[\overline{U}\right] < 3 + \frac{2\mu_c}{\alpha_3}$ then

$$D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T \in \mathrm{Sym} \Rightarrow \left[\overline{U}\in\mathrm{Sym} \Leftrightarrow \overline{R} = \mathrm{polar}(F)\right]. \tag{2.2.17}$$

Otherwise, $D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$ alone does not imply $\overline{U}\in\mathrm{Sym}$. In other words: symmetry of the Cauchy stresses $T = \frac{1}{\det[F]} F\,S_2\,F^T$ does not imply that microrotations coincide with continuum rotations.

Proof. The proof is given in (2.8.1). This discrepancy between the fulfilment of the Boltzmann axiom and the symmetry of the microstretch $\overline{U}$ does not appear in the infinitesimal linear case, see (2.4.62). ■

[17] In the analytical section we will see that $\mu_c > 0$ implies that $\varphi \in W^{1,1}(\Omega,\mathbb{R}^3)$ irrespective of $\overline{R}$, thus **excluding fracture**.

[18] The preferred value $\mu_c = 0$ for the macroscopic case can as well be motivated by the following consideration: Consider the Green strains $F^TF - 1\!\!1 = (\overline{U} - 1\!\!1)^T(\overline{U} - 1\!\!1) + 2\,\mathrm{sym}(\overline{U} - 1\!\!1)$. Therefore $\frac{\mu}{4}\|F^TF - 1\!\!1\|^2 = \mu\|\,\mathrm{sym}\,\overline{U} - 1\!\!1\|^2 + O(\|\overline{U} - 1\!\!1\|^3)$. Hence $\mu_c = 0$ provides the correct first order approximation to a classical St. Venant-Kirchhoff material. With $\mu_c = 0$ we exclusively recover the fact of the classical continuum theory that W isotropic implies symmetry of the Biot stress tensor: $D_U W(U) \in \mathrm{Sym}$. If we expand $\overline{R} = 1\!\!1 + \overline{A} + \ldots$ with $\overline{A} \in \mathfrak{so}(3)$ and write $F = 1\!\!1 + \nabla u$, then the Cosserat effects dissappear to first order for $\mu_c = 0$. In this sense, $\mu_c = 0$ is close to classical elasticity.

We mention that an argument relating to the general case of W_{mp} taken as an isotropic scalar valued function of $\overline{U}$ has been given e.g. in [San99, p.29] and [SB95]. No conditions on the coefficients or the magnitude of $\operatorname{tr}[\overline{U}]$ are involved, which raises some questions. However, the conclusion in [San99, p.29] is true in some cases [19], which may be seen for

$$\begin{aligned} W_{\mathrm{mp}}(\overline{U}) &= \mu\|\overline{U}-1\!\!1\|^2+\mu\left(\det[\overline{U}]+\frac{1}{\det[\overline{U}]}-2\right)^2=\mu\,\|\overline{U}-1\!\!1\|^2+\mu\left(\det[F]+\frac{1}{\det[F]}-2\right)^2\\ D_{\overline{U}}W^{\mathrm{shear}}_{\mathrm{mp}}(\overline{U})\cdot\overline{U}^T &= 2\mu\,\left(\overline{U}-2 1\!\!1\right)\overline{U}^T\,. \end{aligned} \tag{2.2.18}$$

Since $\det[\overline{U}]=\det[\overline{R}^T F]=\det[F]$ is independent of $\overline{R}$, balance of angular momentum is only affected through the shear contribution and independent of the volumetric response. Therefore the symmetry condition in the second equation in (2.2.12) reduces to $2\mu\,\overline{U}\,\overline{U}^T-2\mu\,\overline{U}^T\in\mathrm{Sym}$, which implies already $\overline{U}\in\mathrm{Sym}$. The same reasoning applies to (2.2.11). Incidentally, this could be a reasonable assumption for a generalized finite Cosserat micropolar theory as well: why should the microrotations $\overline{R}$ affect the volumetric response of the material on the macroscopic level at all? We would be led to assume that the stretch energy has the form

$$W_{\mathrm{mp}}(\overline{U})=W^{\mathrm{shear}}_{\mathrm{mp}}(\overline{U})+h(\det[\overline{U}])\,, \tag{2.2.19}$$

decoupling shear behaviour from volumetric response.

The result of Lemma 2.2.1 is noteworthy: It shows that **symmetry** of $\overline{U}$ is an **independent** additional **assumption** generally not implied by balance of angular momentum (2.2.5) in the absence of an internal length scale (arbitrary large samples) in the finite regime.

2.2.6 Classification of elastic Cosserat micropolar models

Let us summarize at this stage the proposed geometrically exact finite elastic Cosserat micropolar model: the task is to find $(\varphi,\overline{R})\in\mathbb{R}^3\times\mathrm{SO}(3,\mathbb{R})$ such that

$$\begin{aligned} &\int_\Omega W_{\mathrm{mp}}(\overline{U})+W_{\mathrm{curv}}(\mathfrak{K})-\langle f,\varphi\rangle-\langle M,\overline{R}\rangle\,\mathrm{dV}-\int_{\Gamma_S}\langle N,\varphi\rangle\,\mathrm{dS}-\int_{\Gamma_C}\langle M_c,\overline{R}\rangle\,\mathrm{dS}\mapsto\min\,.\ \text{w.r.t.}\ (\varphi,\overline{R}),\\ &\overline{U}=\overline{R}^TF,\quad F=\nabla\varphi,\quad \mathfrak{K}=\overline{R}^T\mathrm{D_x}\overline{R},\quad \overline{R}_{|_\Gamma}=\overline{R}_{\mathrm{d}},\quad \varphi_{|_\Gamma}=g_{\mathrm{d}}\\ &W_{\mathrm{mp}}(\overline{U})=\mu\,\|\operatorname{sym}(\overline{U}-1\!\!1)\|^2+\mu_c\,\|\operatorname{skew}(\overline{U})\|^2+\frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\overline{U}-1\!\!1)\right]^2\\ &W_{\mathrm{curv}}(\mathfrak{K})=\mu\frac{L_c^{1+p}}{12}\,\left(1+\alpha_4\,L_c^q\,\|\mathfrak{K}\|^q\right)\left(\alpha_5\,\|\operatorname{sym}\mathfrak{K}\|^2+\alpha_6\,\|\operatorname{skew}\mathfrak{K}\|^2+\alpha_7\operatorname{tr}[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}\,. \end{aligned} \tag{2.2.20}$$

In [Ste97] a classification of isotropic micropolar theories is given. We refer to the geometrically exact case therein and put our models in this framework at the same time extending it.

0. **rigidity in torsion**: $\mu_c=\infty$, $L_c=\infty$. Classical infinitesimal elasticity in tension. $L_c=\infty$ keeps the microrotations constant and $\mu_c=\infty$ implies that continuum- and microrotations coincide.

1. **gradient type, constrained Cosserat micropolar theory** (or **indeterminate couple-stress theory**, a special case of an **elastic material of grade two**,[Tou62, Tou62, Min64, Gri60, Koi64, Eri68, Cap85]): $\mu_c=\infty$, $\mathfrak{K}=\operatorname{polar}(\nabla\varphi)^T\mathrm{D_x}\operatorname{polar}(\nabla\varphi)$, $\overline{U}=U=\operatorname{polar}(\nabla\varphi)^T\nabla\varphi$, variation of the action functional I only with respect to φ: the field equations are of fourth order, microrotations coincide with macrorotations, local contribution to the infinitesimal Cauchy stress tensor σ^{loc} and local contribution[20] to the finite second Piola-Kirchhoff stress tensor $S_2^{\mathrm{loc}}:=F^{-1}\,D_FW_{\mathrm{mp}}(\overline{U})$ are **symmetric**, "trièdre caché" in the original Cosserat terminology [CC09, p.30], only the non-local part (called the hyperstress) is responsible for the overall antisymmetric stresses. The antisymmetric part of the total stress is therefore not determined by the local value of the deformation field alone.

[19] It would be useful to obtain general necessary and sufficient conditions on the free energy function W such that $[\overline{U}\in\mathrm{Sym}\Leftrightarrow\overline{R}=\operatorname{polar}(F)]\ \Leftrightarrow\ D_{\overline{U}}W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T\in\mathrm{Sym}$. A further problem is then to characterize the classical continuum as a certain limit of the finite Cosserat model for vanishing internal length. We might want to conjecture that the classical Boltzmann continuum is the Γ-limit [Mas92] of (2.2.2) for $L_c\to0$ and $\mu_c=0$, with appropriate boundary conditions $\overline{R}_{|_\Gamma}=\operatorname{polar}(\nabla\varphi)$ preventing the situation in (2.8.85). A proof of this conjecture is beyond the scope of this investigation.

[20] For us, stress denotes the sum of local and nonlocal stresses in the force balance equations. Usually, what we call local stress is denoted simply with (constitutive) stress whereas our nonlocal stress is said to be the hyperstress in the case of the constrained gradient model.

2. **regularized gradient theory**: $\mu_c > 0, \mu_c \to \infty$, $\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R}$, independent variation with respect to $(\varphi, \overline{R})$, field equations of second order, independent microrotations, a subclass of case (3), $\mu_c \to \infty$ as a penalty parameter, yields in the limit constrained gradient theory (indeterminate couple stress theory), provides a 'cheap' numerical approximation to (1).

3. **first order Cosserat micropolar**: $\mu_c > 0$, $\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R}$, independent variation with respect to $(\varphi, \overline{R})$, field equations of second order, independent microrotations, strong local coupling of first order between continuum rotations and microrotations, infinitesimal (constitutive) Cauchy stress tensor σ and finite second Piola-Kirchhoff stress tensor $S_2 := F^{-1}\, D_F W_{\mathrm{mp}}(\overline{U})$ are **non-symmetric**, "trièdre mobiles" in the original Cosserat terminology. Appropriate for rather rigid microscopic specimens, fracture excluded since $\mu_c > 0$.

3.1 **traditional infinitesimal Cosserat micropolar**: $\mu_c > 0$, $\kappa = \mathrm{D_x}\overline{A}$, $\overline{R} = \exp(\overline{A})$, $\overline{A} \in \mathfrak{so}(3)$, independent variation w.r.t. displacement and infinitesimal microrotations $(u, \overline{A})$, linearization of (3). The next three cases are our own contribution:

4. **second order Cosserat micropolar** (or **relaxed micropolar theory**): $\mu_c = 0$, $\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R}$, independent variation with respect to $(\varphi, \overline{R})$, field equations of second order, independent microrotations, weak non-local coupling of second order, infinitesimal (linearized) Cauchy stress tensor σ is still **symmetric**, second Piola-Kirchhoff stress tensor $S_2 := F^{-1} D_F W_{\mathrm{mp}}(\overline{U})$ is **non-symmetric**. The antisymmetric part of the stresses is determined. Appropriate for macroscopic specimens, in principle allowing for **fracture**.

4.1 **second order consistent Cosserat micropolar**: as case (4), but independent Dirichlet boundary condition for the microrotations $\overline{R}_{|_\Gamma} = \overline{R}_{\mathrm{d}}$ replaced by **consistent coupling** requirement $\overline{R}_{|_\Gamma} = \mathrm{polar}(\nabla\varphi)_{|_\Gamma}$.

4.2 **finite elasticity with free rotations and microstructure**: $\mu_c = 0$, $L_c = 0$, independent variation w.r.t. $(\varphi, \overline{R})$, no internal length scale. **Symmetry** of the second Piola-Kirchhoff tensor S_2 is a **local side condition** coming from balance of angular momentum which does not imply that $\overline{R} = \mathrm{polar}(\nabla\varphi)$. Local minimization of rotations. Weak solutions of classical finite elasticity are automatically stationary solutions of this minimization problem. In this sense encompassing classical finite elasticity.

5. **classical finite elasticity**: $\mu_c = 0$, $L_c = 0$, independent variation only w.r.t. φ, no internal length scale. The second Piola-Kirchhoff tensor S_2 is automatically **symmetric**.

6. **classical infinitesimal elasticity**: $\mu_c = 0$, $L_c = 0$, variation w.r.t. φ, no internal length scale. The infinitesimal Cauchy stress σ is **symmetric**.

One may be inclined to think that case (1) is closest to classical elasticity. This is not true. To the contrary, the influence of the curvature part on the deformation is much more pronounced since the spatial variation of the continuum rotations is directly penalized. Such a model tends to systematically maximize the material length scale effects, see e.g. [Eri99, p.222] where stress concentration factors are computed for the different cases based on the infinitesimal theory. Use of (1) as a model in its own right has been put into question on theoretical grounds [Eri68, p.698] and rejected by Koiter [Koi64] as well who, however, prematurely concluded that couple stresses altogether played no prominent role. If such a model is intended to approximate classical linear elasticity, then the appearing length scale L_c must be chosen significantly smaller than the length scale L_c, which appears in the Cosserat micropolar models with independent rotations.

If we assume that classical infinitesimal elasticity is a correct approximation to material behaviour under very small loads but that couple stresses may nevertheless occur in a material [TN65, p.398], we prefer for applications within the elastic range defined on a macroscopic level the case (4) of weak nonlocal coupling ($\mu_c = 0$) without excluding the other (more microscopically) cases from our mathematical analysis.

It is clear that if couple stresses are assumed to be second order effects[21], then they should not appear in the infinitesimal treatment in the first place which is exactly what we obtain for $\mu_c = 0$ subsequently. Experimental evidence [Eri99, p.165] suggests already that $\mu_c \approx 0.0039\,\mu$ for the infinitesimal theory, orders of magnitude smaller than the classical shear modulus. This value is consistent with results [DFC98] obtained from calculations on discrete networks with rigid substructure. In the same paper, it is shown that if the representative volume element is increased (macroscopic case) then $\mu_c \to 0$ while the geometrical size effect is still present.[22]

[21] "In classical elasticity couple stresses are to be interpreted as a non-local effect intimately connected with the range of the atomic forces. The couple stresses are of a higher order in this range than force-stresses and can therefore usually be neglected."[HK65]

[22] In the same volume [DDC98] a numerical investigation of contact couples for granular media arrives at the conclusion: "The effect of contact couples appears only by a second order term which is not considered by the (infinitesimal) Cosserat approach." (infinitesimal) my addition. In [Bar94, Fig.2] it is shown that for strains up to 4 percent the effects of particle rotations in idealized granular materials are practically absent, whereas for higher strains particle rotation significantly decreases the failure stress. The role of particle rotations is especially important in shear bands.(ibidem)

The case of $\mu_c > 0$ might be however, suitable for computations on a microscale where internal material length scales are of the order of the geometrical dimensions of the specimen. This is e.g. the case in models for single crystals where the microrotations conceptually should closely follow the lattice vectors for a proper physical definition of them. Then $\mu_c \gg 1$ may be advisable since we do not expect fracture.

We can gain some feeling as regards the influence of the Cosserat couple modulus μ_c and the characteristic length L_c on the finite Cosserat model by looking at extremal values: $L_c = 0$ corresponds to the physically possible limit case of arbitrary large samples, $L_c = \infty$ corresponds to the limit case of arbitrary small samples.

	$L_c = 0$	$L_c = 0.1$	$L_c = \infty$
$\mu_c = 0$	**finite elasticity with free rotations and microstructure**, case (4.2), encompassing classical **finite elasticity** model, case (5)	new **second order Cosserat** model, case (4), linear: case (6)	classical **infinitesimal elasticity** model, case (6)
$\mu_c = \frac{\mu}{2}$	in certain! cases: classical **finite elasticity** model, case (5)	finite **first order Cosserat** model, case (3), linear: case (3.1)	**linear elasticity** in **torsion** (7), **inconsistent** torsional tangent **modulus**, classical **infinitesimal elasticity** in **tension**
$\mu_c = \infty$	classical **finite elasticity** model, case (5)	traditional **indeterminate couple stress** theory, gradient constrained model, case (1)	totally **rigid** behaviour in **torsion**, case (0), classical **infinitesimal elasticity** in tension

The only row where each entry does not conflict with experiments is the first one. It is to be observed that $L_c = \infty$ linearizes the behaviour with respect to a given rigid configuration. [23]

For completeness we state the **finite gradient constrained Cosserat micropolar (indeterminate couple stress theory)** problem as well, formally corresponding to $\mu_c = \infty$. Given the boundary value $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$, $\nabla g_{\mathrm{d}} \in \mathrm{GL}^+(3, \mathbb{R})\, a.e.$ we look for the deformation $\varphi : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$ satisfying

$$
\begin{aligned}
I(\varphi) &= \int_\Omega W_{\mathrm{mp}}(\mathrm{polar}(\nabla\varphi)^T \nabla\varphi) + W_{\mathrm{curv}}(\mathrm{polar}(\nabla\varphi)^T \mathrm{D_x}\,\mathrm{polar}(\nabla\varphi)) - \langle f, \varphi\rangle - \langle M, \mathrm{polar}(\nabla\varphi)\rangle\,\mathrm{dV} \\
&\quad - \int_{\Gamma_S} \langle N, \varphi\rangle\,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \mathrm{polar}(\nabla\varphi)\rangle\,\mathrm{dS} \mapsto \min . \text{ w.r.t. } \varphi, \\
W_{\mathrm{mp}}(U) &= \mu\,\|U - 1\!\!1\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[U - 1\!\!1]^2 \\
S_2^{\mathrm{loc}} &= F^{-1} D_F W_{\mathrm{mp}}(U) \in \mathrm{Sym}\,,\ \text{constitutive stress} \\
W_{\mathrm{curv}}(\mathfrak{K}) &= \mu\frac{L_c^{1+p}}{12}\,(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}\|^q)\left(\alpha_5\,\|\,\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\,\|\,\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}}\,, \\
\varphi_{|_\Gamma} &= g_{\mathrm{d}}(t)\,, \quad \mathrm{polar}(\nabla\varphi)_{|_\Gamma} = \mathrm{polar}(\nabla g_{\mathrm{d}})_{|_\Gamma}\,.
\end{aligned}
\tag{2.2.21}
$$

Balance of angular momentum is, as the consequence of invariance of the action under rigid rotations, automatically satisfied and is but a defining equation for the moment stress tensor and not recorded here. Observe that the first order, local contribution $W_{\mathrm{mp}}(U)$ is **uniformly convex** in U, the continuum right stretch tensor, but that $F \mapsto W_{\mathrm{mp}}(U)$ fails to be Legendre-Hadamard elliptic with respect to F and is as such **not quasiconvex** but satisfies the **Baker-Ericksen inequalities**, cf. section 2.8.2. Whether or not the external couples M, M_c

[23] Uniform traction and uniform compression do not activate rotations, hence the classical identification of the Lamé constants is achieved independent of μ_c. Uniform traction alone allows already to determine the Young modulus E and the Poisson ratio ν [Cia88, p.126]. Contrary to [Gau82, p.411] we do not see the possibility to define a specific "micropolar Young modulus" or "micropolar Poisson ratio".

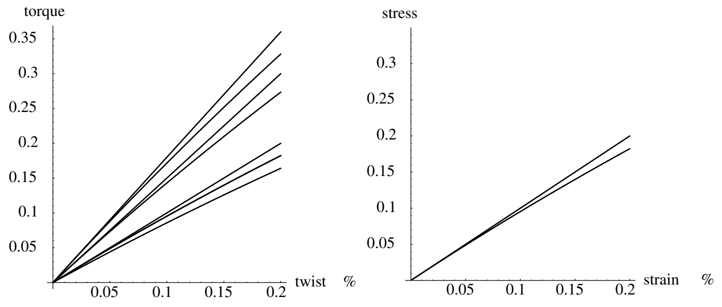

Figure 2.1: Left: Qualitative influence of μ_c and L_c on the **elastic** behaviour in **torsion**. From bottom to top: finite elasticity (5), second order Cosserat (4), infinitesimal elasticity (6), first order Cosserat (3), linearized first order Cosserat (3´), indeterminate couple stress theory (1), linearized indeterminate couple stress theory, rigid behaviour (vertical axes) (0). Right: finite elasticity and infinitesimal linear elasticity coincide in tension for ideally homogeneous bulk material. No Cosserat effects. If inhomogeneity is present, a small Cosserat effect will appear, lower curve. If $\mu_c > 0$, the elastic response in **torsion** would be **stiffer** than expected from calculations with linear elasticity based on material parameters obtained from measurements in tension, uniform traction and uniform compression already for arbitrary small twist. The smaller μ_c, the larger one may choose L_c in order to still approximate classical infinitesimal elasticity.

should be non-zero is a modelling choice. Observe as well that this special elastic material of grade two **does not** completely control the second derivatives of the deformation φ which shows, that a simple compactness argument does not suffice to overcome the nonlinearity and non-quasiconvexity in the first order stretch term. At first sight it is therefore not obvious why such a model can have a regularizing effect in the finite case.

Remark 2.2.2 (Consistent Dirichlet boundary coupling conditions for the rotations)
For all presented models with internal length scale the microrotations on the part of the boundary Γ can in principle be specified arbitrarily. This implies five degrees of freedom: 3 components of the deformation φ and two orthogonal vectors, the third vector of the rotation is then defined, we call this the **rigid Dirchlet case**. *However, if we want to describe a basically classical situation, where only $\varphi_{|\Gamma} = g_{\mathrm{d}|_\Gamma}$ is specified, we may as well impose a* **strong consistent coupling condition**

$$\overline{R}_{|\Gamma} = \mathrm{polar}(\nabla\varphi)_{|\Gamma}\,, \tag{2.2.22}$$

which prevents non-compatibility between micro- and macrorotations on the Dirichlet boundary Γ and allows for otherwise impossible homogeneous solutions. It leads to three essential degrees of freedom at the Dirichlet boundary and disposes us from the need to motivate any independent boundary condition for $\overline{R}$. In addition, the Boltzmann axiom is not violated on Γ. It is mathematically possible to leave the microrotations entirely free on Γ, however, this does not seem to correspond to any physical situation in a real body.

2.3 Non-dissipative Cosserat micropolar elasto-plasticity

2.3.1 Extension to finite micropolar elasto-plasticity

Now we extend the formulation of micropolar elasticity to cover finite plasticity. It should be clear that there exists various ways of obtaining such an extension, for an overview of the competing models we refer to the instructive survey article [FS03]. Incidentally, the Cosserats themselves [CC09, p.5] already envisaged the application of their general theory to plasticity and fracture. For conciseness we take $\alpha_5 = \alpha_6 = 1$, $\alpha_7 = 0$ for the curvature term in this part.

In a first preliminary step we consider the elastic problem defined over a transformed domain $\Omega_\xi = \Theta(\Omega)$ where Θ is a diffeomorphism. With respect to Ω_ξ we assume the micropolar decomposition

$$\nabla_\xi\varphi(\xi) = F_\xi = \overline{R}_\xi \cdot \overline{U}_\xi\,, \tag{2.3.23}$$

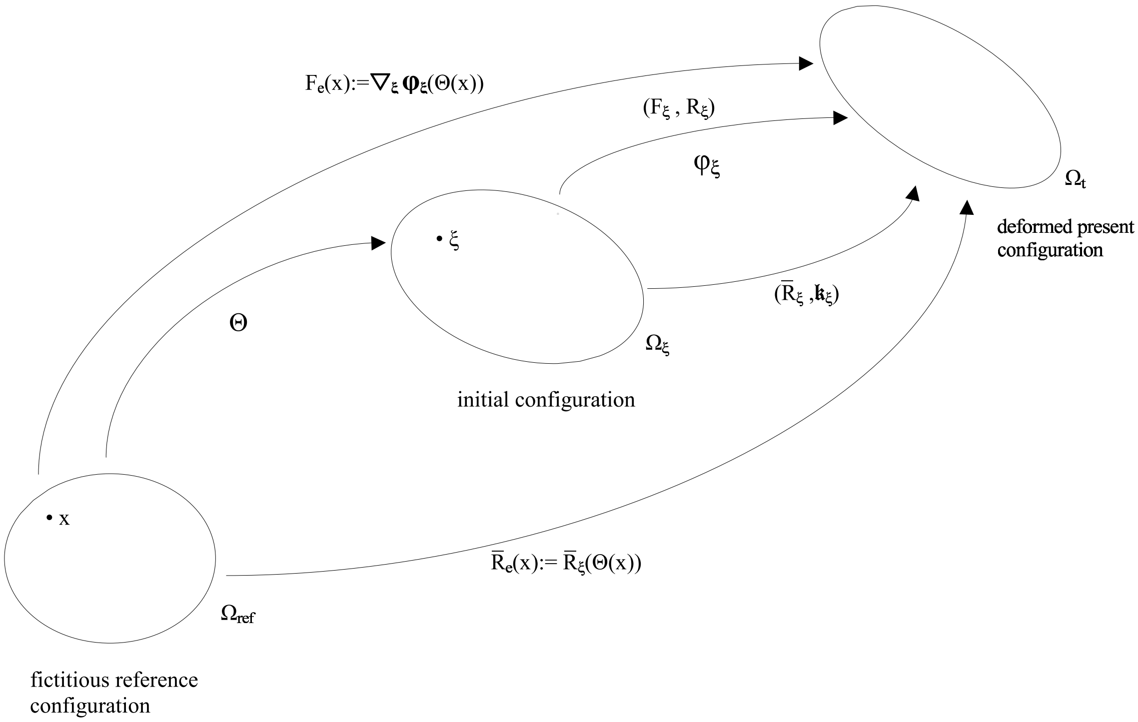

Figure 2.2: Schematic picture of the transformed elastic Cosserat kinematics.

such that the Cosserat problem on Ω_ξ reads

$$
\begin{aligned}
I_\xi(\varphi_\xi, \overline{R}_\xi) &= \int_{\Omega_\xi} W_{\mathrm{mp}}(\overline{R}_\xi^T \nabla_\xi \varphi_\xi) + W_{\mathrm{curv}}(\overline{R}_\xi^T \mathrm{D}_\xi \overline{R}_\xi) - \langle f_\xi, \varphi_\xi \rangle - \langle M_\xi, \overline{R}_\xi \rangle \, \mathrm{d}\xi \\
&\quad - \int_{\Gamma_{S,\xi}} \langle N_\xi, \varphi_\xi \rangle \, \mathrm{dS}_\xi - \int_{\Gamma_{C,\xi}} \langle M_{\xi,c}, \overline{R}_\xi \rangle \, \mathrm{dS}_\xi \mapsto \min . \text{ w.r.t. } (\varphi_\xi, \overline{R}_\xi), \qquad (2.3.24) \\
\overline{R}_{\xi|_{\Gamma_\xi}} &= \mathrm{polar}(\nabla g_\xi)_{|_{\Gamma_\xi}} , \quad \varphi_{\xi|_{\Gamma_\xi}} = g_\xi(\xi) \, .
\end{aligned}
$$

Now we transform the problem to the fictitious reference configuration Ω: the transformation of variables formula and Nansons relation yields

$$
\begin{aligned}
&\int_{\Omega} W_{\mathrm{mp}}(\overline{U}_\xi) \det[\nabla\Theta] + W_{\mathrm{curv}}(\mathfrak{K}_\xi) \det[\nabla\Theta] - \langle f, \varphi \rangle \det[\nabla\Theta] - \langle M, \overline{R}_\xi \rangle \det[\nabla\Theta] \, \mathrm{dV} \\
&\quad - \int_{\Gamma_S} \langle N, \varphi \rangle \, \| \operatorname{Cof} \nabla\Theta . \vec{n} \| \, \mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{R}_\xi \rangle \, \| \operatorname{Cof} \nabla\Theta . \vec{n} \| \, \mathrm{dS} \mapsto \min . \text{ w.r.t. } (\varphi, \overline{R}_\xi), \\
\overline{R}_{\xi_{|_\Gamma}} &= \mathrm{polar}(\nabla g_{\mathrm{d}} \cdot \nabla\Theta^{-1})_{|_\Gamma} , \quad \varphi_{|_\Gamma} = g_\xi(\Theta(x)) = g_{\mathrm{d}} \qquad (2.3.25) \\
W_{\mathrm{mp}}(\overline{U}_\xi) &= \mu \, \| \mathrm{sym}(\overline{U}_\xi - 1\!\!1) \|^2 + \mu_c \, \| \mathrm{skew}(\overline{U}_\xi) \|^2 + \frac{\lambda}{2} \operatorname{tr} \left[\mathrm{sym}(\overline{U}_\xi - 1\!\!1) \right]^2 \\
W_{\mathrm{curv}}(\mathfrak{K}_\xi) &= \mu \frac{L_c^{1+p}}{12} \, (1 + \alpha_4 \, L_c^q \, \| \mathfrak{K}_\xi \|^q) \, \| \mathfrak{K}_\xi \|^{1+p} \\
\nabla_\xi \varphi_\xi &= \nabla\varphi \cdot \nabla\Theta^{-1}, \quad \overline{U}_\xi = \overline{R}_\xi^T \nabla_\xi \varphi_\xi, \quad \nabla\Theta \in \mathrm{GL}^+(3, \mathbb{R}) \\
\mathfrak{K}_\xi &= \overline{R}_\xi^T \mathrm{D}_\xi \overline{R}_\xi = \overline{R}_\xi^T(\Theta(x)) \left[\mathrm{D}_{\mathrm{x}} [\overline{R}_\xi(\Theta(x))] \cdot \nabla\Theta^{-1} \right] .
\end{aligned}
$$

Conceptually, the plasticity model is obtained by **relaxing** the **compatibility** of $\nabla\Theta$: we replace $\nabla\Theta$ with F_p, identify $F_e(x)$ as $F_\xi = \nabla_\xi \varphi_\xi(\Theta(x))$, set $\overline{R}_e(x) := \overline{R}_\xi(\Theta(x))$ and need to supply a thermodynamical admissible flow rule for F_p. This is tantamount to consider directly the multiplicative decomposition of the elastic part of the deformation gradient

$$
F_e = \overline{R}_e \cdot \overline{U}_e \, , \qquad (2.3.26)
$$

which defines micropolar elastic rotations $\overline{R}_e$ and micropolar elastic stretch $\overline{U}_e$.

The thermodynamical admissible flow rule for F_p can be obtained as follows. Consider the rate of change of the elastic energy density only due to the time variation of the incompatible configuration F_p (variation of the "domain" of definition F_p): after some simple but tedious manipulations we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\left[W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] + W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p]\right] \\
&= \langle \det[F_p]\left(\overline{U}_e^T D_{\overline{U}_e} W_{\mathrm{mp}}(\overline{U}_e) - W_{\mathrm{mp}}(\overline{U}_e)\,1\!\!1\right), F_p \frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right]\rangle + \\
&+ \langle \det[F_p]\left(\mathfrak{K}_e^T D_{\mathfrak{K}_e} W_{\mathrm{curv}}(\mathfrak{K}_e) - W_{\mathrm{curv}}(\mathfrak{K}_e)\,1\!\!1\right), F_p \frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right]\rangle\,,
\end{aligned} \tag{2.3.27}$$

where it is understood that $\mathfrak{K}_e^T D_{\mathfrak{K}_e} W_{\mathrm{curv}}(\mathfrak{K}_e) = \sum_{i=1}^3 \mathfrak{K}_e^{i,T}\cdot \partial_{\mathfrak{K}_e^i} W_{\mathrm{curv}}(\mathfrak{K}_e^1, \mathfrak{K}_e^2, \mathfrak{K}_e^3)$. A sufficient condition for the second law of thermodynamics to be fulfilled [Nef03a] is to guarantee the **reduced dissipation inequality**

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] + W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p]\right] \le 0\,, \tag{2.3.28}$$

where $F, \overline{R}_e$ are held constant. This can be achieved by choosing the **left-invariant** flow rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right] &\in -F_p^{-1}\cdot \mathfrak{f}(\Sigma_E), \quad \Sigma_E = \Sigma_{E,\mathrm{mp}} + \Sigma_{E,\mathrm{curv}} \\
\Sigma_{E,\mathrm{mp}} &= \overline{U}_e^T D_{\overline{U}_e} W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] - W_{\mathrm{mp}}(\overline{U}_e)\det[F_p]\,1\!\!1 \\
\Sigma_{E,\mathrm{curv}} &= \mathfrak{K}_e^T D_{\mathfrak{K}_e} W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p] - W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p]\,1\!\!1\,,
\end{aligned} \tag{2.3.29}$$

where the **flow function** $\mathfrak{f} : \mathbb{M}^{3\times 3} \mapsto \mathcal{P}(\mathbb{M}^{3\times 3})$ governs the plastic evolution and must satisfy $\forall\, \Sigma :\ \langle \mathfrak{f}(\Sigma), \Sigma\rangle \ge 0$. Such a formulation can be called of **pre-monotone type** in the sense of the classification for infinitesimal elasto-plastic models in [Alb98].

Here Σ_E denotes the **total elastic Eshelby stress tensor** (the driving force behind evolving inhomogeneities in the reference configuration [Mau99]) which may be reduced to $\Sigma_M = F_e^T D_{F_e} W(F_e, R_e)$, the **elastic Mandel stress tensor** in case of a deviatoric flow rule which preserves the incompressibility constraint $\det[F_p] = 1$. The Eshelby stress tensor has two different contributions: $\Sigma_{E,\mathrm{mp}}$ due to macro-stretch and $\Sigma_{E,\mathrm{curv}}$ due to micro torsion-curvature.

In the case of a simple material a similar derivation of the flow rules for multiplicative elasto-plasticity based on the Eshelby tensor has been given in [Nef03b, Nef03a]. Note that the multiplicative decomposition [Lee69, Man73] of the deformation gradient into elastic and plastic parts

$$\nabla\varphi = F = F_e\cdot F_p\,, \tag{2.3.30}$$

is a by-product of the derivation. [24]

In the quasi-static setting we are thus led to study the following system of coupled partial differential and evolution equations for the finite deformation $\varphi : [0,T]\times\overline{\Omega}\mapsto \mathbb{R}^3$, the plastic deformation $F_p : [0,T]\times\overline{\Omega}\mapsto \mathrm{GL}^+(3,\mathbb{R})$ and the independent Cosserat elastic microrotation $\overline{R}_e : [0,T]\times\overline{\Omega}\mapsto \mathrm{SO}(3)$

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] + W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p] - \langle f, \varphi\rangle\det[F_p] - \langle M, \overline{R}_e\rangle\det[F_p]\,\mathrm{dV} - \int_{\Gamma_S}\langle N, \varphi\rangle\,\|\operatorname{Cof} F_p.\vec{n}\|\,\mathrm{dS} \\
&\qquad - \int_{\Gamma_C}\langle M_c, \overline{R}_e\rangle\,\|\operatorname{Cof} F_p.\vec{n}\|\,\mathrm{dS} \mapsto \min.\ \mathrm{w.r.t.}\ (\varphi, \overline{R}_e)\ \text{at constant}\ F_p, \\
&\overline{R}_{e_{|\Gamma}} = \begin{cases}\mathrm{polar}(\nabla g_{\mathrm{d}}\cdot F_p^{-1})_{|\Gamma}, \ \text{rigid} \\ \mathrm{polar}(\nabla\varphi\cdot F_p^{-1})_{|\Gamma}, \ \text{consistent}\end{cases}, \quad \varphi_{|\Gamma} = g_{\mathrm{d}}(t) \\
&W_{\mathrm{mp}}(\overline{U}_e) = \mu\,\|\mathrm{sym}(\overline{U}_e - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}_e - 1\!\!1)\right]^2 \\
&W_{\mathrm{curv}}(\mathfrak{K}_e) = \mu\frac{L_c^{1+p}}{12}\,\left(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}_e\|^q\right)\,\|\mathfrak{K}_e\|^{1+p}
\end{aligned} \tag{2.3.31}$$

[24] While we continue to use the term **multiplicative decomposition** and **intermediate configuration** it is rather an **elastic isomorphism** in the sense of [Ber98]. Some authors use P instead of F_p^{-1}, [CHM02]. Examples for classical finite plasticity formulations may be found in [Sim88, SO85, Mie95, Sim98, SH98, EGR90, CHM02, Mie00, OR99]. Different models have been compared numerically in [NW03]. Note that F_p is not a plastic strain but rather a relaxed configuration: in a neighbourhood of a point, F_p can be a rigid rotation, while the corresponding plastic strain $C_p - 1\!\!1 = F_p^T F_p - 1\!\!1$ vanishes.

$$
\begin{aligned}
&F_e = \nabla\varphi \cdot F_p^{-1}, \quad \overline{U}_e = \overline{R}_e^T F_e, \quad \mathfrak{K}_e = \overline{R}_e^T [\mathrm{D_x}\overline{R}_e(x) \cdot F_p^{-1}] \\
&\frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right] \in -F_p^{-1} \cdot \mathfrak{f}(\Sigma_E), \quad \Sigma_E = \Sigma_{E,\mathrm{mp}} + \Sigma_{E,\mathrm{curv}} \\
&\Sigma_{E,\mathrm{mp}} = \overline{U}_e^T D_{\overline{U}_e} W_{\mathrm{mp}}(\overline{U}_e) \det[F_p] - W_{\mathrm{mp}}(\overline{U}_e) \det[F_p]\, 1\!\!1 \\
&\Sigma_{E,\mathrm{curv}} = \mathfrak{K}_e^T D_{\mathfrak{K}_e} W_{\mathrm{curv}}(\mathfrak{K}_e) \det[F_p] - W_{\mathrm{curv}}(\mathfrak{K}_e) \det[F_p]\, 1\!\!1 \\
&F_p^{-1}(0) = F_{p_0}^{-1}, \quad F_{p_0} \in \mathrm{GL}^+(3,\mathbb{R}).
\end{aligned}
$$

To complete the phenomenological macroscopic elastic-viscoplastic Cosserat micropolar model we specify $\mathfrak{f} = \partial\mathcal{X}$ such that we retrieve the von Mises type 'incompressible' J_2 -visco-plasticity with **elastic domain** $\mathcal{E} := \{\Sigma_E \mid \|\operatorname{dev}(\operatorname{sym}\Sigma_E)\| \le \sigma_{\mathrm{y}}\}$ and yield stress σ_{y}. To this end we take as viscoplastic potential $\mathcal{X} : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ of generalized Norton-Hoff overstress type[25] the following function:

$$
\mathcal{X}(\Sigma_E) = \begin{cases} 0 & \Sigma_E \in \mathcal{E} \\ \frac{\bar{\sigma}_0}{(r+1)(k+1)\ \eta_p}\left(1 + \left[\frac{\|\operatorname{dev}(\operatorname{sym}\Sigma_E)\| - \sigma_{\mathrm{y}}}{\bar{\sigma}_0}\right]^{r+1}\right)^{k+1} - \frac{\bar{\sigma}_0}{(r+1)(k+1)\ \eta_p} & \Sigma_E \notin \mathcal{E}\,, \end{cases} \tag{2.3.32}
$$

where $\eta_p > 0$ is a relaxation time due to essentially plastic processes inside the grains, $r, k > 0$ and $\bar{\sigma}_0$ is a stress like material constant. This definition is consistent with the postulate of maximum plastic dissipation. The parameter r allows to adjust the smoothness of the flow rule when passing the elastic boundary. A typical range for k in engineering applications is $k \in \{0, \ldots, 80\}$. For $k \to \infty$ we recover formally ideal rate-independent plasticity. A simple calculation shows that the corresponding single valued subdifferential is given by

$$
\partial_\Sigma \mathcal{X}(\Sigma_E) = \frac{1}{\eta_p} \cdot \left(1 + \left[\frac{\|\operatorname{dev}\operatorname{sym}\Sigma_E\| - \sigma_{\mathrm{y}}}{\bar{\sigma}_0}\right]_+^{r+1}\right)^k \left[\frac{\|\operatorname{dev}\operatorname{sym}\Sigma_E\| - \sigma_{\mathrm{y}}}{\bar{\sigma}_0}\right]_+^r \frac{\operatorname{dev}\operatorname{sym}\Sigma_E}{\|\operatorname{dev}\operatorname{sym}\Sigma_E\|}\,. \tag{2.3.33}
$$

The resulting model (2.3.31) is as close as possible to classical macroscopic elasto-plastic models, notably we did **not** introduce any plastic microrotation $\overline{R}_p$ together with a multiplicative decomposition of microrotations $\overline{R} = \overline{R_e} \cdot \overline{R_p}$, nor did we split artificially a total curvature $\mathfrak{K}$ into elastic and plastic parts as has been proposed in [DSW93, Ste94a, IW98, GT01, Gra03]. Such a decomposition represents an additional modelling assumption not necessarily related to elastic Cosserat effects. [26]

It is clear that Σ_E will **not** be **symmetric** in general even under **isotropy** conditions. Thus the choice of $\operatorname{sym}(\Sigma_E)$ in the definition of the elastic domain $\mathcal{E}$ sets the **plastic spin** to **zero**, consistent with current classical isotropic macroscopic formulations for polycrystals. It is possible to incorporate hardening effects independent of the Cosserat framework in the standard, local phenomenological fashion.

We mention that for $\mu_c = 0$ we have

$$
\overline{R}_e \operatorname{skew}(D_{\overline{U}_e} W_{\mathrm{mp}}(\overline{U}_e)\, \overline{U}_e^T) \overline{R}_e^T = \operatorname{skew}\left(B_{\mathrm{approx}}\right)\,, \tag{2.3.34}
$$

i.e. the reaction stress in the Cosserat model (2.3.31) is the driving force of the **viscoelastic** evolution in [Nef03a, (P3)].

2.3.2 Elastic-viscoplastic Cosserat model for polycrystals with grain rotations

A formidable challenge for current research is to find tractable continuum models for crystalline materials at the same time capturing their physical essence and being geometrically exact. There are essentially two ways to proceed: either one starts from the better known single crystal case [CO92, OR99, ORS00] and computes a large array of single crystals in mutual contact or one enriches the classical description with new variables taking account of the microstructural evolution in an averaged sense. We follow the second line of thought.

In a polycrystal, single crystal grains are joined together along grain boundaries. The intergranular **grain boundary** is mainly **responsible** for the **elastic** and **viscous** response of the polycrystal while plastic effects are located predominantly inside each grain operating by slip and twinning along glide planes. The absence of any viscous grain boundary in single crystals explains why the plasticity of single crystals is traditionally

[25] In finite plasticity, the question whether or not the plastic flow has a gradient structure seems to be of minor importance as far as mathematical existence results are concerned in sharp contrast to the infinitesimal case [Alb98, HR99]. However, the very feasibility of a time-incremental variational formulation [Mie00, OR99, CHM02] is contingent upon the potential structure.

[26] It is motivated, though, by the experimental observation [FMAH94] that geometrical size effects are becoming increasingly important in the plastic range. These size effects are explained on a microscale as being due to dislocation interactions. It seems therefore more natural to account for them directly by incorporating a dislocation density $\operatorname{Curl} F_p$ into the model and providing non-local flow rules of parabolic type for F_p. We will not pursue this issue here.

modelled as rate-independent. Consistently, for a specimen made of a single crystal, relaxation effects are practically absent whereas the internal surface between grains, where frictional effects are dominant, increases for smaller grain size and leads to pronounced rate-dependent response already below a macroscopic yield limit [Nef03a]. Hence, a polycrystal is much more than a simple assembly of single crystals corroborating the fact that the small scale (single crystal) behaviour can be quite different from the bulk for non-linear heterogeneous material already for small loads.

In addition, depending on the size of the constituting grains, a polycrystal has different elasto-plastic properties. An account of the necessity for macroscopic problems to incorporate internal length scale effects into a model has been recently given in [WCZM02]. Polycrystalline copper has been made six times harder (apparent yield stress ${\sigma_y}^{\text{nano}} \geq 6\, {\sigma_y}^{\text{class}}$) by reducing the grain-diameter dramatically- a consequence of the Hall-Petch relation. This shows the general need for the incorporation of an internal length scale even when viewing the polycrystal macroscopically. [27]

Experimental evidence [DML91, LLT94] shows that the rotations of the individual grains e.g. in polycrystalline aluminium specimens may deviate considerably from the continuum rotation which must be viewed as orthogonal part of the average grain deformation gradient over some representative volume element. This picture lends itself most naturally to a treatment in a Cosserat context: we identify the averaged individual elastic rotations of grains with the elastic Cosserat microrotations and the orthogonal part of the averaged elastic deformation gradient with the elastic continuum rotation.[28]

Remaining in this context, it is clear, that a material, in which a substructure is allowed to rotate rather independently, should not become more rigid than a corresponding classical (equivalent, macroscopic) homogeneous material. Therefore, we conclude again, that $\mu_c = 0$ is a reasonable choice for a polycrystal treated on a macroscopic level.[29] If the related classical homogeneous description has the strain energy

$$W(U) = \mu\,\|U - 1\!\!1\|^2 + \frac{\lambda}{2}\mathrm{tr}\,[U - 1\!\!1]^2 = \mu\,\|\,\mathrm{sym}(U - 1\!\!1)\|^2 + \frac{\lambda}{2}\mathrm{tr}\,[U - 1\!\!1]^2\,, \tag{2.3.35}$$

then after relaxing the constraint on the rotations to coincide with the continuum rotations we would rather expect the overall macroscopic strain energy due to macroscopic stretch to be smaller than the homogeneous one, i.e. $W_{\mathrm{mp}}(\overline{U}) \leq W(U)$. Since in principle $\mathrm{skew}(\overline{U})$ can be large, we conclude consistently $\mu_c = 0$.[30]

In the elasto-plastic theory the consequences on a macroscale induced by letting $\mu_c > 0$ are even more severe than in the elastic case: imagine a cyclic loading history which systematically leaves the elastic region. In general, we will obtain a time dependent inhomogeneous distribution of the plastic deformation $F_p(x)$. For $\mu_c > 0$, the so called **elastic trial step** [31] will be unconditionally stable irrespective of the accumulated spatial inhomogeneities of F_p as long as $F_p \in L^\infty(\Omega, \mathrm{GL}^+(3,\mathbb{R}))$, while $\mu_c = 0$ allows some sort of **elastic fatigue/softening/failure/fracture** since the positive definiteness of the elastic tangent stiffness matrix w.r.t. the reference configuration is affected by the spatial continuity properties of F_p. This apparent softening, namely the decrease of elastic moduli, is a well documented experimental fact. For a polycrystal, we then adopt the following picture: the plastic deformation F_p represents on a macroscopic level the permanent material substructure, to be more precise, the permanent averaged cumulative plastic deformation of the individual grains due to slip and twinning. We may call therefore F_p the average plastic grain transformation and $\overline{R}_e$ the average elastic grain rotation.

2.3.3 Infinitesimal elasto-plastic Cosserat model

If we assume that plastic deformations F_p and elastic microrotations remain small, we can considerably simplify the problem (2.3.31). We expand $\overline{R}_e = 1\!\!1 + \overline{A}_e + \ldots$, $\overline{A}_e \in \mathfrak{so}(3)$, $\|\overline{A}_e\|^2 \ll 1$, $F_p = 1\!\!1 + p + \ldots$, $\|p\|^2 \ll 1$ and choose as elastic domain $\mathcal{E} := \{\Sigma_E |\ \|\,\mathrm{dev}(\mathrm{sym}\,\Sigma_E)\| \leq \sigma_y\}$ [32], then (2.3.31) reduces to the infinitesimal

[27] Whether the incorporation of the necessary material length scale must be done in a Cosserat framework, cannot be decided. But the experimental evidence [FMAH94] suggests that size dependent hardening occurs predominantly under torsion (possible rotations), while in uniaxial tension (no rotations) strain gradients are negligible and length scale effects remain small.

[28] Observe that these two rotations do not coincide in general: the **averaged rotation** is understood to be the **best-approximating single rotation to a rotation field** defined over a representative volume element while the **continuum rotation** is the **orthogonal part** of the **averaged deformation gradient**. In the infinitesimal case, both so defined infinitesimal rotations coincide!

[29] The same argument put differently: Consider a wall made of concrete bricks and cast concrete, respectively. The cast wall will have more rigidity.

[30] "Polycrystalline ... type microstructures behave classically or nearly classically (in their elastic range)." [Lak95, p.22]

[31] or elastic predictor. In an operator-split method it amounts to 'freezing' the plastic evolution and to compute elastic equilibrium.

[32] no **infinitesimal plastic spin**: $p = \varepsilon_p$ and isochoric plasticity: $\mathrm{tr}\,[\varepsilon_p] = 0$ and assume $\langle \varepsilon_p.\vec{n}, \vec{n}\rangle = 0$ on $\Gamma_S \cup \Gamma_C$.

elasto-plastic system in variational form with **non-dissipative** Cosserat effects and reads

$$
\begin{aligned}
&\int_\Omega \mu\,\|\varepsilon-\varepsilon_p\|^2+\mu_c\,\|\operatorname{skew}(\nabla u-\overline{A}_e)\|^2+\frac{\lambda}{2}\operatorname{tr}[\varepsilon]^2+\mu\,\frac{L_c^2}{12}\,\|\nabla\operatorname{axl}(\overline{A}_e)\|^2-\langle f,u\rangle-\langle M,\overline{A}_e\rangle\,\mathrm{dV}\\
&\qquad-\int_{\Gamma_S}\langle N,u\rangle\,\mathrm{dS}-\int_{\Gamma_C}\langle M_c,\overline{A}_e\rangle\,\mathrm{dS}\mapsto\min\,.\quad \text{w.r.t. } (u,\overline{A}_e) \text{ at constant } \varepsilon_p\\
&\varepsilon_e=\varepsilon-\varepsilon_p,\quad \varepsilon(\nabla u(x))=\tfrac{1}{2}(\nabla u^T+\nabla u),\quad \varepsilon_p=\tfrac{1}{2}(p^T+p),\\
&\dot{\varepsilon_p}(t)\in\mathfrak{f}(T_E),\quad T_E=2\mu\,(\varepsilon-\varepsilon_p) \qquad (2.3.36)\\
&u_{|\partial\Omega}(t,x)=g_{\mathrm{d}}(t,x)-x,\quad \overline{A}_{e|_\Gamma}=\operatorname{skew}(\nabla g_{\mathrm{d}}(t,x))_{|\Gamma}\,.
\end{aligned}
$$

The corresponding equilibrium system of equations for pure Dirichlet conditions and without external couples M, M_c is given by (note that $\|\overline{A}_e\|^2=2\|\operatorname{axl}(\overline{A}_e)\|^2$ for $\overline{A}_e\in\mathfrak{so}(3,\mathbb{R})$)

$$
\begin{aligned}
&0=\operatorname{Div}\sigma+f,\quad x\in\Omega\\
&\sigma=2\mu\,(\varepsilon-\varepsilon_p)+2\,\mu_c\,(\operatorname{skew}(\nabla u)-\overline{A}_e)+\lambda\operatorname{tr}[\varepsilon]\cdot 1\!\!1\\
&0=\mu\frac{L_c^2}{12}\,\Delta\operatorname{axl}(\overline{A}_e)+\mu_c\,\operatorname{axl}((\operatorname{skew}(\nabla u)-\overline{A}_e)\\
&\dot{\varepsilon_p}(t)\in\mathfrak{f}(T_E),\quad T_E=2\mu\,(\varepsilon-\varepsilon_p) \qquad (2.3.37)\\
&u_{|\partial\Omega}(t,x)=g_{\mathrm{d}}(t,x)-x,\quad x\in\partial\Omega\,,\quad \overline{A}_{e|\partial\Omega}=\operatorname{skew}(\nabla g_{\mathrm{d}}(t,x))_{|\partial\Omega}\,.\\
&\operatorname{tr}[\varepsilon_p(0)]=0\,,\quad \varepsilon_p(0)\in\operatorname{Sym}(3)\,.
\end{aligned}
$$

It must be observed that this completely reduced set of equations is still intrinsically thermodynamically admissible. The model can also be obtained as limit case of models proposed in [IW98, Bes74, Lip69].

The infinitesimal model has already been completely justified as a non-local regularization ($\mu_c\to 0$) of classical ideal rate-independent plasticity in [NC03] using the methods exploited before in [Che98]. Precisely, it has been proved that (2.3.37) admits a unique global solution. The system (2.3.36) is therefore a reasonable regularization of classical plasticity in the sense that the system remains of second order and the plastic flow part is left unaltered compared to the traditional one.

In [DSW93, p.815] an elasto-plastic model based on the infinitesimal Cosserat theory with dissipative micropolar effects has been investigated. They show that $\mu_c>0$ has a decisive influence [33] on localization effects essentially excluding mode II shear failure. Since our reduced model is non-dissipative it is difficult however, to transfer this insight directly. This remark finishes the modelling part of this contribution.

[33] Since the infinitesimal model is neither observer-invariant nor frame-indifferent it is not possible to conclude on physical grounds that $\mu_c>0$ is necessary for the general theory. A model which is, strictly speaking, physically irrelevant cannot lead to stringent conclusions on the physical significance of some parameter. The infinitesimal model has only merits as a first approximation. We might wonder therefore about the status of mathematical investigations of the infinitesimal elasto-plastic models and the effort still put into the investigations of their intricacies (rate-independent limit), which are either irrelevant for the geometrically exact model or misleading. We conclude that their investigation is mainly of high inner - mathematical interest. This remark applies mutatis mutandis to the numerical treatment of classical infinitesimal elasto-plasticity. Here refined time-integration algorithms of higher order are investigated without leading to consequences for the numerical treatment of the exact theory, which is faced with completely different problems.

2.4 Mathematical analysis

2.4.1 Statement of the finite elastic Cosserat problem in variational form

Let us now return to the purely elastic Cosserat case. The problem has been posed in a variational form. The task is to find a pair $(\varphi, \overline{R}) \in \mathbb{R}^3 \times \mathrm{SO}(3,\mathbb{R})$ of deformation φ and **independent microrotation** $\overline{R}$ satisfying

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{curv}}(\mathfrak{K}) - \langle f, \varphi\rangle - \langle M, \overline{R}\rangle \,\mathrm{dV} - \int_{\Gamma_S} \langle N, \varphi\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{R}\rangle \,\mathrm{dS} \mapsto \min.\ \text{w.r.t.}\ (\varphi, \overline{R}),\\
&\overline{U} = \overline{R}^T F, \quad \overline{R}_{|_\Gamma} = \overline{R}_{\mathrm{d}}, \quad \varphi_{|_\Gamma} = g_{\mathrm{d}} \qquad (2.4.38)\\
&W_{\mathrm{mp}}(\overline{U}) = \mu\, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\, \|\mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2} \mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2\\
&W_{\mathrm{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12}\, (1 + \alpha_4\, L_c^q\, \|\mathfrak{K}\|^q)\, \left(\alpha_5\, \|\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\, \|\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\, \mathrm{tr}\,[\mathfrak{K}]^2\right)^{\frac{1+p}{2}},\\
&\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R} = \left(\overline{R}^T\nabla(\overline{R}.e_1), \overline{R}^T\nabla(\overline{R}.e_2), \overline{R}^T\nabla(\overline{R}.e_3)\right), \quad \text{third order } \textbf{curvature tensor}.
\end{aligned}$$

The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ depends on the deformation gradient $F = \nabla\varphi$ and microrotations $\overline{R}$ together with their space derivatives. Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}}, \overline{R}_{\mathrm{d}}$ for displacements and microrotations, respectively, are prescribed while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions N are applied with $\Gamma \cap \Gamma_S = \emptyset$. The external volume force is f and M takes on the role of external volume couples. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where external surface couples M_c are applied with $\Gamma \cap \Gamma_C = \emptyset$. The parameters μ, $\lambda > 0$ are the Lamé constants of classical elasticity, $\mu_c \geq 0$ is called the **Cosserat couple modulus** and $L_c > 0$ introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal. If not stated otherwise, we assume that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$.

2.4.2 The different cases

We distinguish five completely different situations:

I: $\boldsymbol{\mu_c > 0,\ \alpha_4 \geq 0,\ p \geq 1,\ q \geq 0}$, unconditional elastic macro-stability, local first order Cosserat micropolar, unqualified existence, microscopic specimens, non-zero Cosserat couple modulus. Fracture excluded.

II: $\boldsymbol{\mu_c = 0,\ \alpha_4 > 0,\ p \geq 1,\ q > 1}$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Fracture excluded.

III: $\boldsymbol{\mu_c = \infty,\ \alpha_4 \geq 0,\ p \geq 1,\ q \geq 0}$, unconditional elastic macro-stability, the constrained gradient Cosserat micropolar problem (indeterminate couple stress model (2.2.21)). Compatible Dirichlet boundary conditions: $\varphi_{|_\Gamma} = g_{\mathrm{d}}$, $\mathrm{polar}(\nabla\phi)_{|_\Gamma} = \mathrm{polar}(\nabla g_{\mathrm{d}})_{|_\Gamma}$.

IV: $\boldsymbol{\mu_c = 0,\ \alpha_4 = 0,\ 0 < p \leq 1,\ q = 0}$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Since possibly $\varphi \notin W^{1,1}(\Omega, \mathbb{R}^3)$, due to lack of elastic coercivity, including **fracture** in multiaxial situations.

V: $\boldsymbol{\mu_c = 0, L_c = 0}$, elastic pre-stability, finite elasticity with free rotations and microstructure. Weak solutions of finite elasticity are stationary points of this minimization problem. Allowing for **sharp interfaces**.

We refer to $0 < p < 1$, $q \geq 0$ as the **sub-critical case**, $p = 1$, $q \geq 0$ as the **critical case** and $p \geq 1$, $q > 1$ as the **super-critical case**. We will mathematically treat the first three cases.

2.4.3 The coercivity inequality

The decisive analytical tool for the treatment of case II (super-critical) is the following non-trivial coercive inequality:

Theorem 2.4.1 (Extended 3D-Korn's first inequality)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with non vanishing 2-dimensional Lebesgue measure. Define $H_\circ^{1,2}(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$ and let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$. Moreover suppose that $\mathrm{Curl}\, F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. *Then*

$$\exists\, c^+ > 0\ \forall\, \phi \in H_\circ^{1,2}(\Omega, \Gamma): \quad \|\nabla\phi F_p^{-1}(x) + F_p^{-T}(x)\nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\, \|\phi\|^2_{H^{1,2}(\Omega)}.$$

Proof. The proof has been presented in [Nef02]. Note that for $F_p = \nabla\Theta$ we would only have to deal with the classical Korn's inequality evaluated on the transformed domain $\Theta(\Omega)$. However, in general, F_p is **incompatible** giving rise to a **non-riemannian manifold** structure. Compare to [CG01] for an interpretation and the physical relevance of the quantity $\operatorname{Curl} F_p$. ■

Motivated by the investigations in [Nef02], it has been shown recently by my colleague W. Pompe [Pom03] that the extended Korn's inequality can be viewed as a special case of a general class of coercive inequalities for quadratic forms. He was able to show that indeed $F_p \in C(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$ is sufficient for (2.4.1) to hold without any condition on the compatibility.

However, taking the special structure of the extended Korn's inequality again into account, work in progress suggests that continuity is not really necessary: instead $F_p \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ and $\operatorname{Curl} F_p \in L^{3+\delta}(\Omega)$ should suffice, whereas $F_p \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ alone is not sufficient, see the counterexample presented in [Pom03].

In view of the important role of the extended Korn's first inequality let us agree in saying that a material is **elastically pre-stable**, whenever

$$\begin{aligned}
&\exists\, H \in \mathbb{M}^{3\times 3},\ H \neq 0: \quad D_F^2 W(x,F).(H,H) = 0 \qquad (2.4.39)\\
&\exists\, c^+ > 0\ \exists\, G \in \mathrm{GL}^+(3,\mathbb{R})\ \forall\, H \in \mathbb{M}^{3\times 3}:\ D_F^2 W(x,F).(H,H) \geq c^+ \|G(x)^T H + H^T G(x)\|^2 .
\end{aligned}$$

In this terminology, infinitesimal classical elasticity is pre-stable with $G = 1\!\!1$ due to the classical Korn's first inequality and the extended Korn's first inequality links the smoothness of G to the positive definiteness of the elastic tangent stiffness tensor.

2.4.4 The geometrically exact elastic Cosserat model

The following results are the first existence theorems for geometrically exact elastic Cosserat models known to the author:[34]

Theorem 2.4.2 (Existence for 3D-finite elastic Cosserat model: case I.)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega,\mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S,\mathbb{R}^3)$ together with $M \in L^1(\Omega,\mathbb{M}^{3\times 3})$ and $M_c \in L^1(\Gamma_C,\mathbb{M}^{3\times 3})$. Then (2.4.38) with material constants conforming to case I admits at least one minimizing solution pair $(\varphi,\overline{R}) \in H^1(\Omega,\mathbb{R}^3) \times W^{1,1+p}(\Omega,\mathrm{SO}(3,\mathbb{R}))$.

Proof. We apply the direct methods of variations and consider for simplicity $N, M, M_c = 0$. With the prescription of $(g_{\mathrm{d}}, \overline{R}_{\mathrm{d}})$ it is clear that $I < \infty$ for some pair $(\varphi,\overline{R})$. Observe first that the micropolar curvature term $\mathfrak{K}$ controls $\overline{R} \in W^{1,1+p}(\Omega,\mathrm{SO}(3,\mathbb{R}))$, since $\|\mathfrak{K}\| = \|\overline{R}^T \mathrm{D_x}\overline{R}\| = \|\mathrm{D_x}\overline{R}\|$, pointwise and $\alpha_5, \alpha_6 > 0$.

Moreover, $\mathrm{SO}(3,\mathbb{R})$ is weakly closed in the topology of $W^{1,1+p}(\Omega)$. We omit to show that I is bounded below: this will turn out not to be necessary. We may choose decreasing (infimizing) sequences of pairs $(\varphi^k, \overline{R}^k)$. The curvature contribution together with the appropriate boundary conditions and Poincarés inequality yields boundedness of $\overline{R}^k \subset W^{1,1+p}(\Omega,\mathrm{SO}(3,\mathbb{R}))$. We may extract a subsequence again denoted by $\overline{R}^k$ converging strongly in $L^{1+p}(\Omega)$ to an element $\tilde{\overline{R}} \in W^{1,1+p}(\Omega,\mathrm{SO}(3,\mathbb{R}))$ since $p > 0$ by assumption. Because $\mu_c > 0$, it is immediate that $\nabla\varphi^k = F^k$ is **bounded** in $L^2(\Omega,\mathbb{M}^{3\times 3})$, **independent** of $\overline{R}^k$ on account of

$$\begin{aligned}
W_{\mathrm{mp}}(\overline{R}^{k,T} F^k) &\geq \mu_c\, \|\overline{R}^{k,T} F^k - 1\!\!1\|^2 = \mu_c\, \left(\|F^k\|^2 - 2\langle \overline{R}^{k,T} F^k, 1\!\!1\rangle + 3\right)\\
&\geq \mu_c \left(\|F^k\|^2 - 2\sqrt{3}\|F^k\| + 3\right) , \qquad (2.4.40)
\end{aligned}$$

and

$$\begin{aligned}
\infty &> \int_\Omega W_{\mathrm{mp}}(\overline{U}_k) + W_{\mathrm{curv}}(\mathfrak{K}_k) - \langle f, \varphi_k\rangle \,\mathrm{dV} \geq \int_\Omega W_{\mathrm{mp}}(\overline{U}_k) - \langle f, \varphi_k\rangle \,\mathrm{dV}\\
&\geq \int_\Omega W_{\mathrm{mp}}(\overline{U}_k)\,\mathrm{dV} - \|f\|_{L^2(\Omega)}\, \|\varphi_k\|_{H^{1,2}(\Omega)} \qquad (2.4.41)\\
&\geq \mu_c\, \|\nabla\varphi_k\|^2 - 2\sqrt{3}\,\mu_c\, \|\varphi_k\|_{H^{1,2}(\Omega)} - \|f\|_{L^2(\Omega)}\, \|\varphi_k\|_{H^{1,2}(\Omega)} + 3\mu_c\\
&\geq \mu_c\, \|\nabla u_k\|^2 - C_1\, \|u_k\|_{H^{1,2}(\Omega)} + C_2 \geq \mu_c\, c^+\, \|u_k\|^2_{H^{1,2}(\Omega)} - C_1\, \|u_k\|_{H^{1,2}(\Omega)} + C_2 ,
\end{aligned}$$

[34] The proposed finite results determine the macroscopic deformation $\varphi \in H^1(\Omega,\mathbb{R}^3)$ and not more. This means that discontinuous macroscopic deformations by cavities or the formation of holes are not excluded (possible mode I failure). If $\mu_c > 0$ fracture is effectively ruled out, which is unrealistic.

where we made use of the appropriate boundary conditions for $\varphi^k = x + u_k(x)$, and applied Poincarés inequality to u_k since it has zero boundary values on Γ. This yields the boundedness of φ^k in $H^1(\Omega, \mathbb{R}^3)$. Hence we may extract a subsequence, not relabelled, such that $\varphi^k \rightharpoonup \tilde{\varphi} \in H^1(\Omega, \mathbb{R}^3)$. Furthermore, we may always obtain a subsequence of $(\varphi^k, \overline{R}^k)$ such that $\overline{U}_k = \overline{R}^{k,T} F^k$ converges weakly in $L^2(\Omega)$ to an element $\tilde{\overline{U}}$ on account of the boundedness of the stretch energy and $\mu_c > 0$.

For $p \geq 1$ we have as well that $\overline{R}^k$ converges indeed strongly in $L^2(\Omega)$ to an element $\tilde{\overline{R}} \in H^{1,2}(\Omega, \mathrm{SO}(3, \mathbb{R}))$. Thus $\overline{R}^{k,T} F^k$ converges weakly to $\tilde{\overline{R}}^T F$ in $L^1(\Omega)$. The weak limit in $L^1(\Omega)$ must coincide with the weak limit of $\overline{U}_k$ in $L^2(\Omega)$. Hence, $\tilde{\overline{U}} = \tilde{\overline{R}}^T \nabla\tilde{\varphi}$.

Since the total energy is convex in $(\overline{U}, \mathfrak{K})$ and $(\nabla\varphi, D\overline{R})$, we get

$$\begin{aligned} I(\tilde{\varphi}, \tilde{\overline{R}}) &= \int_\Omega W_{\mathrm{mp}}(\tilde{\overline{U}}) + W_{\mathrm{curv}}(\tilde{\mathfrak{K}}) - \langle f, \tilde{\varphi} \rangle \, \mathrm{dV} \\ &\leq \liminf_{k\to\infty} \int_\Omega W_{\mathrm{mp}}(\overline{U}_k) + W_{\mathrm{curv}}(\mathfrak{K}_k) - \langle f, \varphi_k \rangle \, \mathrm{dV} = \lim_{k\to\infty} I(\varphi^k, \overline{R}^k) \,, \end{aligned} \tag{2.4.42}$$

which implies that the limit pair is a minimizer. Note that the limit microrotations $\tilde{\overline{R}}$ may fail to be continuous if $p \leq 2$ (non-existence or limit case of Sobolev embedding). Moreover, uniqueness cannot be ascertained, since $\mathrm{SO}(3, \mathbb{R})$ is a nonlinear manifold (and the considered problem is indeed nonlinear), such that convex combinations of rotations are not rotations in general. Since the functional I is differentiable the minimizing pair is a stationary point and therefore a solution of the field equations (2.2.12). Note again that the limit microrotations are trivial in $L^\infty(\Omega)$ but may fail to be continuously distributed in space. That under these unfavourable circumstances a minimizing solution may nevertheless be found is entirely due to $\mu_c > 0$ and $p \geq 1$. ■

We continue with the super-critical case appropriate for macroscopic situations and close to classical elasticity. For simplicity we restrict attention to the case with zero loads.

Theorem 2.4.3 (Existence for 3D-finite elastic Cosserat model: case II.)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3, \mathbb{R}))$. Then (2.4.38) with material constants conforming to case II admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3, \mathbb{R}))$.

Proof. We repeat the argument of case I. However, the boundedness of infimizing sequences is not immediately clear. Boundedness of the rotations $\overline{R}^k$ holds true in the space $W^{1,1+p+q}(\Omega, \mathrm{SO}(3, \mathbb{R}))$ with $1 + p + q > N = 3$, hence we may extract a subsequence, not relabelled, such that $\overline{R}^k$ converges strongly to $\hat{\overline{R}} \in C^0(\overline{\Omega}, \mathrm{SO}(3, \mathbb{R}))$ in the topology of $C^0(\overline{\Omega}, \mathrm{SO}(3, \mathbb{R}))$ on account of the Sobolev-embedding theorem. Along such strongly convergent sequence of rotations, the corresponding sequence of deformations φ^k is also bounded in $H^1(\Omega, \mathbb{R}^3)$. However, this is not due to a basically simple pointwise estimate as in case I, but only true after integration over the domain: at face value we only control certain mixed symmetric expressions in the deformation gradient. More precisely, we have

$$\begin{aligned} \infty &> \int_\Omega W_{\mathrm{mp}}(\overline{U_k}) + W_{\mathrm{curv}}(\mathfrak{K}_k) \, \mathrm{dV} \geq \int_\Omega W_{\mathrm{mp}}(\overline{U}_k) \, \mathrm{dV} \\ &\geq \int_\Omega \frac{\mu}{4} \|\overline{R}_k^T \nabla\varphi_k + \nabla\varphi_k^T \overline{R}_k - 2\mathbb{1}\|^2 \, \mathrm{dV} \\ &\geq \int_\Omega \frac{\mu}{8} \|\overline{R}_k^T \nabla u_k + \nabla u_k^T \overline{R}_k\|^2 \, \mathrm{dV} - C_1 \, \|u_k\|_{H^{1,2}(\Omega)} + C_2 \\ &= \int_\Omega \frac{\mu}{8} \|(\overline{R}_k - \hat{R} + \hat{R})^T \nabla u_k + \nabla u_k^T (\overline{R}_k - \hat{R} + \hat{R})\|^2 \, \mathrm{dV} - C_1 \, \|u_k\|_{H^{1,2}(\Omega)} + C_2 \\ &\geq \int_\Omega \frac{\mu}{8} \underbrace{\|\hat{R}^T \nabla u_k + \nabla u_k^T \hat{R}\|^2}_{\text{combinations of derivatives}} \, \mathrm{dV} - C_3 \, \|\hat{R} - \overline{R}_k\|_\infty \, \|u_k\|^2_{H^{1,2}(\Omega)} \\ &\qquad - (C_1 + 2\,\|\hat{R} - \overline{R}_k\|_\infty) \, \|u_k\|_{H^{1,2}(\Omega)} + C_2 \\ &\geq (\frac{\mu}{8} c_K - C_3 \, \|\hat{R} - \overline{R}_k\|_\infty) \, \|u_k\|^2_{H^{1,2}(\Omega)} - (C_1 + 2\,\|\hat{R} - \overline{R}_k\|_\infty) \, \|u_k\|_{H^{1,2}(\Omega)} + C_2 \,, \end{aligned} \tag{2.4.43}$$

where we made use of the appropriate boundary conditions for $\varphi^k = x + u_k$ and applied the extended Korn's inequality (2.4.1) in the improved version of [Pom03] yielding the positive constant c_K for the continuous microrotation $\hat{R}$. Since $\|\hat{R} - \overline{R}_k\|_\infty \to 0$ we conclude the boundedness of u_k in $H^1(\Omega)$. Hence, φ_k is bounded in $H^1(\Omega)$. Now we obtain that $\overline{U_k} \rightharpoonup \tilde{\overline{U}} = \hat{\overline{R}}^T \nabla\tilde{\varphi}$ by construction with the notations as in case I. The remainder proceeds as in case I. This finishes the argument. The limit microrotations $\hat{\overline{R}}$ are indeed found to be continuous. ■

Theorem 2.4.4 (Existence for 3D-finite elastic Cosserat model with nonlinear volume part)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M \in L^1(\Omega, \mathbb{M}^{3\times 3})$ and $M_c \in L^1(\Gamma_C, \mathbb{M}^{3\times 3})$. Assume furthermore that W_{mp} has the form

$$W_{\mathrm{mp}}(\overline{U}) = \mu\,\|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}\,\overline{U}\|^2 + \beta_1\,\|\mathrm{Cof}\,\overline{U}\|^2 - \beta_2\,\ln\det[\overline{U}] + \beta_3\,(\det[\overline{U}] - 1)^2\,, \qquad (2.4.44)$$

with $\beta_1, \beta_2, \beta_3 > 0$. Then (2.4.38) with material constants conforming to case I (if $\mu_c > 0$) or conforming to case II (if $\mu_c = 0$) admits at least one minimizing solution pair $(\varphi, \overline{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$ and $\det[\nabla\varphi] > 0$ a.e.

Proof. The argument of case I/II can be modified. We note first that the additional terms involving $\beta_i, i = 1,2,3$ are in fact independent of $\overline{R}$ and can be reduced to their dependence on F. The additional terms have no influence on the shear failure. Second, the additional terms are **polyconvex** [Bal77] w.r.t. F. Third, $W_{\mathrm{mp}} \to \infty$ for $\det[F] \to 0$. In addition to the properties of minimizing sequences in case II we obtain that $\mathrm{Cof}\,F_k \rightharpoonup \mathrm{Cof}\,F \in L^2(\Omega)$, $\det[F_k] \rightharpoonup \det[F] \in L^1(\Omega)$ due to separate control of the cofactor and determinant in the energy. The result follows by standard arguments. The minimizer may not be a solution of the corresponding Euler-Lagrange equations. Note that replacing $\mu\,\|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2$ with $\beta_0\,\|\overline{U}\|^2$ would decouple the problem and remove the possibility of elastic shear failure. ■

Corollary 2.4.5 (Existence for 3D-constrained Cosserat model: case III)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\mathrm{polar}(\nabla g_{\mathrm{d}}) \in W^{1,1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M \in L^1(\Omega, \mathbb{M}^{3\times 3})$ and $M_c \in L^1(\Gamma_C, \mathbb{M}^{3\times 3})$. Then problem (2.2.21) admits at least one minimizing solution $\varphi \in H^{1,2}(\Omega, \mathbb{R}^3)$.

Proof. We slightly modify the consistent coupling condition to mean $\overline{U} \in \mathrm{Sym}$. This is the incorporated in the admissibility class for the variational problem. The proof mimics now case I since the sequence of infimizing rotations R_k is constrained such that $\overline{U}_k = \overline{R}_k^T F_k \in \mathrm{Sym}$ for the corresponding sequence of deformation gradients F_k. Since the constraint $\overline{U} \in \mathrm{Sym}$ is weakly closed, the proof is finished. ■

Remark 2.4.6
Complete higher regularity of φ in the constrained Cosserat model, i.e. $\varphi \in H^{2,2}(\Omega, \mathbb{R}^3)$ cannot be ascertained in general since we only control certain second derivatives of φ in the curvature term. One might wonder therefore, whether the additional C^1-continuity in treating the fourth order indeterminate couple stress problem numerically is worth the effort.

It is not immediately clear to us how to extend the method of proof to cover the elastic-plastic Cosserat case also. This topic will be left open for future research.

2.4.5 Partially linearized elastic Cosserat theory

If we assume small microrotations, i.e. $\overline{R} = 1\!\!1 + \overline{A} + \ldots$, $\overline{A} \in \mathfrak{so}(3)$, $\|\overline{A}\| \ll 1$, then the micropolar stretch tensor may be expanded as follows

$$\overline{U} = \overline{R}^T F = (1\!\!1 + \overline{A} + \ldots)^T(1\!\!1 + \nabla u) \approx 1\!\!1 + \nabla u - \overline{A} - \overline{A}\nabla u + \ldots\,, \qquad (2.4.45)$$

where u is the (unrestricted) displacement but not the infinitesimal displacement from which we conclude the approximative expression of the stretch energy

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{small}}(\nabla u, \overline{A}) &= \mu\,\|\mathrm{sym}\,\nabla u - \mathrm{sym}(\overline{A}\nabla u)\|^2 + \mu_c\,\|\mathrm{skew}(\nabla u) - \overline{A} - \mathrm{skew}(\overline{A}\nabla u)\|^2 \\ &\quad + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\nabla u) - \mathrm{sym}(\overline{A}\nabla u)\right]^2. \end{aligned} \qquad (2.4.46)$$

The value $\mu_c = 0$ is still admissible, since the problem does not decouple, but if $\mu_c = 0$, the local coupling takes place only in the second order contribution $\overline{A}\nabla u$.

Since $\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R} = (1\!\!1+\overline{A}+\ldots)^T\,\mathrm{D_x}(1\!\!1+\overline{A}+\ldots) \approx \mathrm{D_x}\overline{A}+\overline{A}\,\mathrm{D_x}\overline{A}+\ldots$ to first order we get as approximation for the curvature energy [35], based on (2.2.7) the expression

$$\begin{aligned} W_{\mathrm{curv}}^{\mathrm{small}}(\mathfrak{K}) &= \mu\frac{L_c^2}{12}\left(\alpha_5\,\|\operatorname{sym}\mathrm{D_x}\overline{A}\|^2 + \alpha_6\,\|\operatorname{skew}\mathrm{D_x}\overline{A}\|^2 + \alpha_7\operatorname{tr}\left[\mathrm{D_x}\overline{A}\right]^2\right) \\ \overline{A}_{|_\Gamma} &= \overline{A}_{\mathrm{d}}\,, \end{aligned} \tag{2.4.47}$$

together with the consistently reduced boundary condition for small rotations. For a mathematical treatment, the decisive simplification afforded by (2.4.47) is the treatment of $\overline{A}$ on the linear manifold $\mathfrak{so}(3)$ of skew-symmetric matrices instead of $\overline{R} \in \mathrm{SO}(3)$. The corresponding equation of balance of angular momentum

$$\begin{aligned} &\operatorname{skew}\big(-2\mu[\operatorname{sym}\nabla u - \operatorname{sym}(\overline{A}\nabla u)]\nabla u^T \\ &\quad - 2\mu_c[\operatorname{skew}\nabla u - \overline{A} - \operatorname{skew}(\overline{A}\nabla u)](1\!\!1 + \nabla u)^T \\ &\quad -\lambda\langle\operatorname{sym}\nabla u - \operatorname{sym}(\overline{A}\nabla u), 1\!\!1\rangle\,\nabla u^T\big) - \operatorname{skew}\left(\operatorname{Div} D_{\mathrm{D_x}\overline{A}}W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D_x}\overline{A})\right) - \operatorname{skew}(M) = 0_{\mathbb{M}^{3\times 3}}\,, \end{aligned} \tag{2.4.48}$$

can be written equivalently as

$$\frac{L_c^2}{12}\operatorname{Div}\mathfrak{M}.\mathrm{D_x}\overline{A} - \operatorname{skew}(M) = \widehat{M}(\overline{A},\nabla u) := \operatorname{skew}\left(\ldots\right), \tag{2.4.49}$$

with a (rearranging) linear operator $\mathfrak{M} : \mathfrak{T}(3) \mapsto \mathfrak{T}(3)$ and

$$\widehat{M}(\overline{A}+\hat{\overline{A}},X) = \widehat{M}(\overline{A},X) + \widehat{M}(\hat{\overline{A}},X), \quad \|\widehat{M}(\overline{A},X)\| \le (1 + C^+(\overline{A}))\cdot\|X\|^2\,. \tag{2.4.50}$$

It is readily seen that (2.4.49) is a uniformly Legendre-Hadamard elliptic system with constant coefficients in $\overline{A}$ which is linear at given ∇u.

For a model based on (2.4.46) and (2.4.47) the only mathematically interesting case left is the critical case $\mu_c = 0, q = 0, p = 1$ since otherwise the theorems treating the exact finite situations already apply. However, if $\mu_c = 0$ we have to make up for the loss of pointwise control in the stretch and the loss of regularity of $(1\!\!1 + \overline{A})$ if $\mathrm{D_x}\overline{A} \in L^2(\Omega)$ only. This suggests a slight modification of the problem on the level of the corresponding equilibrium system. We replace (regularize) $\widehat{M}$ in (2.4.49) with

$$\widehat{M}_\sharp^{ij}(\overline{A},X) := \begin{cases} \widehat{M}^{ij}(\overline{A},X) & |\widehat{M}^{ij}(\overline{A},X)| \le K-1 \\ \tilde{M}^{ij}(\overline{A},X) & K-1 \le |\widehat{M}^{ij}(\overline{A},X)| \le K \\ K & |\widehat{M}^{ij}(\overline{A},X)| > K\,, \end{cases} \tag{2.4.51}$$

such that $\widehat{M}_\sharp^{ij}(\overline{A},X)$ is smooth to any order we need. It means physically that the components of the reaction stresses (couple stresses) are assumed to be essentially bounded by a constant $K > 0$. This is a consistent requirement with the other simplifying assumptions already made. The complete problem reads therefore:

$$\begin{aligned} &\int_\Omega W_{\mathrm{mp}}^{\mathrm{small}}(\nabla u,\overline{A}) - \langle f,u\rangle - \langle M,\overline{A}\rangle\,\mathrm{dV} \mapsto \min.\ \text{w.r.t. } u \text{ at given } \overline{A} \\ &\qquad u_{|_{\partial\Omega}}(x) = g_{\mathrm{d}}(x) - x\,, \quad \overline{A}_{|_{\partial\Omega}} = \overline{A}_{\mathrm{d}} \\ &\frac{L_c^2}{12}\operatorname{Div}\mathfrak{M}.\mathrm{D_x}\overline{A} = \widehat{M}_\sharp(\overline{A},\nabla u) - \operatorname{skew}(M)\,. \end{aligned} \tag{2.4.52}$$

We can prove the following result:

Theorem 2.4.7 (Existence for 3D-small rotation elastic Cosserat model)
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary of class C^3 and assume for the boundary data $g_{\mathrm{d}} \in H^3(\Omega,\mathbb{R}^3)$ and $\overline{A}_{\mathrm{d}} \in H^{2,2}(\Omega,\mathfrak{so}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and $\operatorname{skew}(M) \in L^2(\Omega,\mathfrak{so}(3,\mathbb{R}))$. Then the system (2.4.52) with $\mu_c \ge 0, \alpha_6 \ge 0$ admits a, perhaps non-unique, solution pair $(u,\overline{A}) \in H^3(\Omega,\mathbb{R}^3) \times H^2(\Omega,\mathfrak{so}(3))$. The solution is smoother if the data are smoother.

[35] The expansion of the curvature shows that we need not introduce a smallness assumption on the curvature itself, if we already assume that rotations are small.

Proof. Since the exponent in the curvature energy is only two, we cannot mimic the variational argument of the last section since we have no easy way to obtain a strongly convergent subsequence $\overline{A}_k$ in a topology as strong as $C^0(\overline{\Omega})$. Instead, we define the following iteration scheme: given $(u_j, \overline{A}_j) \in H^{m+1}(\Omega, \mathbb{R}^3) \times H^m(\Omega, \mathfrak{so}(3))$ compute $(u_{j+1}, \overline{A}_{j+1}) \in H^{m+1}(\Omega, \mathbb{R}^3) \times H^m(\Omega, \mathfrak{so}(3))$ such that

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}^{\mathrm{small}}(\nabla u_{j+1}, \overline{A}_j) - \langle f, u_{j+1}\rangle - \langle M, \overline{A}_j\rangle \,\mathrm{dV} \mapsto \min.\ \text{w.r.t. } u_{j+1} \text{ at given } \overline{A}_j \\
&\qquad u_{j+1}{}_{|_{\partial\Omega}}(x) = g_{\mathrm{d}}(x) - x\,, \quad \overline{A}_{j+1}{}_{|_{\partial\Omega}} = \overline{A}_{\mathrm{d}} \\
&\qquad \frac{L_c^2}{12}\,\mathrm{Div}\,\mathfrak{M}.\mathrm{D_x}\overline{A} = \widehat{M}_\sharp(\overline{A}_j, \nabla u_{j+1}) - \mathrm{skew}(M)\,.
\end{aligned} \tag{2.4.53}$$

We proceed to show that the sequence $(u_j, \overline{A}_j)_{j=1}^\infty$ is bounded in $H^{m+1}(\Omega, \mathbb{R}^3) \times H^m(\Omega, \mathfrak{so}(3))$ independent of j. To this end we note first that the sequence is uniquely determined if $\overline{A}_j \in C(\overline{\Omega}, \mathfrak{so}(3))$ (extended Korn's first inequality). Based on sharp L^2-elliptic regularity [Ebe02] for systems with variable coefficients (Dirichlet conditions everywhere on $\partial\Omega$) for both equations separately yields for some yet unspecified pair $(m, r) \in \mathbb{N} \times \mathbb{N}$ the estimates [Nef03a]:

$$\begin{aligned}
\|u_{j+1}\|_{m+2,2} &\le C(\Omega, \|\overline{A}_j\|_{m+1,2}) \cdot \left(\|g\|_{m+2,2} + \|f\|_{m,2} + \|\overline{A}_j\|_{m+1,2}\right) \\
\|\overline{A}_{j+1}\|_{r+2,2} &\le C(\Omega) \cdot \left(\|\overline{A}_{\mathrm{d}}\|_{r+2,2} + \|\widehat{M}_\sharp(\overline{A}_j, \nabla u_{j+1})\|_{r,2} + \|\,\mathrm{skew}(M)\|_{r,2}\right),
\end{aligned} \tag{2.4.54}$$

if the solutions are unique, respectively. The constant in the first estimate is a polynomial in $\|\overline{A}_j\|_{m+1,2}$ and bounded above if the Legendre-Hadamard ellipticity constant of the related acoustic tensor is bounded away from zero. The algebraic estimate

$$D^2_{\nabla u} W_{\mathrm{mp}}^{\mathrm{small}}(\nabla u, \overline{A}).(\nabla\phi, \nabla\phi) \ge \mu\,\|(1\!\!1 + \overline{A})^T \nabla\phi + \nabla\phi^T(1\!\!1 + \overline{A})\|^2\,, \tag{2.4.55}$$

implies (cf. (2.2.15)) that

$$\begin{aligned}
D^2_{\nabla u} W_{\mathrm{mp}}^{\mathrm{small}}(\nabla u, \overline{A}).(\xi\otimes\eta, \xi\otimes\eta) &\ge 2\,\mu\,\|(1\!\!1 + \overline{A})^T \xi\otimes\eta\|^2 \\
&\ge 2\mu\lambda_{\min}\left((1\!\!1 + \overline{A})(1\!\!1 + \overline{A})^T\right)\|\xi\|^2\,\|\eta\|^2\,.
\end{aligned} \tag{2.4.56}$$

Since $(1\!\!1 + \overline{A})(1\!\!1 + \overline{A})^T = 1\!\!1 - \overline{A}^2$ and $\langle(1\!\!1 - \overline{A}^2).v, v\rangle = \|v\|^2 + \|\overline{A}.v\|^2 \ge \|v\|^2$, we conclude that $\lambda_{\min}\left((1\!\!1 + \overline{A})(1\!\!1 + \overline{A})^T\right) \ge 1$ and the ellipticity constant of the force balance equation is indeed uniform.

If $\alpha_6 > 0$, then the balance of angular momentum equation has a unique solution and $(2.4.54)_2$ is true as such. If $\alpha_6 = 0$, then we control only $\mathrm{sym}\,\mathrm{D_x}\overline{A}$. However, this is still enough to guarantee a unique solution. [36]

For $r = 0$ we obtain therefore that the sequence $\overline{A}_j$ is bounded independent of j in the space $H^2(\Omega, \mathfrak{so}(3)) \subset C^{2-\frac{3}{2}}(\overline{\Omega}, \mathfrak{so}(3))$ which implies for $m = 1$ that the sequence u_j is bounded independent of j in the space $H^3(\Omega, \mathbb{R}^3)$. Extracting strongly convergent subsequences and letting $j \to \infty$ shows that the limit $(u_\infty, \overline{A}_\infty)$ is a solution of the modified field equation (2.4.52). This ends the argument. Note that due to the definition of $\widehat{M}_\sharp$, we are able to increase the regularity of the solution if the data are smoother. ■

Appreciating the method of proof, we see that the partially linearized model together with quadratic curvature energy is appropriate only for exceptionally 'smooth' situations. The additional provision of sufficiently

[36] For $\overline{A} \in \mathfrak{so}(3, \mathbb{R})$ we have

$$\overline{A} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}, \quad \mathrm{axl}(\overline{A}) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad \nabla\,\mathrm{axl}(\overline{A}) = \begin{pmatrix} \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \\ \gamma_x & \gamma_y & \gamma_z \end{pmatrix}, \quad \mathrm{sym}\,\nabla\,\mathrm{axl}(\overline{A}) = \begin{pmatrix} \alpha_x & \frac{\alpha_y+\beta_x}{2} & \frac{\alpha_z+\gamma_x}{2} \\ \frac{\alpha_y+\beta_x}{2} & \beta_y & \frac{\beta_z+\gamma_y}{2} \\ \frac{\alpha_z+\gamma_x}{2} & \frac{\beta_z+\gamma_y}{2} & \gamma_z \end{pmatrix},$$

$$\|\,\mathrm{sym}\,\nabla\,\mathrm{axl}(\overline{A}\|^2 = \alpha_x^2 + \beta_y^2 + \gamma_z^2 + \frac{(\alpha_y+\beta_x)^2}{2} + \frac{(\alpha_z+\gamma_x)^2}{2} + \frac{(\beta_z+\gamma_y)^2}{2} \tag{2.4.57}$$

$$\|\,\mathrm{sym}\,\mathrm{D_x}\overline{A}\|^2 = \|\,\mathrm{sym}\,\nabla\overline{A}.e_1\|^2 + \|\,\mathrm{sym}\,\nabla\overline{A}.e_2\|^2 + \|\,\mathrm{sym}\,\nabla\overline{A}.e_3\|^2\,. \tag{2.4.58}$$

Now it is easy to see that for some $c^+ > 0$ it holds $\|\,\mathrm{sym}\,\mathrm{D_x}\overline{A}\|^2 \ge c^+\,\|\,\mathrm{sym}\,\nabla\,\mathrm{axl}(\overline{A})\|^2$ since $\|\,\mathrm{sym}\,\mathrm{D_x}\overline{A}\|^2 = 0$ implies $\|\,\mathrm{sym}\,\nabla\,\mathrm{axl}(\overline{A})\|^2 = 0$, algebraically. Hence, the standard Korn's inequality applied to $\|\,\mathrm{sym}\,\nabla\,\mathrm{axl}(\overline{A})\|^2$ yields unique existence.

Note that with the permutation $P = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in \mathrm{O}(3)$ the **infinitesimal micropolar curvature tensor** is defined as $\kappa := \nabla P.\,\mathrm{axl}(\overline{A}) = P\cdot\nabla\,\mathrm{axl}(\overline{A})$ and it holds that $P.\,\mathrm{axl}(\mathrm{skew}(\nabla u)) = \frac{1}{2}\,\mathrm{curl}\,u$ and $\mathrm{tr}\,[\nabla P.\,\mathrm{axl}(\mathrm{skew}(\nabla u))] = \frac{1}{2}\,\mathrm{Div}\,\mathrm{curl}\,u = 0$. As a consequence, if we were to take $W_{\mathrm{mp}}^{\mathrm{infin}}(\kappa) = \frac{L_c^2}{12}\|\kappa\|^2$, then the corresponding infinitesimal couple-stress tensor $D_\kappa W_{\mathrm{mp}}^{\mathrm{infin}}(\kappa)$ is trace-free in the indeterminate couple stress theory. We make no explicit use of κ.

smooth boundary and data and restriction to the pure Dirichlet case allows us to conclude the continuity of the infinitesimal microrotations $\overline{A}$ circumventing the direct use of Sobolev embedding theorems. Considering the reductions made, it is nevertheless our belief that the model (2.4.52) captures the essential features of the geometrically exact Cosserat micropolar framework in contrast to the subsequent infinitesimal models. However, it remains open whether a consistent boundary coupling in the sense of requiring $\overline{A}_{|\partial\Omega} = \operatorname{skew}(\nabla u)_\Gamma$ is possible for this model.

Rather for historical reasons and completeness we reconsider the classical infinitesimal Cosserat micropolar problem. Existence results have been obtained e.g. in [Ies71, Duv70, HH69, Ghe74a, Ghe74b].

2.4.6 Infinitesimal linear elastic Cosserat theory

If we finally assume infinitesimal microrotations and infinitesimal displacements, the micropolar stretch tensor may again be expanded as follows

$$\overline{U} = \overline{R}^T F = (1\!\!1 + \overline{A} + \ldots)^T (1\!\!1 + \nabla u) \approx 1\!\!1 + \nabla u - \overline{A} - \overline{A}\nabla u + \ldots . \tag{2.4.59}$$

Neglecting consistently the quadratic term $\overline{A}\nabla u$ in (2.4.46) yields the approximate expression for the stretch energy [37] (2.2.6)

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u - \overline{A}) &= \mu \, \|\operatorname{sym}(\nabla u - \overline{A})\|^2 + \mu_c \, \|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym}(\nabla u - \overline{A})\right]^2 \\ &= \mu \, \|\operatorname{sym} \nabla u\|^2 + \mu_c \, \|\operatorname{skew}(\nabla u) - \overline{A}\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym}(\nabla u)\right]^2 , \end{aligned} \tag{2.4.60}$$

and for the curvature term

$$W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D_x}\overline{A}) = \mu \frac{L_c^2}{12} \left(\alpha_5 \, \|\operatorname{sym} \mathrm{D_x}\overline{A}\|^2 + \alpha_6 \, \|\operatorname{skew} \mathrm{D_x}\overline{A}\|^2 + \alpha_7 \operatorname{tr}\left[\mathrm{D_x}\overline{A}\right]^2 \right) . \tag{2.4.61}$$

Two observations are essential. First, if $\mu_c = 0$, the infinitesimal problem completely decouples - the infinitesimal microrotations $\overline{A}$ have no influence whatsoever on the macroscopic behaviour of the infinitesimal displacements. We believe that this is a deficiency of the infinitesimal problem without much physical significance for the geometrically exact model. It has led perhaps to the erroneous common belief that $\mu_c > 0$ is regarded to be essential [38] for any true Cosserat micropolar theory as well.

And second, the choice $\alpha_6 = 0$ is possible, contrary to the finite case, since coercivity of the reduced curvature expression can still be concluded on account of the classical Korn's first inequality applied to $\operatorname{sym} \mathrm{D_x}\overline{A}$ as already shown. We mention also that contrary to the finite case there is no gap: balance of angular momentum without internal length scale and $\mu_c > 0$ yields

$$D_{\overline{A}} W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u, \overline{A}) \in \mathrm{Sym} \Leftrightarrow D_{\overline{A}} W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u, \overline{A}) = 0 \Leftrightarrow \operatorname{skew}(\nabla u) = \overline{A} . \tag{2.4.62}$$

This implies already that **infinitesimal continuum- and microrotations coincide**, and this in turn is equivalent to the symmetry of the infinitesimal Cauchy stress σ or the **Boltzmann axiom**.

Hence the infinitesimal case rather inhibits our understanding of the general finite Cosserat micropolar problem.

The infinitesimal micropolar model in variational form is given by

$$\begin{aligned} &\int_\Omega \mu \, \|\operatorname{sym} \nabla u\|^2 + \mu_c \, \|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym} \nabla u\right]^2 + \mu \, \frac{L_c^2}{12} \, \|\nabla \operatorname{axl}(\overline{A})\|^2 - \langle f, u\rangle - \langle M, \overline{A}\rangle \, \mathrm{dV} \\ &\qquad - \int_{\Gamma_S} \langle N, u\rangle \, \mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{A}\rangle \, \mathrm{dS} \mapsto \min . \quad \text{w.r.t. } (u, \overline{A}) \\ &u_{|\partial\Omega}(t,x) = g_{\mathrm{d}}(t,x) - x, \quad x \in \partial\Omega , \quad \overline{A}_{|\partial\Omega} = \operatorname{skew}(\nabla g_{\mathrm{d}}(t,x))_{|\partial\Omega} . \end{aligned} \tag{2.4.63}$$

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[37] Traditionally, the infinitesimal model is obtained not as linearization but by directly assuming a split of the displacement gradient into infinitesimal micropolar stretch and infinitesimal microrotations: $\nabla u = \overline{\varepsilon} + \overline{A}$, where $\overline{\varepsilon}$ is not necessarily symmetric.

[38] In many treatments of the infinitesimal theory, e.g. [Kup79, p.34] the assumption $D^2 W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u - \overline{A}).(H,H) \geq c^+ \|H\|^2$ is explicitly introduced as being motivated on physical grounds rather then being necessary for a meaningful treatment of the infinitesimal Cosserat micropolar theory. It would exclude classical infinitesimal elasticity as a special case. Sometimes, a so called **coupling number** $N^2 = \frac{\mu_c}{\mu + \mu_c} \in [0,1]$ is introduced, which allows to compare different material moduli. $N = 0$ corresponds to classical infinitesimal elasticity, $N = 1$ corresponds to infinitesimal indeterminate couple stress theory.

The corresponding equilibrium system of equations for pure Dirichlet conditions and without external couples M, M_c is given by (note that $\|\overline{A}\|^2 = 2\|\operatorname{axl}(\overline{A})\|^2$ for $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$)

$$\begin{aligned}
0 &= \operatorname{Div}\sigma + f, \quad x \in \Omega \\
\sigma &= 2\mu \operatorname{sym}\nabla u + 2\mu_c\,(\operatorname{skew}(\nabla u) - \overline{A}) + \lambda \operatorname{tr}[\operatorname{sym}\nabla u]\cdot 1\!\!1 \qquad (2.4.64)\\
0 &= \mu\,\frac{L_c^2}{12}\,\Delta \operatorname{axl}(\overline{A}) + \mu_c \operatorname{axl}((\operatorname{skew}(\nabla u) - \overline{A})\,.
\end{aligned}$$

If we consider nonetheless the only nontrivial case left open, $\mu_c > 0$, it is standard to prove that the corresponding minimization problem admits a unique minimizing pair $(u, \overline{A}) \in H^1(\Omega,\mathbb{R}^3)\times H^1(\Omega,\mathfrak{so}(3))$.

Theorem 2.4.8 (Existence for 3D-infinitesimal elastic Cosserat model)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^1(\Omega,\mathbb{R}^3)$ and $\overline{A}_{\rm d} \in H^1(\Omega,\mathfrak{so}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S,\mathbb{R}^3)$ together with $M \in L^2(\Omega,\mathfrak{so}(3,\mathbb{R}))$ and $M_c \in L^2(\Gamma_C,\mathfrak{so}(3,\mathbb{R}))$. Then the model (2.4.63) admits a unique minimizing solution pair $(u,\overline{A}) \in H^1(\Omega,\mathbb{R}^3)\times H^1(\Omega,\mathfrak{so}(3))$. The solution is smoother if the data are smoother.

Proof. We are again in a position to apply the direct methods of variations. Without loss of generality consider $N, M, M_c = 0$ and $\mu \geq \mu_c > 0$. First we observe that infimizing sequences $(u_k, \overline{A}_k)$ exist and

$$\begin{aligned}
\infty &> \int_\Omega W_{\rm mp}^{\rm infin}(\nabla u_k - \overline{A}_k) + W_{\rm curv}^{\rm small}(\mathrm{D_x}\overline{A}_k) - \langle f, u_k\rangle\,\mathrm{dV} \geq \int_\Omega \mu_c\,\|\nabla u_k - \overline{A}_k\|^2\,\mathrm{dV} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)} \qquad (2.4.65)\\
&= \int_\Omega \mu_c\,\|\operatorname{sym}(\nabla u_k - \overline{A}_k)\|^2 + \mu_c\,\|\operatorname{skew}(\nabla u_k - \overline{A}_k)\|^2\,\mathrm{dV} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)}\\
&\geq \int_\Omega \mu_c\,\|\operatorname{sym}\nabla u_k\|^2\,\mathrm{dV} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)} \geq \mu_c\,c_K\,\|u_k\|^2_{H^1(\Omega)} - \|f\|_{L^2}\,\|u_k\|_{H^1(\Omega)}\,,
\end{aligned}$$

showing that u_k is bounded in $H^1(\Omega)$. We have used that sym is orthogonal to skew and the classical Korn's first inequality together with the boundary conditions for u_k. Moreover, again by the classical Korn's first inequality (if $\alpha_6 = 0$) or directly pointwise, we obtain boundedness of $\overline{A}_k$ in $H^1(\Omega,\mathfrak{so}(3))$. We can choose a subsequence of $(u_k, \overline{A}_k)$ converging strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$. By overall convexity of the energy density in $(\nabla u, \mathrm{D_x}\overline{A})$ the limit pair is a minimizer.

For the uniqueness we consider the second derivative of the total strain energy $W = W_{\rm mp}^{\rm infin} + W_{\rm curv}^{\rm small}$ with respect to the complete argument

$$\begin{aligned}
&D^2_{(\nabla u,\overline{A})} W(\nabla u - \overline{A}).((\nabla\phi,\delta\overline{A}),(\nabla\phi,\delta\overline{A})) \geq \mu_c\|\nabla\phi - \delta\overline{A}\|^2\\
&= \mu_c\|\operatorname{sym}\nabla\phi\|^2 + \mu_c\|\operatorname{skew}(\nabla\phi - \delta\overline{A})\|^2 \geq \mu_c\|\operatorname{sym}\nabla\phi\|^2\,. \qquad (2.4.66)
\end{aligned}$$

By the classical Korn's first inequality we obtain uniform positivity of the second derivative upon integration. [39] The functional is strictly convex, the solution is unique.

Since the resulting field equations of force balance and balance of angular momentum are linear, uniformly elliptic with constants coefficients the standard elliptic regularity theory applies such that for pure Dirichlet boundary conditions the solution is the smoother the smoother the data. ∎

Theorem 2.4.9 (Existence for 3D-infinitesimal elastic Cosserat model with consistent coupling)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^1(\Omega,\mathbb{R}^3)$ and $\operatorname{skew}(\nabla g_{\rm d}) \in H^1(\Omega,\mathbb{M}^{3\times 3})$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S,\mathbb{R}^3)$ together with $M \in L^2(\Omega,\mathfrak{so}(3,\mathbb{R}))$ and $M_c \in L^2(\Gamma_C,\mathfrak{so}(3,\mathbb{R}))$. Then the model (2.4.63) with **consistent coupling**

$$\forall\,\varepsilon > 0: \quad \overline{A}^\varepsilon{}_{|_\Gamma} = \operatorname{skew}(\nabla u^\varepsilon(t,x))_{|_\Gamma}\,, \qquad (2.4.67)$$

admits a unique minimizing solution pair $(u,\overline{A}) \in H^1(\Omega,\mathbb{R}^3)\times H^1(\Omega,\mathfrak{so}(3))$. The solution is smoother if the data are smoother. Here, $\overline{A}^\varepsilon$ is a suitable regularization by mollification which allows to evaluate $\overline{A}^\varepsilon$ at the boundary Γ.

[39] Assume $(\phi,\delta\overline{A}) \neq 0$, then $D^2_{(\nabla u,\overline{A})} W(\nabla u - \overline{A}).((\nabla\phi,\delta\overline{A}),(\nabla\phi,\delta\overline{A})) > 0$.

Proof. Observe that with g_{d} as specified the energy is finite for consistent coupling in the strong sense. Next, note that infimizing sequences $(u_k, \overline{A}_k)$, satisfying strong consistent coupling exist and are bounded since the displacement can be controlled independent of $\overline{A}_k$ and the additional L^2 control of $\|\operatorname{skew}(\nabla u) - \overline{A}_k\|$ shows that $\overline{A}_k$ is as well L^2-controlled. Hence $(u_k, \overline{A}_k) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathfrak{so}(3))$ are bounded, independent of k. Extracting weakly convergent subsequences and using that the relaxed consistent coupling condition is also weakly closed for any $\varepsilon > 0$ finishes the argument. The role played by exact rotations in the finite case is replaced by the role of $\mu_c > 0$. ■

The corresponding **infinitesimal gradient constrained Cosserat micropolar** model (infinitesimal indeterminate couple stress model) has the form (simplified curvature term: $\alpha_5 = \alpha_6 = 1$, $\alpha_7 = 0$)

$$\begin{aligned}
&\int_\Omega \mu\|\operatorname{sym}\nabla u\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}\nabla u\right]^2 + \mu\,\frac{L_c^2}{12}\,\|\mathrm{D_x}\operatorname{skew}(\nabla u)\|^2 - \langle f, u\rangle\,\mathrm{dV}\\
&\qquad - \int_{\Gamma_S} \langle N, u\rangle\,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \operatorname{skew}(\nabla u)\rangle\,\mathrm{dS} \mapsto \min.\ \text{w.r.t. } u \qquad (2.4.68)\\
&\sigma = \sigma^{\mathrm{loc}} + \sigma^{\mathrm{hyper}}\\
&\sigma^{\mathrm{loc}} = 2\,\mu\operatorname{sym}(\nabla u) + \lambda\operatorname{tr}\left[\operatorname{sym}(\nabla u)\right]\cdot 1\!\!1 \in \operatorname{Sym},\ \text{constitutive stress}\\
&\sigma^{\mathrm{hyper}} = -2\mu\,\frac{L_c^2}{12}\operatorname{axl}^{-1}\left(\operatorname{Div}\nabla\operatorname{axl}(\operatorname{skew}(\nabla u))\right) \in \mathfrak{so}(3,\mathbb{R})\\
&u_{|\partial\Omega}(x) = g_{\mathrm{d}}(x) - x\,, \quad \operatorname{skew}(\nabla u)_{|\partial\Omega} = \operatorname{skew}(\nabla g_{\mathrm{d}})_{|\partial\Omega}\,.
\end{aligned}$$

Using the same methods as before we obtain

Theorem 2.4.10 (Existence for 3D-infinitesimal gradient case)
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary of class C^1 and assume for the boundary data $g_{\mathrm{d}} \in H^2(\Omega, \mathbb{R}^3)$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M_c \in L^2(\Gamma_C, \mathfrak{so}(3))$. Then the model (2.4.68) admits a unique minimizing solution $u \in H^1(\Omega) \cap \{\nabla \operatorname{curl} u \in L^2(\Omega)\}$.

Proof. As before. See also [Duv70]. ■

2.5 On the choice of the curvature energy contribution

In the finite regime we have various choices for the assumed dependence of the elastic energy density on gradients of the microrotations. It seems that there is no universally correct term available, rather the applications aimed at dictate in some respect this form. We list several of these possibilities which can be useful:

1. The case exhibiting similarity to **plates** and **shells**:

$$W_{\mathrm{curv}}(\mathfrak{K}) = \mu\frac{L_c^{1+p}}{12}\,(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}\|^q)\left(\alpha_5\,\|\operatorname{sym}\mathfrak{K}\|^2 + \alpha_6\,\|\operatorname{skew}\mathfrak{K}\|^2 + \alpha_7\operatorname{tr}\left[\mathfrak{K}\right]^2\right)^{\frac{1+p}{2}}, \qquad (2.5.69)$$

for $\mu_c = \alpha_4 = \alpha_6 = 0$, $p = 1$, one component of $\mathfrak{K}$, namely $\overline{R}^T\nabla\overline{R}.e_3$ appears in the resulting curvature contribution obtained after expanding the underlying stretch energy with respect to deformations of the midsurface $m(x, y)$ of a plate assuming that $\varphi(x, y, z) = m(x, y) + z\,\overline{R}.e_3$ and integrating through the thickness. For infinitesimal rotations this may be reduced to

$$W_{\mathrm{curv}}(\mathrm{D_x}\overline{A}) = \mu\frac{L_c^2}{12}\left(\alpha_5\,\|\operatorname{sym}\mathrm{D_x}\overline{A}\|^2 + \alpha_6\,\|\operatorname{skew}\mathrm{D_x}\overline{A}\|^2 + \alpha_7\operatorname{tr}\left[\mathrm{D_x}\overline{A}\right]^2\right), \qquad (2.5.70)$$

which yields pointwise control of $\mathrm{D_x}\overline{A}$ for $\alpha_5, \alpha_6 > 0$. If $\alpha_6 = 0$, non-local control of $\|\mathrm{D_x}\overline{A}\|_{L^2(\Omega)}$ is possible on account of Korn's classical first inequality.

2. The case for **almost rigid material** behaviour: i.e. $\nabla\varphi \approx \overline{R} \in \mathrm{SO}(3,\mathbb{R})$, then $\mathfrak{K} = \overline{R}^T\mathrm{D_x}\overline{R} \approx \operatorname{Sym}$, such that the former may be reduced to

$$W_{\mathrm{curv}}(\mathfrak{K}) = \mu\frac{L_c^{1+p}}{12}\,(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}\|^q)\left(\alpha_5\,\|\mathfrak{K}\|^2 + \alpha_7\operatorname{tr}\left[\mathfrak{K}\right]^2\right)^{\frac{1+p}{2}}, \qquad (2.5.71)$$

where pointwise control of the torsion-curvature tensor $\mathfrak{K}$ is possible.

3. The case accounting for **dislocations**: a measure accounting for incompatibility is the torsion tensor $\overline{R}^T \operatorname{Curl} \overline{R} \in \mathbb{M}^{3\times 3}$. Orthogonal to this expression is $\overline{R}^T.\operatorname{Div} \overline{R} \in \mathbb{R}^3$ (Helmholtz decomposition), a measure for the curvature. Thus W_{curv} may as well be assumed to consist of two independent parts
$$W_{\text{tors,div}} = \mu \frac{L_c^{1+p}}{12} \left(\alpha_8 \|\overline{R}^T \operatorname{Curl} \overline{R}\|^2 + \alpha_9 \|\overline{R}^T.\operatorname{Div} \overline{R}\|^2 \right)^{\frac{1+p}{2}} . \tag{2.5.72}$$
Non-local control of all first order derivatives of $\overline{R}$ is possible, if $\alpha_8, \alpha_9 > 0$. For infinitesimal incompatibility, $\overline{R} = 1\!\!1 + \overline{A} + \dots$ and $\overline{R}^T \operatorname{Curl} \overline{R} \approx \operatorname{Curl} \overline{A}$ such that $\alpha_8 > 0, \alpha_9 \geq 0$ is sufficient for pointwise(!) control of $\mathrm{D_x} \overline{A}$ on account of the fact that $\operatorname{Curl} \overline{A}$ for $\overline{A} \in \mathfrak{so}(3, \mathbb{R})$ controls all first derivatives of $\overline{A}$. [40] While it is clear that we cannot control completely $\|\mathrm{D_x} \overline{R}\|$ if $\alpha_9 = 0$ in the finite case we might still stand a chance to show that $\alpha_8 > 0, \alpha_9 = 0$ is already sufficient for the variational problem. Counting derivatives in the finite case, we see that 9 independent entries in $\operatorname{Curl} \overline{R}$ together with 3 independent entries in $\operatorname{Div} \overline{R}$ control already the norm of a total of 27 derivatives of R. This is the object of ongoing research.

4. The traditional representation based on the **reduced second order curvature tensor** $\hat{\mathfrak{K}}$:
$$W_{\text{curv}}(\hat{\mathfrak{K}}) = \mu \frac{L_c^{1+p}}{12} \left(\alpha_5 \, \| \operatorname{sym} \hat{\mathfrak{K}}\|^2 + \alpha_6 \, \| \operatorname{skew} \hat{\mathfrak{K}}\|^2 + \alpha_7 \operatorname{tr} \left[\hat{\mathfrak{K}}\right]^2 \right)^{\frac{1+p}{2}} , \tag{2.5.73}$$
where $\hat{\mathfrak{K}} := \left(\operatorname{axl}(\overline{R}^T \partial_x \overline{R}) | \operatorname{axl}(\overline{R}^T \partial_y \overline{R}) | \operatorname{axl}(\overline{R}^T \partial_z \overline{R}) \right) \in \mathbb{M}^{3\times 3}$. For small curvature this is further reduced to $(\operatorname{axl}(\partial_x \overline{A}) | \operatorname{axl}(\partial_y \overline{A}) | \operatorname{axl}(\partial_y \overline{A})) = \nabla \operatorname{axl}(\overline{A}) = P \cdot \kappa$ with κ the **infinitesimal micropolar curvature tensor** and the permutation matrix P as in (2.4.57).

2.6 The quasiconvex hull and relaxation

Since we have investigated the salient regularizing features of the elastic Cosserat approach which in principle should furnish a mesh-independent solution for related numerical implementations it seems worthwhile to contrast this with another well known successful computational method [Lam02], based on quasiconvexity, to obtain mesh-independent results for stress-strain relations. [41]

The elastic free energy density $W : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ is **quasiconvex**, if
$$\forall \hat{F} \in \mathbb{M}^{3\times 3} : \; |\Omega| \cdot W(\hat{F}) \leq \int_\Omega W(\hat{F} + \nabla\phi(x)) \, \mathrm{dV} \quad \forall \phi \in C_0^\infty(\Omega, \mathbb{R}^3) . \tag{2.6.74}$$
This implies notably that the homogeneous deformation is a global minimizer to its own boundary conditions. For a given free energy density $W : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ the quasiconvex hull QW is defined as the largest quasiconvex function below W, i.e.
$$QW := \sup\{\tilde{W} \leq W \; : \; \tilde{W} \text{ is quasiconvex}\} . \tag{2.6.75}$$
If W is locally bounded and Borel measurable, another equivalent characterisation [Dac89] is
$$QW(F) := \inf \{ \frac{1}{|\Omega|} \int_\Omega W(F + \nabla\phi(x)) \, \mathrm{dV} \; : \; \phi \in C_0^\infty(\Omega, \mathbb{R}^3) \} , \tag{2.6.76}$$
and the infimum is independent of Ω. The resulting **relaxed** function is always Legendre-Hadamard elliptic,
$$D_F^2[QW](F).(\xi \otimes \eta, \xi \otimes \eta) \geq 0 , \tag{2.6.77}$$
and the quasiconvex hull QW is weakly lower semi continuous.

In most cases it is not possible to derive analytical formulae for QW, however, there is one notable exception. If $W(F) = \hat{W}(U)$ is a convex function in U and bounded below, then a general result in [Pip94] implies[42] that
$$QW(F) = \inf_{P \in \mathrm{PSym}} \hat{W}(U + P) . \tag{2.6.78}$$

[40] In fact, for $A \in \mathfrak{so}(3, \mathbb{R})$ it holds that $\mathrm{D_x} A = L.\operatorname{Curl} A$, with L a linear mapping.

[41] In many practical cases, not a complete mesh-independent result for stresses and deformations is needed, but only a mesh-independent result for the stresses is looked for. The relaxed functional serves this restricted purpose. Average values of stresses are computed, the geometry of deformation is only considered in a probabilistic sense.

[42] The proof in [Pip94] is based on $W(F) = \hat{W}(C)$, $\hat{W}$ convex in C. With appropriate changes it carries over to the representation in U. We remark that the loss of ellipticity of $\|C - 1\!\!1\|^2$ is already occurring in uniaxial compression. Clearly a deficiency of the Green strains not shared by the Biot strains which loose ellipticity in biaxial situations only.

Let us apply this result to $W(F) = \|U - 1\!\!1\|^2$ which is easily seen to be not Legendre-Hadamard elliptic over the whole range of admissible deformations $F \in \mathrm{GL}^+(3, \mathbb{R})$ but to satisfy the Baker-Ericksen inequalities. The loss of ellipticity is due to the behaviour of the orthogonal part $R = \mathrm{polar}(F)$ of F. We obtain

$$QW(F) = \sum_{\lambda_i \geq 1} |\lambda_i - 1|^2 \,, \tag{2.6.79}$$

where $\lambda_i \geq 0$ are the eigenvalues of U. The resulting function is Legendre-Hadamard elliptic but not uniformly, since compression has zero energy. Moreover, the linearization of the relaxed density $QW(F)$ does not coincide with the linearization of $W(F)$ which is uniformly Legendre-Hadamard elliptic in a neighbourhood of $F = 1\!\!1$. It is well known that in a neighbourhood of the identity, the nonlinear unmodified problem is well-posed [Val88], also under compression. It is therefore apparent that replacing W by its quasiconvex hull would modify the physical nature of the problem. Special care should therefore be exerted when taking QW also as a means of regularization, especially when the analytical form of W is not known, as is often the case in incremental elasto-plasticity.

2.7 Discussion

A finite Cosserat model has been introduced and is consistently extended to elasto-viscoplasticity where Cosserat effects remain non-dissipative. Various reduced forms of the model are introduced. For elasticity, it is motivated that the Cosserat ansatz with **independent** rotations is especially suited in conjunction with energies quadratic in the micropolar stretch tensor $\overline{U} = \overline{R}^T F$, in which case the rotations are indeed the only essential nonlinearity left in the problem.

Our constitutive analysis suggests that the Cosserat ansatz with weak local coupling (Cosserat couple modulus $\mu_c = 0$) leading to a stretch energy density for small elastic strains of the form $W_{\mathrm{mp}}(\overline{U}) = \mu \, \|\, \mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \frac{\lambda}{2} \, \mathrm{tr} \left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2$ provides just the correct amount of regularization needed for classical macroscopic energies quadratic in the continuum stretch since loss of Legendre-Hadamard ellipticity in the equation of linear momentum can be traced back to the presence of the continuum rotations and is removed by taking independent variations with respect to these rotations.

A competing method of regularization, namely the computation of the quasiconvex hull QW_{mp} of the energy, which also restores the Legendre-Hadamard ellipticity leads to unphysical behaviour under compression in certain cases: the material shows no resistance under compression. Hence, the quasiconvexification is a useful computational tool to reduce the influence of localizations but can hardly be regarded as generally admissible without admitting a modification of the underlying physics which is however, sometimes intended, especially in elasto-plasticity, where complete local relaxation is arguably the optimal response. In the elastic case, we prefer to augment the underlying physics by introducing a length scale.

A delicate interplay between stretch and curvature terms allows under reasonable physical assumptions to establish the existence of minimizers or stationary points of the corresponding elastic action functional. The sub-critical case (including the true fracture case IV) $\mu_c = 0$, $\alpha_4 > 0$, $0 < p < 1$, $q \geq 0$ in (2.2.7) and the critical case $\mu_c = 0$, $\alpha_4 > 0$, $p = 1$, $q = 0$ in (2.2.7) and $\mu_c = 0$, $\alpha_8 > 0$, $\alpha_9 = 0$, $p \geq 1$ in (2.5.72) are not settled and leave a wide field of challenging purely mathematical problems.

Depending on the applications aimed at, the Cosserat couple modulus μ_c should either be very large (microscopic specimens) or zero (macroscopic specimens). The mathematical analysis reflects this dichotomy.

The different reduced elastostatic Cosserat micropolar models have thus been shown to be completely justified. The finite macroscopic elastic-plastic case where Cosserat effects are assumed to be non-dissipative, however, is completely left open.

As another result of our investigation we note that extreme care should be exerted when determining material constants already for isotropic Cosserat micropolar models: the data fit should preferably be based on the geometrically exact model with the same number and type of parameters and not on the infinitesimal model which degenerates for admitted values of material parameters which however, are to be fitted. If this is done it might turn out that for specific macroscopic situations we are left with the problem of finding just the characteristic internal length L_c while the Cosserat couple modulus μ_c, already orders of magnitudes smaller than the classical shear modulus μ in many cases, drops completely out. In the macroscopic case, we favour therefore an essentially reduced **three-parameter isotropic Cosserat micropolar** theory with **independent** microrotations: the two classical Lamé constants μ, λ, one additional internal characteristic length scale $L_c > 0$ and $(\mu_c, \alpha_4, \alpha_5, \alpha_6, \alpha_7, p, q) = (0, 1, 1, 1, 0, 1, 1)$, thus discarding all other unnecessary material constants. This far reaching reduction will also facilitate renewed experimental identification of the length scale L_c and its precise relation to the shear band width, obscured in the infinitesimal theory.

To conclude, we believe that the choice $\mu_c = 0$ represents a refreshing departure from traditional linear approaches; it reconciles experimental evidence on a macroscale and the possibility of fracture with the Cosserat model and leads to interesting new mathematical problems. It shows to furnish a natural method to physically regularize certain shear failure problems and it introduces experimentally observed second order size effects which seem to be present in nearly all materials. It is therefore hoped that the presented development will stimulate further mathematical research in this important field for the applications.

2.8 Appendix A

2.8.1 The Boltzmann axiom without internal length

Lemma 2.8.1 (Limit rotations with zero internal length scale)
Define W_{mp} *as*

$$W_{\mathrm{mp}}(\overline{U}) = \alpha_1 \,\|\,\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c \,\|\,\mathrm{skew}(\overline{U})\|^2 + \alpha_3 \,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2 \tag{2.8.80}$$

as in (2.2.6). *If* $\alpha_1 = \mu_c$ *and* $\mathrm{tr}\left[\overline{U}\right] < 3 + \frac{2\mu_c}{\alpha_3}$ *then* $D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T \in \mathrm{Sym} \Rightarrow \left[\overline{U} \in \mathrm{Sym} \Leftrightarrow \overline{R} = \mathrm{polar}(F)\right]$. *Otherwise,* $D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$ *alone does not imply* $\overline{U} \in \mathrm{Sym}$.

Proof. An argument relating to the general case of W_{mp} taken as an isotropic scalar valued function of $\overline{U}$ has been given e.g. in [San99, p.29] and [SB95]. However, no conditions on the coefficients or the magnitude of $\mathrm{tr}\left[\overline{U}\right]$ are involved, which raises some questions. Note first that $D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$ is equivalent to

$$\mathrm{skew}\left((\alpha_1 - \mu_c)\,\overline{U}^T\overline{U}^T - 2\,\alpha_1\,\overline{U}^T + 2\,\alpha_3\,\mathrm{tr}\left[\overline{U} - 1\!\!1\right]\overline{U}^T\right) = 0\,. \tag{2.8.81}$$

We write $\overline{U} = \mathrm{sym}\,\overline{U} + \mathrm{skew}\,\overline{U} = S + A$. This yields in three steps

$$\begin{aligned}
\mathrm{skew}\left((\alpha_1 - \mu_c)\,(S + A)^T(S + A)^T - 2\,\alpha_1\,(S + A)^T + 2\,\alpha_3\,\mathrm{tr}\,[S - 1\!\!1]\,(S + A)^T\right) &= 0\\
\mathrm{skew}\left((\alpha_1 - \mu_c)\,(S^TS^T + S^TA^T + A^TS^T + A^TA^T) - 2\,\alpha_1\,A^T + 2\,\alpha_3\,\mathrm{tr}\,[S - 1\!\!1]\,A^T\right) &= 0\\
\mathrm{skew}\left((\alpha_1 - \mu_c)\,(SA^T + A^TS) - 2\,\alpha_1\,A^T + 2\,\alpha_3\,\mathrm{tr}\,[S - 1\!\!1]\,A^T\right) &= 0\,,
\end{aligned} \tag{2.8.82}$$

and since $A^2 \in \mathrm{Sym}(3)$ for $A \in \mathfrak{so}(3, \mathbb{R})$, this leads to the system

$$-(\alpha_1 - \mu_c)\,(SA + AS) + 2\,(\alpha_1 + \alpha_3\,\mathrm{tr}\,[S - 1\!\!1])\,A = 0\,, \tag{2.8.83}$$

which represents 3 linear equations for three unknowns in $A \in \mathfrak{so}(3, \mathbb{R})$. If $\alpha_1 = \mu_c$ and $\mathrm{tr}\left[\overline{U}\right] < 3 + \frac{2\mu_c}{\alpha_3}$ then $A = 0$ necessarily. A small algebraic argument shows that $SA + AS = 0$, $A \in \mathfrak{so}(3, \mathbb{R})$, $S \in \mathrm{Sym}$ implies $A = 0$ if $(d_1 + d_2)\,(d_2 + d_3)\,(d_1 + d_3) \neq 0$ for d_i the eigenvalues of S. For the second statement, set $\mu_c = \mu = \alpha_1$, $\alpha_3 = \frac{\lambda}{2} = \mu$, which is in the range of actual material behaviour of macroscopic crystalline solids, and consider accordingly

$$W_{\mathrm{mp}}(\overline{U}) = \mu\,\|\overline{U} - 1\!\!1\|^2 + \mu\,\mathrm{tr}\left[\overline{U} - 1\!\!1\right]^2, \quad D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^T = 2\mu\,\overline{U}\,\overline{U}^T - 2\mu\,\overline{U}^T + 2\mu\,\mathrm{tr}\left[\overline{U} - 1\!\!1\right]\overline{U}^T\,. \tag{2.8.84}$$

Now take

$$\overline{U} = \begin{pmatrix} 1 & \alpha & 0\\ -\alpha & 1 & 0\\ 0 & 0 & 2 \end{pmatrix} \in \mathrm{GL}^+(3, \mathbb{R}) \text{ if } |\alpha| \ll 1 \quad \text{and} \quad \overline{U} \notin \mathrm{Sym}\,. \tag{2.8.85}$$

The corresponding microrotation $\overline{R}$ and deformation φ such that $\overline{U} = \overline{R}^T\nabla\varphi$ can be easily deduced. This implies $D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^T \in \mathrm{Sym}$ and balance of linear momentum is satisfied for zero loads since the quantities $(\overline{R}, \nabla\varphi)$ are homogeneous if appropriate boundary conditions are specified but $\overline{U} \notin \mathrm{Sym}$. ∎

2.8.2 Macroscopic elastic shear failure

It is appropriate to define what we mean by **shear failure** in classical isotropic elasticity. Let $W(F)$ be the free elastic energy density of the bulk material. If for some $F \in \mathrm{GL}^+(3, \mathbb{R})$

$$\exists\,\xi, \eta \in \mathbb{R}^3: \quad D^2W(F).(\xi \otimes \eta, \xi \otimes \eta) < 0\,, \tag{2.8.86}$$

we say that the material fails or looses **Legendre-Hadamard ellipticity (LH)**, also called a **material instability**. [43] This failure can give rise to highly localized deformation patterns, subsumed under the notion of **microstructure**. Related is the possible emergence of discontinuous deformations since **Hadamard´s jump relations** are violated. However, loss of ellipticity may already occur for deformations which are not related to shear, e.g. uniaxial situations and pure dilations. Thus we say that W suffers from **genuine elastic shear failure** whenever

$$\begin{aligned}
&\exists\, F \in \mathrm{GL}^+(3, \mathbb{R}) \quad \exists\,\xi, \eta \in \mathbb{R}^3: \quad D^2W(F).(\xi \otimes \eta, \xi \otimes \eta) < 0\,, \quad \text{but}\\
&\forall\, F \in \mathrm{diag}(\lambda_1^+, \lambda_2^+, \lambda_3^+) \quad \forall\,\xi, \eta \in \mathbb{R}^3: \quad D^2W(F).(\xi \otimes \eta, \xi \otimes \eta) \geq 0\,.
\end{aligned} \tag{2.8.87}$$

It seems that failure of a material on a macroscale other than shear failure is unphysical and rather due to the idiosyncrasy of the constitutive equations, as long as the bulk is modelled as elastic. In fact, Legendre-Hadamard ellipticity for $F = \mathrm{diag}(\lambda_1^+, \lambda_2^+, \lambda_3^+)$

[43] Material instability should be carefully distinguished from **geometrical instabilities** occurring in buckling or necking and which are fully consistent with Legendre-Hadamard ellipticity. In this sense, **polyconvex** materials are **unconditionally** materially stable and certainly appropriate for rubber and soft-tissues [SN02, HN03].

implies immediately the **Baker-Ericksen (BE)** inequalities [MH83, p.19] and genuine elastic shear failure happens, if BE is satisfied but LH is violated.[44]

In this sense the following non exhaustive list of free energy terms should be avoided since they are not only failing under shear (already BE is not satisfied):

$$\|C-\mathbb{1}\|^2,\ \langle C-\mathbb{1},\mathbb{1}\rangle^2,\ \langle \ln C,\mathbb{1}\rangle^2,\ \langle \ln C,\mathbb{1}\rangle^2+\|\operatorname{dev}\ln C\|^2,\ \langle \ln U,\mathbb{1}\rangle^2,\\ -\ln\det[F]+(\ln\det[F])^2,\ \|\frac{C}{\det[C]^{1/3}}-\mathbb{1}\|^2\,. \tag{2.8.88}$$

Of course, combination with other terms could remove the problem. Terms which genuinely fail only in shear are e.g. $\|U-\mathbb{1}\|^2$, $\langle U-\mathbb{1},\mathbb{1}\rangle^2$ and $\|\frac{U}{\det[U]^{1/3}}-\mathbb{1}\|^2$, $\operatorname{tr}\left[\frac{U}{\det[U]^{1/3}}-\mathbb{1}\right]^2$.

2.8.3 Analytical investigation of incompressible elastic simple shear

In order to elucidate the proposed theory and to be able to validate numerical solutions we consider first the deformation of an **incompressible** homogeneous unit cube in simple shear at the upper and lower faces. Let $\Omega=[0,1]\times[0,1]\times[0,1]$ be the unit cube and impose the boundary conditions $g(x_1,x_2,0)=(x_1,x_2,0)^T$, $g(x_1,x_2,1)=(x_1+\gamma,x_2,x_3)^T$, $0\le x_1,x_2\le 1$. The parameter $\gamma\ge 0$ is the amount of maximal shear at the upper face per unit length. In order to arrive at an analytically tractable one-dimensional problem we wish to find **energy minimizing** deformations of the form

$$\varphi(x_1,x_2,x_3)=\begin{pmatrix}x_1+u(x_1,x_3)\\ x_2\\ x_3\end{pmatrix},\ \nabla\varphi(x_1,x_2,x_3)=F=\begin{pmatrix}1+u_{x_1}(x_1,x_3) & 0 & u_{x_3}(x_1,x_3)\\ 0&1&0\\0&0&1\end{pmatrix}, \tag{2.8.89}$$

with $u(x_1,0)=0$, $u(x_1,1)=\gamma$. Effectively, we require, that initially horizontal material planes remain horizontal.[45] Incompressibility requires that $\det[F]=1$ and implies $u_{x_1}(x_1,x_3)=0$. The boundary conditions show then that u must be constant in x_1-direction. Hence the reduced kinematics

$$\varphi(x_1,x_2,x_3)=\begin{pmatrix}x_1+u(x_3)\\ x_2\\ x_3\end{pmatrix},\ \nabla\varphi(x_1,x_2,x_3)=F=\begin{pmatrix}1&0&u'(x_3)\\0&1&0\\0&0&1\end{pmatrix}, \tag{2.8.90}$$

with $u(0)=0$, $u(1)=\gamma$ suffices.[46] Accordingly we postulate microrotations $\overline{R}\in\mathrm{SO}(3,\mathbb{R})$ and infinitesimal microrotations $\overline{A}\in\mathfrak{so}(3,\mathbb{R})$ of the type:

$$\overline{R}(x_1,x_2,x_3)=\begin{pmatrix}\cos\alpha(x_3)&0&\sin\alpha(x_3)\\0&1&0\\-\sin\alpha(x_3)&0&\cos\alpha(x_3)\end{pmatrix},\quad \overline{A}(x_1,x_2,x_3)=\begin{pmatrix}0&0&\alpha(x_3)\\0&1&0\\-\alpha(x_3)&0&0\end{pmatrix}. \tag{2.8.91}$$

This implies that $\|\mathrm{D_x}\overline{R}\|^2=\|\mathrm{D_x}\overline{A}\|^2=2\,|\alpha'(x_3)|^2$. In view of symmetry considerations we try to find solutions for the shear profile angle $\alpha:[0,1]\mapsto\mathbb{R}$ of the form $\alpha(1/2+x)=\alpha(1/2-x)$, implying that $\alpha^{(n)}(0)=(-1)^n\,\alpha^{(n)}(0)$. Explicit calculation shows that

$$\overline{R}^TF=\begin{pmatrix}\cos\alpha(x)&0&\cos\alpha(x)\cdot u'(x)-\sin\alpha\\0&1&0\\ \sin\alpha(x)&0&\sin\alpha(x)\cdot u'(x)+\cos\alpha(x)\end{pmatrix},\ \operatorname{sym}\overline{R}^TF=\begin{pmatrix}\cos\alpha(x)&0&\frac{\cos\alpha(x)\cdot u'(x)}{2}\\0&1&0\\ \frac{\cos\alpha(x)\cdot u'(x)}{2}&0&\sin\alpha(x)\cdot u'(x)+\cos\alpha(x)\end{pmatrix},$$
$$\operatorname{skew}\overline{R}^TF=\begin{pmatrix}0&0&\frac{\cos\alpha(x)\cdot u'(x)-2\sin\alpha(x)}{2}\\0&0&0\\-\frac{\cos\alpha(x)\cdot u'(x)-2\sin\alpha(x)}{2}&0&0\end{pmatrix}. \tag{2.8.92}$$

The energy[47] is given by

$$\int_\Omega \mu\,\|\operatorname{sym}\overline{R}^TF-\mathbb{1}\|^2+\mu_c\,\|\operatorname{skew}\overline{R}^TF\|^2+2\mu\,L_c^2|\alpha'(x)|^2\,\mathrm{dV}\mapsto\min.\quad \text{w.r.t. } (u,\alpha). \tag{2.8.93}$$

$u(0)=0,\ u(1)=\gamma,$ Dirichlet boundary conditions for displacement

$\alpha(0)=\alpha(1)=0,$ classical **(rigid)** boundary conditions for microrotations

$0=\operatorname{skew}(\overline{R}^TF)_{\{0,1\}}\ \Leftrightarrow\ 2\tan\alpha(0)=u'(0),\ 2\tan\alpha(1)=u'(1),$ **consistent coupling** boundary conditions

[44] One version of **BE** can be stated as follows: for $\lambda_i>0$ the principal stretches, the free energy $\Phi(\lambda_1,\lambda_2,\lambda_3):=W(U)$ is separately convex in λ_i. No mathematical existence results based only on BE are known. Note also that BE is enough to effectively exclude phase-transformations, modelled with multi-well potentials.

[45] These are not the most general possible deformations. The most general deformations are of the form $\varphi(x_1,x_2,x_3)=(x_1+u(x_1,x_3),x_2,x_3+v(x_1,x_3))^T$.

[46] The considered problem is therefore the exact formulation of the **simple glide** in e_1-direction with amount γ at the upper face of an infinite layer of material with unit height fixed at the bottom.

[47] The energy corresponds to the class studied in Theorem 2.4.4. Note that by the analytical methods proposed in section (4) we already know that minimum energy configurations in the finite case exist for both types of boundary conditions. In the one-dimensional case the coercivity $|\alpha'(x)|^2$ is enough to guarantee strong convergence of minimizing sequences of microrotations $(\sin\alpha_k,\cos\alpha_k)$ in the space of continuous functions due to Sobolev embedding theorems.

in the finite case[48] and

$$\int_\Omega \mu\,\|\operatorname{sym}F-1\!\!1\|^2+\mu_c\,\|\operatorname{skew}F-\overline{A}\|^2+2\mu\,L_c^2|\alpha'(x)|^2\,\mathrm{dV}\mapsto\min.\quad\text{w.r.t. }(u,\alpha). \tag{2.8.94}$$

$$\begin{aligned}
&u(0)=0,\ u(1)=\gamma\\
&\alpha(0)=\alpha(1)=0,\quad\text{classical rigid boundary conditions}\\
&0=\left(\operatorname{skew}(F)-\overline{A}\right)_{\{0,1\}}\ \Leftrightarrow\ 2\alpha(0)=u'(0),\ 2\alpha(1)=u'(1),\quad\text{reduced consistent coupling boundary conditions},
\end{aligned}$$

in the infinitesimal case. The classical elasticity formulation in the finite case[49] is recovered, if $\mu_c=0$, $L_c=0$, $\operatorname{skew}(\overline{R}^TF)=0\Leftrightarrow 2\tan\alpha(x)=u'(x)$ and in the infinitesimal case, if $\mu_c=0$, $L_c=0$, $\left(\operatorname{skew}(F)-\overline{A}\right)=0\Leftrightarrow 2\alpha(x)=u'(x)$ and independent variation is performed w.r.t. to the displacement u only.

In terms of (u',α) the finite energy expression is

$$\begin{aligned}
W_{\text{finite}}(u',\alpha)&=\mu\left(2(\cos\alpha-1)^2+\frac{u'^2}{2}+\frac{\sin^2\alpha u'^2}{2}+2(\cos\alpha-1)\sin\alpha u'\right)\\
&\quad+\mu_c\left(\frac{\cos^2\alpha u'^2}{2}-2\cos\alpha\sin\alpha u'+2\sin^2\alpha\right)+2\mu\,L_c^2|\alpha'|^2\,,
\end{aligned}\tag{2.8.95}$$

and the **first order reduction** $\cos\alpha\approx1$, $\sin\alpha\approx\alpha$ for small α yields the corresponding infinitesimal energy expression

$$W_{\text{small}}(u',\alpha)=\mu\,\frac{u'^2}{2}+\mu_c\left(\frac{u'^2}{2}-2\alpha\cdot u'+2\alpha^2\right)+2\mu\,L_c^2|\alpha'|^2=\mu\,\frac{u'^2}{2}+2\mu_c\left(\frac{u'}{2}-\alpha\right)^2+2\mu\,L_c^2|\alpha'|^2\,. \tag{2.8.96}$$

The second derivative of the energy in the infinitesimal case w.r.t. (u',α) is given by

$$D^2_{(u',\alpha)}W_{\text{small}}(u',\alpha).((\delta u,\delta\alpha),(\delta u,\delta\alpha))=\mu\,|(\delta u)'|^2+2\mu_c\,|\frac{(\delta u)'}{2}-\delta\alpha|^2+4\mu\,L_c^2|(\delta\alpha)'|^2\,, \tag{2.8.97}$$

which shows that for **classical rigid Dirichlet boundary conditions** and for **consistent coupling** conditions the **solution** (u,α) of the **infinitesimal** problem is **unique**. Since the homogeneous deformation $u(x)=\gamma\cdot x$ together with constant shear angle $\alpha(x)=\frac{\gamma}{2}$ is always a solution for consistent coupling, it is the unique solution coinciding with the unique solution of the classical infinitesimal elasticity problem with shear stress at the upper face $\tau_{\text{lin}}=\mu\,\gamma$.

It is useful to consider as well a **second order reduction** of the energy: $\cos\alpha\approx1-\frac{\alpha^2}{2}$, $\sin\alpha\approx\alpha-\frac{\alpha^3}{3!}$. Skipping terms higher then order four we get

$$W_{\text{red}}(u',\alpha)=\mu\,\left(\frac{1+\alpha^2}{2}u'^2+\frac{\alpha^4}{2}-\alpha^3\,u'\right)+2\mu_c\,\left(\left(\frac{u'}{2}-\alpha\right)^2-\frac{\alpha^2\,u'^2}{4}+\frac{2}{3}\alpha^3u'-\frac{\alpha^4}{3}\right)+2\mu\,L_c^2|\alpha'|^2\,. \tag{2.8.98}$$

The corresponding reduced classical elasticity problem is obtained by

$$W_{\text{red}}^{\text{class}}(u'):=W_{\text{red}|_{\mu_c=0,L_c=0}}(u',\frac{u'}{2})=\mu\,\left(\frac{u'^2}{2}+\frac{u'^4}{32}\right)\,. \tag{2.8.99}$$

The equilibrium equations in the fully finite case are obtained by taking free variations w.r.t. (u,α). Force balance amounts to

$$\begin{aligned}
0&=\frac{\mathrm{d}}{\mathrm{d}t}_{t=0}W_{\text{finite}}(u+t\phi,\alpha)\\
0&=\big[-\mu\,u''(x)-\mu\left(2\sin\alpha(x)\cos\alpha(x)\alpha'(x)u'(x)+\sin^2\alpha(x)\,u''(x)\right)-2\mu\left((\cos\alpha(x)-1)\cos\alpha(x)\alpha'(x)-\sin^2\alpha(x)\alpha'(x)\right)\\
&\quad-\mu_c\left(-2\cos\alpha(x)\sin\alpha\alpha'(x)u'(x)+\cos^2\alpha(x)u''(x)\right)-2\mu_c\left(-\sin^2\alpha(x)\alpha'(x)+\cos^2\alpha(x)\alpha'(x)\right)\big]\cdot\phi\,,
\end{aligned}\tag{2.8.100}$$

and balance of angular momentum is obtained from variation w.r.t. α:

$$\begin{aligned}
0&=\frac{\mathrm{d}}{\mathrm{d}t}_{t=0}W_{\text{finite}}(u,\alpha+t\,\delta\alpha)\\
&=-4\mu(\cos\alpha(x)-1)\,\sin\alpha\,\delta\alpha+\frac{\mu}{2}\,2\sin\alpha(x)\,\cos\alpha(x)u'(x)^2\,\delta\alpha+2\mu(\cos\alpha(x)-1)\,\cos\alpha(x)\,u'(x)\,\delta\alpha(x)\\
&+2\mu(-\sin\alpha(x)\,\sin\alpha(x)u'(x)\delta\alpha(x)+4\mu\,L_c^2\alpha'(x)(\delta\alpha)'(x)\\
&+\mu_c\left(\frac{2\cos\alpha(x)(-\sin\alpha(x))}{2}\delta\alpha\,u'(x)^2-2\cos\alpha(x)\cos\alpha(x)\delta\alpha(x)+2\sin\alpha(x)\sin\alpha(x)\delta\alpha u'(x)+4\sin\alpha(x)\cos\alpha(x)\delta\alpha\right)\,.
\end{aligned}\tag{2.8.101}$$

In the finite case, we evaluate the generated tangential shear stresses τ at the upper face where maximal shear occurs. They are

$$\tau=\langle S_1(F(1),\overline{R}(1)).e_3,e_1\rangle=\langle\overline{R}(1)\left(2\mu\ \operatorname{sym}(\overline{R}^TF-1\!\!1)+2\mu_c\ \operatorname{skew}(\overline{R}^TF)\right).e_3,e_1\rangle\,. \tag{2.8.102}$$

For consistent coupling conditions, the homogeneous deformation $u(x)=\gamma\,x$ and $\tan\alpha(0)=\tan\alpha(x)=\frac{\gamma}{2}$ is always a solution and leads to a **nonlinear, strictly monotone** shear response at the upper face of

$$\begin{aligned}
\tau&=\mu u'(1)+\mu\sin^2\alpha(1)u'(1)+2\mu\sin\alpha(1)(\cos\alpha(1)-1)\\
&=\mu\gamma+\mu\sin^2(\arctan\frac{\gamma}{2})\,\gamma+2\mu\sin(\arctan\frac{\gamma}{2})\,(\cos(\arctan\frac{\gamma}{2})-1)\,.
\end{aligned}\tag{2.8.103}$$

[48]It is a delicate matter to specify independent boundary conditions for the microrotations α. Somehow it requires to know the solution of the boundary value problem in advance. In this sense such a classical rigid boundary condition is nothing but a first guess, perhaps useful as start value in a numerical scheme.

[49]The consistent reduction of this requirement for small α is $\alpha=\frac{u'}{2}-\frac{u'^2}{24}+\ldots$.

For the finite problem, at this point, we do not know whether the homogeneous response is the only one possible or realizes the minimum energy.

In order to investigate this point further, we simplify the two equilibrium equations by considering the second order expansions $\cos\alpha(x) = 1 - \frac{\alpha^2}{2} + \ldots$, $\sin\alpha = \alpha - \frac{\alpha^3}{6} + \ldots$ and keep terms up to order three in the variables (u', α). After partial integration such a reduction coincides with the system of equilibrium equations based directly on the reduced energy W_{red}:

$$
\begin{aligned}
0 &= u'' + \frac{2(\mu-\mu_c)\,\alpha\alpha' u'}{\mu(1+\alpha^2)+\mu_c(1-\alpha^2)} + \frac{(4\mu_c-3\mu)\,\alpha^2\alpha'}{\mu(1+\alpha^2)+\mu_c(1-\alpha^2)} - \frac{2\mu_c\alpha'}{\mu(1+\alpha^2)+\mu_c(1-\alpha^2)} \\
0 &= \mu\left(\frac{1}{2}\alpha^3 + \frac{1}{4}\alpha u'^2 - \frac{3}{4}\alpha^2 u' - L_c^2\alpha''\right) - \mu_c\left(\frac{1}{4}\alpha u'^2 + \frac{2}{3}\alpha^3 + \mu_c\alpha^2 u' + \left(\frac{u'}{2} - \alpha\right)\right) \\
& u(0) = 0,\ u(1) = \gamma \\
& \alpha(0) = \alpha(1) = 0, \quad \text{classical rigid boundary conditions} \\
& 2\alpha(0) = u'(0),\ 2\alpha(1) = u'(1), \quad \text{reduced consistent boundary conditions}\,.
\end{aligned} \tag{2.8.104}
$$

Observe that the homogeneous deformation remains a solution of the reduced system for consistent coupling.

It is natural to **require** that the solution of the coupled boundary value problem, notably the **shear profile angle** α, is in fact **independent of** the **shear modulus** μ and the **Cosserat couple modulus** μ_c as in classical elasticity. This condition can only be met with $\mu_c = 0$ or $\mu_c = \mu$.

If $\mu_c = 0$ the corresponding system of balance equations based consistently on W_{red} reduces further to

$$
\begin{aligned}
0 &= \frac{2\alpha\alpha'}{1+\alpha^2}\,u' - \frac{3\alpha^2\alpha'}{1+\alpha^2} + u'' \\
0 &= \frac{1}{2}\alpha^3 + \frac{1}{4}\alpha u'^2 - \frac{3}{4}\alpha^2 u' - L_c^2\alpha'' \\
& u(0) = 0,\ u(1) = \gamma \\
& \alpha(0) = \alpha(1) = 0, \quad \text{classical rigid boundary conditions} \\
& 2\alpha(0) = u'(0),\ 2\alpha(1) = u'(1), \quad \text{consistent coupling boundary conditions}\,.
\end{aligned} \tag{2.8.105}
$$

Let us integrate the first equation of (2.8.105) at given α by means of the variations of constants formula. This shows that

$$
u'(x) = e^{-\int_0^x \frac{2\alpha(t)\alpha'(t)}{1+\alpha^2(t)}\,\mathrm{dt}} \cdot \left(u'(0) + \int_0^x \frac{3\alpha^2\alpha'(t)}{1+\alpha^2(t)}\, e^{\int_0^t \frac{2\alpha(s)\alpha'(s)}{1+\alpha^2(s)}\,\mathrm{ds}}\right)\mathrm{dt} = \frac{1+\alpha^2(0)}{1+\alpha^2(x)}\left(u'(0) + \frac{[\alpha^3(x)-\alpha^3(0)]}{1+\alpha^2(0)}\right)\,. \tag{2.8.106}
$$

The last equation is the **integrated form** of the **force balance** equation. The Dirichlet boundary conditions for u imply the additional integral condition

$$
\gamma = \gamma - 0 = u(1) - u(0) = \int_0^1 u'(x)\,\mathrm{dx} = \int_0^1 \frac{1+\alpha^2(0)}{1+\alpha^2(x)}\left(u'(0) + \frac{[\alpha^3(x)-\alpha^3(0)]}{1+\alpha^2(0)}\right)\mathrm{dx}\,. \tag{2.8.107}
$$

Let us consider the limit case $L_c = 0$, disregarding the possible boundary values for α in a first approach. This corresponds to the case (4.2) **finite elasticity with free rotations and microstructure** of the classification (2.2.20). The second (now algebraic) equation of (2.8.105) can then be solved for α with the result that α can take on only three distinct values

$$
\alpha(x) \in \{0,\ \frac{u'(x)}{2},\ u'(x)\}\,. \tag{2.8.108}
$$

Reinserting the result into (2.8.106) shows that we have altogether only two different values for $u'(x)$ at our disposition, determined by three equations (two coinciding)

$$
[1+\alpha^2(0)]u'(0) - \alpha^3(0)] = \begin{cases} u'(x) & \text{insert } \alpha(x) = 0 \\ u'(x) + \frac{1}{8}\,u'(x)^3 & \text{insert } \alpha(x) = \frac{u'(x)}{2} \\ u'(x) & \text{insert } \alpha(x) = u'(x) \end{cases}\,. \tag{2.8.109}
$$

Let us choose $u'(0)$, $\alpha(0)$ such that

$$
[1+\alpha^2(0)]u'(0) - \alpha^3(0)] = \gamma + \delta^+, \quad \delta^+ > 0\,. \tag{2.8.110}
$$

The value $\delta^+ > 0$ to be determined subsequently. Taking $\alpha(0) = \frac{u'(0)}{2}$ in (2.8.110) implies that $u'(0) + \frac{1}{8}\,u'(0)^3 = \gamma + \delta^+$ and (2.8.109) reduces to

$$
\gamma + \delta^+ = \begin{cases} u'(x) \\ u'(x) + \frac{1}{8}\,u'(x)^3 \end{cases}\,. \tag{2.8.111}
$$

We take $\delta^+ > 0$ as the unique solution of the equation

$$
(\gamma - \delta^+) + \frac{1}{8}(\gamma - \delta^+)^3 = \gamma + \delta, \quad \text{if } \delta^+ = c^+\cdot\gamma \Leftrightarrow (1-c^+)^3 = \left(\frac{4}{\gamma}\right)^2 c^+\,. \tag{2.8.112}
$$

For $\gamma \to \infty$ we have $\delta^+ \to \gamma$, $c^+ \to 1$ and $\delta^+ = \delta^+(\gamma)$ is a monotone increasing function of γ. This implies

$$
u'(x) = \begin{cases} \gamma_1 = \gamma + \delta^+ \\ \gamma_2 = \gamma - \delta^+ \end{cases}\,, \tag{2.8.113}
$$

and notably $u'(0) = \gamma - \delta^+$. Now consider the family of straight lines

$$
v_1(x) = \gamma_1\,x, \quad v_2(x) = \gamma_1\,x + (\gamma - \gamma_1), \quad v_3(x) = \gamma_2\,x, \quad v_4(x) = \gamma_2\,x + (\gamma - \gamma_2)\,. \tag{2.8.114}
$$

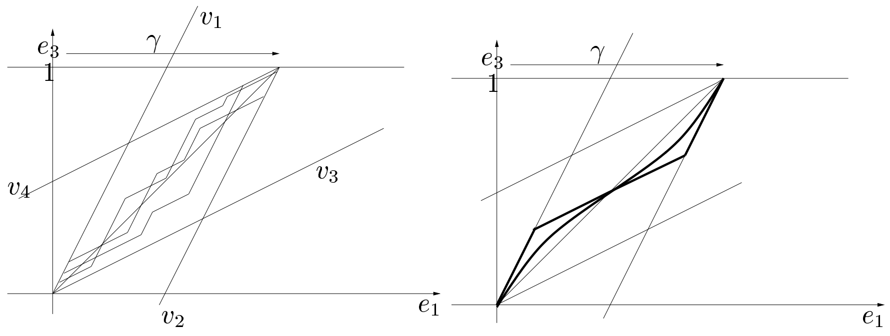

Figure 2.3: Left: Constructed equilibrium microstructure solution (2.8.114) in simple glide for $\mu_c = 0$, $L_c = 0$. Right: Point symmetric microstructure with least number of weak discontinuities (2.8.115) and expected regularized profile for $L_c > 0$ with consistent coupling.

A family of weak solutions of (2.8.105) with $L_c = 0$ and $\alpha(0) = \frac{u'(0)}{2}$, $u'(0) = \gamma - \delta^+$ is given as a continuous combination of piecewise affine functions with slopes parallel to $v_i, i = 1, \dots 4$ satisfying $u(0) = 0$, $u(1) = \gamma$. This is the expected **microstructure**. The constructed solutions u are absolutely continuous, but do not belong to $H^2([0,1], \mathbb{R})$.

Any point symmetric solution w.r.t. (1/2) must have $\alpha(1) = \frac{u'(1)}{2}$, $u'(1) = \gamma - \delta^+$. Symmetry, however, is not enough to single out a unique response. We choose that solution, which has the least number of weak discontinuity points. It is given by

$$u(x) = \begin{cases} (\gamma - \delta^+)\, x & 0 \le x \le \frac{1}{2} - \frac{1}{4} \\ (\gamma + \delta^+)\, x - \frac{\delta^+}{2} & \frac{1}{2} - \frac{1}{4} \le \frac{1}{2} + \frac{1}{4} \\ (\gamma - \delta^+)\, x + \delta^+ & \frac{1}{2} + \frac{1}{4} \le x \le 1 \end{cases}, \quad \alpha(x) = \frac{u'(x)}{2} = \begin{cases} \gamma - \delta^+ & 0 \le x \le \frac{1}{2} - \frac{1}{4} \\ \gamma + \delta^+ & \frac{1}{2} - \frac{1}{4} \le \frac{1}{2} + \frac{1}{4} \\ \gamma - \delta^+ & \frac{1}{2} + \frac{1}{4} \le x \le 1 \end{cases}, \tag{2.8.115}$$

showing the expected (sharp) S-type symmetric shear profile. [50]

For the reduced formulation we evaluate the generated shear stress response at the upper face. They are

$$\tau_{\text{red}} = D_{u'} W_{\text{red}}(u'(1)) = \mu\, \left(u'(1) + \alpha^2(1) u'(1) - \alpha^3(1)\right) = \mu\, \left(u'(1) + \left(\frac{u'(1)}{2}\right)^2 u'(1) - \left(\frac{u'(1)}{2}\right)^3\right) \Rightarrow$$

$$\tau_{\text{red}}^{\text{hom}} = \mu\, (1 + \frac{1}{8}\gamma^2)\, \gamma, \qquad \tau_{\text{red}}^{\text{micro}} = \mu\, (1 + c^+(\gamma))\, \gamma\,, \tag{2.8.116}$$

with $c^+(\gamma) \in (0,1)$ from (2.8.112). This shows that the response with **microstructure** due to free rotations is always **weaker** (energetically favourable) than the still possible homogeneous response. [51]

In order to determine L_c for a given material we consider the same material in different sample sizes with edge length $L_i > 1$ and perform corresponding shear experiments. Due to scaling relations the different sizes can be transformed to the unit cube leading to a modified internal length $\frac{L_c}{L_i}$. The solution on the unit cube depends only on γ and $\frac{L_c}{L_i}$. Hence the sequence of experiments leads to best-fitting values $\frac{L_c}{L_i} = C_i$, which implies that $L_c = \frac{1}{n} \sum_{i=1}^{n} L_i\, C_i$ is a good candidate for the real characteristic length. Note that no knowledge of size-independent material parameters is necessary to obtain a value for L_c if $\mu_c = 0$.

Let us contrast the foregoing result with a similar analysis of the infinitesimal Cosserat model with necessarily $\mu_c > 0$. The system of balance equations is now given by

$$\begin{aligned} u''(x) &= 2 \frac{\mu_c}{\mu + \mu_c}\, \alpha'(x) = 2\, N^2\, \alpha'(x) \\ \mu\, L_c^2 \alpha''(x) &= \mu_c\, \left(\alpha(x) - \frac{u'(x)}{2}\right) \\ & u(0) = 0,\; u(1) = \gamma \\ & \alpha(0) = \alpha(1) = 0, \quad \text{classical rigid boundary conditions} \\ & 2\alpha(0) = u'(0),\; 2\alpha(1) = u'(1), \quad \text{consistent coupling boundary conditions}\,, \end{aligned} \tag{2.8.117}$$

with the traditional **Cosserat coupling number** $N^2 = \frac{\mu_c}{\mu_c + \mu}$. Observe again that the homogeneous solution is the unique solution for consistent coupling.

In order to find the unique nontrivial solution for rigid conditions, we integrate the first equation of (2.8.117) and get

$$2\, N^2[\alpha(x) - \alpha(0)] = u'(x) - u'(0), \quad \text{and} \quad 2\, N^2 \int_0^1 \alpha(x)\, \mathrm{dx} - 2\, N^2 \alpha(0) = u(1) - u(0) - u'(0) = \gamma - u'(0)\,, \tag{2.8.118}$$

[50] The constructed solution satisfies the Boltzmann axiom: $\alpha = \frac{u'}{2}$ everywhere and the corresponding microstretch is symmetric to first order, yet the solution does not coincide with the classical solution.

[51] The weak discontinuities inherent in this microstructure can be seen as a precursor to **fracture**. The road to fracture starts with homogeneous solutions, which turn into smooth inhomogeneous solutions $u \in H^2$, which degenerate into solutions with weak discontinuities $u \in H^1 \setminus H^2$, which finally fail along glide planes with $u \notin W^{1,1}$.

where we have used the Dirichlet boundary conditions for the displacement u. This shows

$$u'(0) = \gamma - 2\,N^2 \int_0^1 \alpha(x)\,\mathrm{dx}, \quad (\text{use } (2.8.118)_2 \text{ and } \alpha(0) = 0) \;\Rightarrow \tag{2.8.119}$$

$$u'(x) = \gamma + 2\,N^2 \left(\alpha(x) - \int_0^1 \alpha(x)\,\mathrm{dx}\right), \quad \text{classical rigid boundary conditions,} \Rightarrow u'(0) = u'(1) < \gamma\,.$$

Inserting the result for $u'(x)$ into balance of angular momentum and rearranging yields the linear second order differential equation

$$\alpha''(x) - \frac{N^2}{L_c^2}\alpha(x) = -\frac{\mu_c}{\mu\,L_c^2}\frac{\gamma}{2} + \frac{N^2\,\mu_c}{L_c^2\mu}\int_0^1 \alpha(x)\,\mathrm{dx}\,. \tag{2.8.120}$$

Differentiating w.r.t. x once more yields the linear third order equation

$$\alpha'''(x) - \frac{N^2}{L_c^2}\alpha'(x) = 0 \Leftrightarrow -N^2\,\alpha'(x) + L_c^2\,\alpha'''(x) = 0\,. \tag{2.8.121}$$

The general solution of this differential equation in view of the exerted point symmetry of α w.r.t. $\frac{1}{2}$ is given by

$$\alpha(x) = \beta_1 \cdot \cosh\left(\frac{N}{L_c}[x - \frac{1}{2}]\right) + \beta_2\,. \tag{2.8.122}$$

For the rigid Dirichlet case we use now $\alpha(0) = \alpha(1) = 0$. This implies that

$$0 = \alpha(0) = \beta_1 \cdot \cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) + \beta_2 \Rightarrow \quad \alpha(x) = \beta_1 \cdot \left[\cosh\left(\frac{N}{L_c}[x - \frac{1}{2}]\right) - \cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)\right]\,. \tag{2.8.123}$$

We calculate

$$\alpha''(x) = \beta_1 \cdot \frac{N^2}{L_c^2}\cosh\left(\frac{N}{L_c}[x - \frac{1}{2}]\right) \tag{2.8.124}$$

$$\int_0^1 \alpha(x)\,\mathrm{dx} = -\beta_1\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) + \beta_1\frac{L_c}{N}\left[\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right) - \sinh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)\right] = 2\beta_1\frac{L_c}{N}\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right) - \beta_1\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)$$

Inserting the results into (2.8.120) we obtain

$$\beta_1 = \frac{-\gamma}{2\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) - 4NL_c\,\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}\,, \tag{2.8.125}$$

which yields the micropolar shear profile

$$\alpha(x) = -\frac{\left[\cosh\left(\frac{N}{L_c}[x-\frac{1}{2}]\right) - \cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)\right]\cdot\gamma}{2\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) - 4NL_c\,\beta_1\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right)} = \frac{\left[\cosh\left(\frac{N}{L_c}[x-\frac{1}{2}]\right) - \cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)\right]\cdot\,\gamma}{-2\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) + 4NL_c\,\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}\,, \tag{2.8.126}$$

with the non-uniform behaviour $0 \le \alpha(x) \to \frac{\gamma}{2}$ for $0 < x < 1$ and $N, L_c \to 0$. A physically acceptable (smooth) S-type symmetric shear profile is characterized by a steepest tangent of u at $\frac{1}{2}$. This corresponds to a maximum of α at $\frac{1}{2}$. The sign of the denominator in the last formula is decisive: it should be negative. Therefore

$$0 > -2\cosh\left(\frac{N}{L_c}[-\frac{1}{2}]\right) + 4NL_c\,\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right) \Leftrightarrow 2N\,L_c < \coth\left(\frac{N}{L_c}[\frac{1}{2}]\right) \Leftrightarrow 1 > 2NL_c\tanh\left(\frac{N}{2L_c}\right) \tag{2.8.127}$$

which is always satisfied since $\tanh(x) < x, x > 0$ and $0 < N^2 < 1$. The tangential shear stresses are given by

$$\begin{aligned}\tau_{\text{small}} &= \langle\sigma.e_3, e_1\rangle = \langle[2\mu\,\text{sym}(F - 1\!\!1) + 2\mu_c(\text{skew}(F - 1\!\!1) - \overline{A})].e_3, e_1\rangle = (\mu + \mu_c)\cdot u'(1) - 2\mu_c\,\alpha(1)\\ &= (\mu + \mu_c)\cdot u'(0) = (\mu + \mu_c)\left(\gamma - 2N^2\int_0^1 \alpha(x)\,\mathrm{dx}\right) = \mu\left(\frac{1}{1 - 2NL_c\tanh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}\right)\cdot\gamma\,.\end{aligned} \tag{2.8.128}$$

Expansion shows that for $N > 0, L_c \to \infty$ (ever smaller samples)[52] it results in the limit $\tau_{\text{small}} = (\mu + \mu_c)\cdot\gamma$ and $u(x) = \gamma\,x$, the evaluated stresses τ_{small} are increased due to $\mu_c > 0$ and the incompatibel rigid boundary prescription. For $0 < N \ll 1$, $L_c > 0$ we observe that $\tau_{\text{small}} \approx (\mu + \mu_c)\cdot\gamma$ and $u(x) \approx \gamma\,x$. In this case, it can be seen that the **Cosserat couple modulus** $\mu_c > 0$ is in fact also **a measure of the influence of boundary conditions on the solution** and therefore **not a material parameter**.[53] Similarly, it can be shown that $N \to 0$, $L_c > 0$ is possible and results in the classical response $\tau_{\text{small}} = \mu\cdot\gamma$. Finally, $N > 0$, $L_c \to 0$ approaches the classical result as well. In all cases the micropolar response for rigid Dirichlet data is **stiffer** than the corresponding homogeneous classical response.

[52] Only a formal limit: the smallest sample size should be larger than the chosen $L_c > 0$ of the unit cube, i.e. the smallest sample size must be larger than the smallest constituents of the material given as unit cube. Hence, if L_c has any physical meaning, we should have $0 \le L_c < 1$.

[53] Consider any other independent (artificial) Dirichlet boundary condition for the shear angle $0 \le \alpha(0) = \alpha(1) = a < \frac{\gamma}{2}$. The solution u will produce a different shear stress response $\tau_{\text{small}} = \mu\,u'(1) + 2\mu_c\,(\frac{u'(1)}{2} - a)$ which, for different L_c, necessitates a modification of μ_c for the same material. In our example, this inconsistency can be avoided for consistent coupling but persists in the general case.

The computed micropolar displacement for rigid Dirichlet data is given by

$$\begin{aligned} u(x) &= \int_0^x u'(s)\,\mathrm{ds} = \int_0^x \gamma + 2N^2\alpha(s) - 2N^2\int_0^1 \alpha(x)\,\mathrm{dx}\,\mathrm{ds} = \gamma\,x + 2N^2\int_0^x \alpha(s)\,\mathrm{ds} - 2N^2\,x\int_0^1\alpha(x)\,\mathrm{dx} \\ &= \gamma\left(\frac{\cosh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}{\cosh\left(\frac{N}{L_c}[\frac{1}{2}]\right) - 2NL_c\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}\,x - NL_c\frac{\sinh\left(\frac{N}{L_c}[x-\frac{1}{2}]\right) - \sinh\left(\frac{N}{L_c}[-\frac{1}{2}]\right)}{\cosh\left(\frac{N}{L_c}[\frac{1}{2}]\right) - 2NL_c\sinh\left(\frac{N}{L_c}[\frac{1}{2}]\right)}\right). \end{aligned} \tag{2.8.129}$$

The complete solution of the problem in terms of the displacement u is now a function of γ, $\frac{N}{L_c}$ and $N\cdot L_c$. Consider the same material given in different sample sizes of cubes with edge length $L_i > 1$. Due to scaling relations, we may transform the different sample sizes to the unit cube resulting in a modified internal length $\frac{L_c}{L_i}$ but identical values γ, N by the assumption that μ_c, hence N is a **material parameter independent of size**. Performing a corresponding shear experiment on each sample size we obtain best-fitting values of $\frac{N}{(L_c/L_i)} = \hat{C}_i$ and $N\,\frac{L_c}{L_i} = \hat{D}_i$. If the infinitesimal micropolar model is correct this implies that $N^2 = \hat{C}_i\cdot\hat{D}_i$ independent[54] of i. A striking consequence of this development is that the assumed size-independent material parameter N cannot be determined without prior knowledge of the characteristic length L_c in contrast to the other elastic constants and vice-versa: the characteristic internal length L_c can only be determined once N is known. This is a problematic feature shared by all micropolar models with $\mu_c > 0$.

As for the homogeneous solution for consistent coupling: The tangential stresses are given by

$$\tau_{\text{small}}^{\text{hom}} == (\mu+\mu_c)\cdot u'(1) - 2\mu_c\,\alpha(1) = (\mu+\mu_c)\cdot u'(0) - 2\mu_c\alpha(0) = \mu\,\gamma\,. \tag{2.8.130}$$

Now we consider the **infinitesimal indeterminate couple stress** response in simple shear. The variational problem is easily obtained from (2.8.94) by identifying $\alpha(x) = \frac{u'(x)}{2}$ and taking free variations w.r.t. u only. This results in the problem

$$\int_\Omega \mu\,\|\operatorname{sym}F - 1\!\!1\|^2 + \frac{\mu}{2}\,L_c^2|u''(x)|^2\,\mathrm{dV} \mapsto \min\,.\quad \text{w.r.t. } u,\quad u(0)=0,\, u(1)=\gamma\,, \tag{2.8.131}$$

$$\sigma = \sigma^{\text{loc}} + \sigma^{\text{hyper}},\quad \tau_{\text{small}}^{\text{indet}} = \mu\,u'(x) - \mu\,L_c^2\,u'''(x)\,,$$

and the Euler-Lagrange equation of **fourth order** is given by

$$\begin{aligned} -u''(x) + L_c^2\,u^{(4)}(x) &= 0, && u(0)=0,\, u(1)=\gamma \\ u'(0) = u'(1) &= 0 && \text{classical rigid condition} \\ u''(0) = u''(1) &= 0 && \text{natural boundary condition}\,. \end{aligned} \tag{2.8.132}$$

This equation coincides with (2.8.121) if we identify again $u' = \frac{\alpha}{2}$ and take $N\equiv 1$.

The general solution of (2.8.132) is given by

$$u(x) = b_1\,\sinh\left(\frac{1}{L_c}[x-\frac{1}{2}]\right) + b_2\,\cosh\left(\frac{1}{L_c}[x-\frac{1}{2}]\right) + b_3\,(x-\frac{1}{2}) + b_4\,. \tag{2.8.133}$$

Natural boundary conditions imply effectively $u(x) = \gamma\,x$ as unique homogeneous solution with shear stress response $\tau_{\text{small}}^{\text{indet,hom}} = \mu\,\gamma$. For rigid boundary conditions the constants are

$$b_1 = -\frac{\gamma\,L_c}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)},\quad b_2 = 0,\quad b_3 = -\frac{\gamma\,\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)},\quad b_4 = \frac{\gamma}{2} \tag{2.8.134}$$

and the unique solution of the rigid indeterminate couple stress problem is

$$\begin{aligned} u(x) &= \gamma\left(\frac{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}\,x - L_c\frac{\sinh\left(\frac{1}{L_c}[x-\frac{1}{2}]\right) - \sinh\left(\frac{1}{L_c}[-\frac{1}{2}]\right)}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}\right) \\ u'(x) &= \left(\frac{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - \cosh\left(\frac{1}{L_c}[x-\frac{1}{2}]\right)}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}\right)\cdot\gamma\,. \end{aligned} \tag{2.8.135}$$

The term $\frac{u'(x)}{2}$ does coincide with $\alpha(x)$ in (2.8.126) for $N\equiv 1$. The limit $L_c\to 0$ (ever larger samples) is possible, converging pointwise to the homogeneous solution, but the convergence is not uniform due to the appearance of a strong **boundary layer** caused by the incompatible rigid boundary prescription. For large L_c the solution converges to a smooth S-type shear profile.

The shear stress response is given by

$$\begin{aligned} \tau_{\text{small}}^{\text{indet}} &= \mu\,u'(1) - \mu\,L_c^2u'''(1) = -\mu L_c^2\left(\frac{b_1}{L_c^3}\cosh\left(\frac{1}{L_c}[1-\frac{1}{2}]\right) + \frac{b_2}{L_c^3}\cosh\left(\frac{1}{L_c}[1-\frac{1}{2}]\right)\right) \\ &= -\mu\frac{b_1}{L_c}\cosh\left(\frac{1}{L_c}[1-\frac{1}{2}]\right) = \mu\,\frac{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}{\cosh\left(\frac{1}{L_c}[\frac{1}{2}]\right) - 2L_c\sinh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}\,\gamma = \mu\left(\frac{1}{1-2L_c\tanh\left(\frac{1}{L_c}[\frac{1}{2}]\right)}\right)\cdot\gamma\,, \end{aligned} \tag{2.8.136}$$

coinciding with the **stiffer** shear stress response of the infinitesimal micropolar model for $N=1$. Passage to the limit $L_c\to\infty$ (ever smaller samples) is not possible.[55] This implies as a rule for additional boundary conditions: boundary conditions should be such that in principal homogeneous solutions remain possible. The boundary conditions in a three-dimensional problem should not be the cause for nonhomogeneous response! This principle does not apply to plates and shells where boundary conditions appear naturally by a dimensional reduction process and carry physical information.

[54] It appears to us that $N^2 = \hat{C}_i\cdot\hat{D}_i$ independent of i for different sizes is highly questionable.

[55] Since $\tanh x = x - \frac{x^3}{3} + \ldots$, for $L_c\to\infty$, then $\tau_{\text{small}}^{\text{indet}}\to\infty$, a severe shortcoming of the indeterminate couple stress model. This underlines the objections of Koiter against this model.

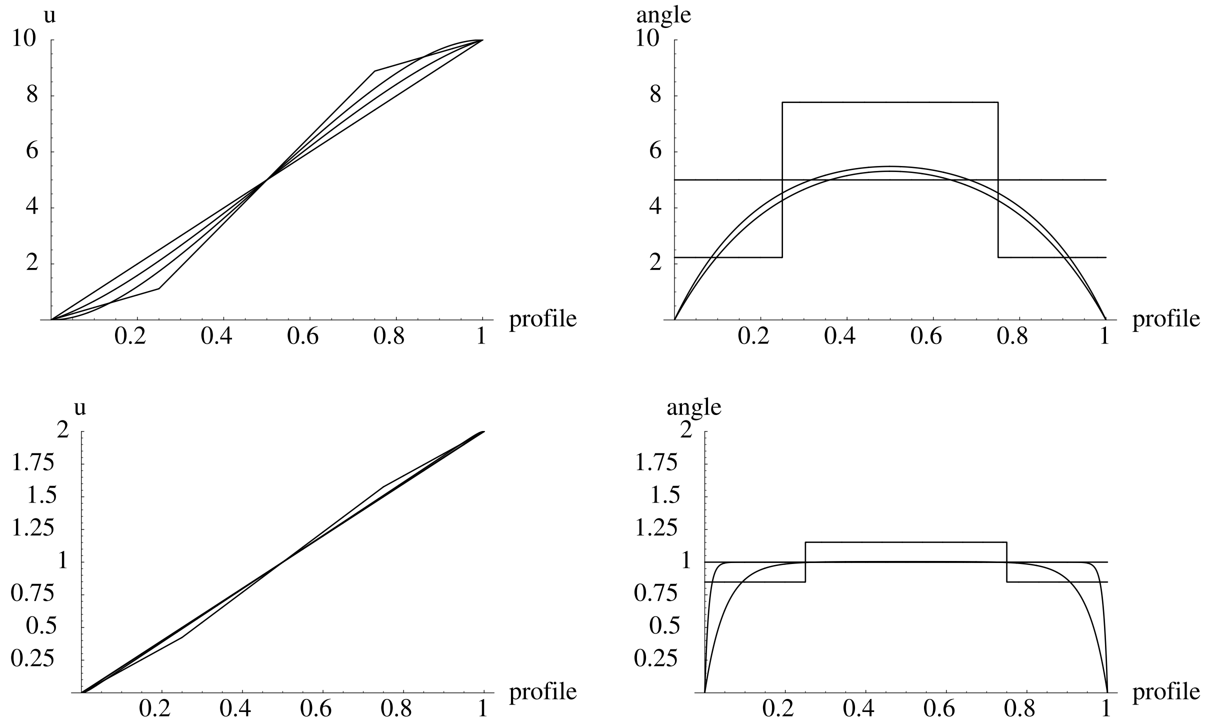

Figure 2.4: Computed shear profile u and corresponding angle α for large (above) and small (below) values of maximal shear γ in simple glide. Different curves: finite reduced problem with free rotations, infinitesimal micropolar and indeterminate couple stress model. $u(x) = \gamma\, x$, $\alpha(x) = \gamma/2$ the homogeneous classical response. The traditional infinitesimal Cosserat models with rigid boundary conditions and $\mu_c > 0$ show a strong **boundary layer**- not shared by the new relaxed model with $\mu_c = 0$ and consistent coupling. The new model allows for **sharp interfaces** for vanishing internal length L_c, indicating the onset of fracture.

2.8.4 Analytical investigations of classical incompressible elastic Biot-material in simple shear

Now we consider the same situation of simple glide in a classical elasticity context. The energy of the classical elastic Biot material is assumed to be of the form

$$\int_\Omega \mu\, \|\frac{U}{\det[U]^{(1/3)}} - 1\!\!1\|^2 + \beta \left(\det[U] + \frac{1}{\det[U]} - 2\right)^2 \mathrm{dx}\,, \tag{2.8.137}$$

which reduces to

$$\int_\Omega \mu\, \|U - 1\!\!1\|^2 \,\mathrm{dx} = \int_\Omega \mu\, \|R^T F - 1\!\!1\|^2 \,\mathrm{dx} = \int_\Omega \mu\, \|F - R\|^2 \,\mathrm{dx} = \int_\Omega \mu \left(\|F\|^2 - 2\langle F, R\rangle + 3\right) \mathrm{dx} \tag{2.8.138}$$

for incompressible behaviour and $R \in \mathrm{SO}(3, \mathbb{R})$ represents the **continuum rotation**. For the assumed kinematics in simple glide, the continuum rotation has the form

$$R(x_1, x_2, x_3) = \begin{pmatrix} \cos\alpha(x_3) & 0 & \sin\alpha(x_3) \\ 0 & 1 & 0 \\ -\sin\alpha(x_3) & 0 & \cos\alpha(x_3) \end{pmatrix}, \tag{2.8.139}$$

with continuum rotation angle $\alpha \in [0, 2\pi)$. A simple calculation shows that

$$\sin\alpha(x) = \frac{u'(x)}{\sqrt{4 + (u'(x))^2}}, \qquad R(x) = \frac{1}{\sqrt{4 + (u'(x))^2}} \begin{pmatrix} 2 & 0 & u'(x) \\ 0 & \sqrt{4 + u'(x)^2} & 0 \\ -u'(x) & 0 & 2 \end{pmatrix}, \tag{2.8.140}$$

and the (surprisingly?) **convex** total energy is given by

$$\int_0^1 \mu \left(\|F\|^2 - 2\langle F, R(u')\rangle + 3\right) \mathrm{dx} == \int_0^1 \mu \left(\sqrt{4 + u'(x)^2} - 1\right)^2 - \mu \,\mathrm{dx}\,. \tag{2.8.141}$$

The Euler-Lagrange equation is given by

$$\forall \quad \phi \in C_0([0,1],\mathbb{R}): \quad \int_0^1 \frac{\left(\sqrt{4+u'(x)^2}-1\right)}{\sqrt{4+u'(x)^2}}\, u'(x)\,\phi'(x)\,\mathrm{dx} = 0, \qquad \textbf{weak form} \tag{2.8.142}$$

$$u''(x)\cdot\left(1-\frac{1}{\sqrt{4+u'(x)^2}}+\frac{u'(x)^2}{\left(4+u'(x)^2\right)^{(3/2)}}\right)=0, \qquad \textbf{differentiated form}$$

$$u(0)=0,\; u(1)=\gamma,$$

showing that the homogeneous deformation is always a solution and hence, by strict convexity, the **unique solution** with **non-linear, strictly-monotone** shear stress response

$$\tau_{\text{finite}} = 2\,\mu\,\frac{\left(\sqrt{4+\gamma^2}-1\right)}{\sqrt{4+\gamma^2}}\cdot\gamma \quad (=\mu\,\gamma+o(\gamma))\;, \tag{2.8.143}$$

which coincides in fact with the shear response of the finite Cosserat model with consistent coupling evaluated for this homogeneous response. [56]

In contrast, from a three-dimensional viewpoint, the shear energy $\mu\,\|U-\mathbb{1}\|^2$ is **not quasiconvex** and **not Legendre-Hadamard elliptic** but satisfies the **Baker-Ericksen inequalities**. We consider therefore the behaviour of its **quasiconvexification** for the same assumed kinematics. It can be given explicitly. If λ_i are the eigenvalues of U we have

$$QW(F)=\mu\sum_{\lambda_i\geq 1}|\lambda_i-1|^2\,. \tag{2.8.144}$$

In view of the underlying kinematics, the λ_i can be calculated explicitly and we obtain

$$\lambda_1=1,\quad \lambda_2=\frac{4+(u')^2-(u')\sqrt{4+(u')^2}}{2\sqrt{4+(u')^2}},\quad \lambda_3=\frac{4+(u')^2+(u')\sqrt{4+(u')^2}}{2\sqrt{4+(u')^2}}\,, \tag{2.8.145}$$

such that

$$QW(F)=\mu\sum_{\lambda_i\geq 1}|\lambda_i-1|^2=\mu\,|\lambda_3-1|^2=\mu\left(\frac{4+(u')^2+(u')\sqrt{4+(u')^2}}{2\sqrt{4+(u')^2}}-1\right)^2=\mu\left(\frac{1}{2}\sqrt{4+(u')^2}+\frac{\gamma}{2}-1\right)^2. \tag{2.8.146}$$

The resulting formulation is again strictly convex and the homogeneous response is the unique minimizer of the corresponding minimization problem. However, the **shear response** is **considerably weaker** than the unmodified one for the same homogeneous solution, since

$$\tau^{\text{quasi}}_{\text{finite}} = D_{u'}\,[QW(F)]] = \mu\left(\frac{1}{2}\sqrt{4+\gamma^2}+\frac{\gamma}{2}-1\right)\cdot\left(\frac{\gamma}{\sqrt{4+\gamma^2}}+1\right)\,, \tag{2.8.147}$$

We note that in this specific example, quasiconvexification of the three-dimensional problem coincides with the rank-one convexifcation. However, quasiconvexification changes the stress/strain law already in situations, where convexity holds true for the assumed kinematics, i.e. where there is no imminent need for any change due to instabilities. This underlines the care, which has to be exerted when using the quasiconvex hull.

Let us summarize the obtained stress/strain behaviour in uniaxial shear for small amounts of shear. For consistent coupling we have:

$$\mu\,\gamma=\tau_{\text{lin}}=\tau_{\text{small}}=\tau^{\text{indet}}_{\text{small}}>\tau^{\text{quasi}}_{\text{class}}\,,\qquad \mu\,\gamma=\tau_{\text{lin}}<\tau^{\text{micro}}_{\text{red}}<\tau^{\text{hom}}_{\text{finite}}<\tau^{\text{hom}}_{\text{red}}\,, \tag{2.8.148}$$

where $\tau_{\text{lin}}<\tau^{\text{micro}}_{\text{red}}<\tau^{\text{hom}}_{\text{finite}}<\tau^{\text{hom}}_{\text{red}}$ have the same tangent in 0, but $\tau^{\text{quasi}}_{\text{class}}$ is weaker. For rigid Dirichlet data we obtain

$$\mu\,\gamma=\tau_{\text{lin}}<\tau_{\text{small}}<\tau^{\text{indet}}_{\text{small}}\,, \tag{2.8.149}$$

with artificially stiffer behaviour for arbitrary small shear.

2.9 Appendix B

2.9.1 Koiter's remarks on couple stresses

It seems appropriate to cite also Koiter [Koi64]: "The predictions of the (classical elasticity) theory are usually in satisfactory agreement with careful experiments, if the stresses remain within the elastic limit of the material. Unfortunately the theory of elasticity apparently fails, however, to give an adequate description of the behaviour in fatigue of machine parts or other structural elements involving high stress concentrations. This failure can hardly be ascribed completely to inelastic behaviour of the material, because the endurance limit in fatigue is usually well below the macroscopic elastic limit of the material. A more likely explanation is that the classical (finite elasticity) theory is not adequate in the presence of large stress gradients. The latter explanation is entirely plausible in view of the discontinuous polycrystalline structure of actual engineering materials. It is also supported by evidence that the discrepancy between the theoretical predictions and fatigue test results is more marked for materials with a coarse grain structure. It would seem therefore that the idealized model of an elastic continuum is not quite appropriate for the analysis of stress and strain in an actual discontinuous polycrystalline material involving large stress gradients. It need hardly be argued, however, that a detailed analysis of the transmission of loads between the individual grains in a polycrystalline material would pose a formidable problem. Some idealisation, preferably in the form of a continuous model, is highly desirable in order to

[56] To see only after some algebra.

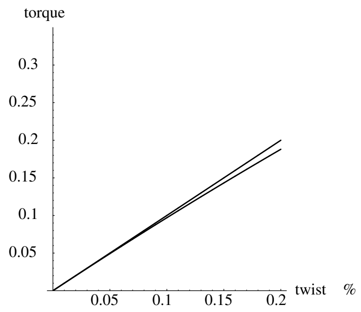

Figure 2.5: Parabolic behaviour for infinitesimal load.

make the problem amenable to analysis. At first sight it might seem that this return to a continuous model would also imply a complete return to the classical theory of elasticity. It should be remembered, however, that we have already alluded to additional assumptions made in the classical theory, apart from the model of a continuum. The assumption in question is that the transmission of loads between the material on both sides of an infinitesimal surface element dS is described completely by a force vector $p\,$dS acting in the center of gravity of the surface element. We emphasize that this is an assumption which can neither be proved directly, nor disproved. It can only be tested by a confrontation of its predictions for measurable quantities with experiments. For most purposes it has indeed proved to be an appropriate assumption, resulting in satisfactory agreement between theoretical predictions and experimental evidence. The lack of agreement between theory and experiment on the effect of stress gradients, however, makes this assumption questionable at least in cases of large stress gradients. A quite natural generalisation of the classical theory of the elastic continuum is thus obtained, if we drop the additional assumption. (...) It turned out, however, that the magnitude of the effect of couple-stresses, required to explain quantitatively the effect of stress gradients in fatigue tests, was such that it could not easily have escaped attention in other careful experiments.“ and he continues [Koi64, p.41]: "We venture to conjecture that the stress gradient effect in fatigue cannot be described satisfactorily by allowing the presence of couple-stresses in an isotropic elastic medium." Brackets my addition.

It must be noted that Koiter came to reject the significant presence of couple stresses because he based his investigations on the so called indeterminate couple stress theory (2.2.21), which tends to maximize the influence of length scale effects. His arguments only show that this special constrained gradient theory cannot be based on experimental evidence. However, the main thrust of his comments remains valid. I have not been aware of Koiters contribution during the preparation of my main arguments, but it squares with my development

2.9.2 Some experimental results: Nonlinear relation near zero stress and size dependence

Usually, in engineering application, several simplifying assumptions are tacitly introduced. The first one is that the elastic behaviour of the structure can be neglected altogether leading to the rigid plasticity models and the second one, that the stress-strain relation for very small stress levels is indeed linear (generalized Hookes law). While the first simplification is evidently not appropriate the last simplification is generally assumed to be valid. However, it cannot be based on evidence. To the contrary, taking the vast amount of precise experimental data for metallic material compiled in [Bel73] seriously, we must conclude that there is no level of stress, such that the stress-strain relation is linear. Instead, practically all materials under infinitesimal loads (in torsion and tension) obey a parabolic relation of the form $\|\sigma\| = E\,\|\varepsilon\| - b\,\|\varepsilon\|^2$ with E the initial zero stress modulus in tension and b a positive constant [Bel73, p.127] and see the footnote in [Nad63, p.10]. There has never been made a serious attempt to understand this parabolic behaviour. Our Cosserat model offers an interpretation: since the initial tangent modulus is bounded, the case $\mu_c = \infty$ can be disposed of for all values of L_c. Because the initial tangent moduli determined from tension experiments are consistent with predictions based on the classical linear theory for torsion experiments, we must set $\mu_c = 0$. The point is that $\mu_c > 0$ would immediately change the elastic behaviour for very small stress levels in torsion. In this sense, μ_c is small strain determined. Whether the couple modulus is really zero, cannot be decided by experiment alone, given the unavoidable scatter in the experiment. In any case it would have to be very small.

The second order effect of rotations will intervene if we keep the geometrically exact structure of the model and lead to reduced tangent moduli in tension and torsion, with a more pronounced reduction in torsion, cf. (2.2.6). A difference of response in tension and torsion for small stress levels has been observed as early as 1857. Wertheim [Wer57] and Bauschinger [Bau81] observed (roughly) **linearity** in **tension** and **nonlinearity** in **torsion**. Moreover, to quote from [Bel73, p.89]: "For all the cylinders, Wertheim noticed that in addition to the nonlinearity of the relation between torque and angle, the amount of the departure from the linear approximation depended upon the length of the specimen. Only when very long specimens were compared could he achieve the independence of length assumed in the elementary theory."

In Wertheims own words [Wer57, p.281] he summarizes: "2. Les angles de torsion temporaires (elastic twist) ne sont pas rigoureusement proportionelles aux moments des couples; ils augment plus rapide que ceux-ci, et cet accroissement de l´angle moyen (empiric modulus in torsion) ... s´etend jusque´a la rupture ... 3. Ces angles temporaires ne sont pas rigoureusement proportionels aux longueurs; ramene a l´unite de longeur, suivant cette loi de proportionalite, on les trouve d´autant plus grand, tout egal du reste, que la piece soumise a l´experience a ete plus courte.

... cette proportionalite (classical infinitesimal linear elasticity) ne saurait etre considere que comme la limite vers laquelle tendent les angles a mesure que diminuent les intensites des couples et les dimension du corps qui sont perpendiculaire a l´axe de torsion."

2.9.3 Alternative representation of the micropolar stretch energy

For a **small elastic strain** theory, which should already cover many cases of physical interest we required that $W_{\mathrm{mp}}(\overline{U})$ is a non negative isotropic quadratic form with

$$W_{\mathrm{mp}}(1\!\!1) = 0, \quad D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})_{|_{\overline{U}=1\!\!1}} = 0\,. \tag{2.9.150}$$

The most general form of W_{mp} consistent with (2.2.5) is

$$\begin{aligned} W_{\mathrm{mp}}(\overline{U}) &= \alpha_1 \,\|\operatorname{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c \,\|\operatorname{skew}(\overline{U} - 1\!\!1)\|^2 + \alpha_3 \operatorname{tr}\left[\operatorname{sym}(\overline{U} - 1\!\!1)\right]^2 \\ &= \frac{\alpha_1 + \mu_c}{2}\,\|\overline{U} - 1\!\!1\|^2 + \frac{\alpha_1 - \mu_c}{2}\,\langle \overline{U} - 1\!\!1, (\overline{U} - 1\!\!1)^T\rangle + \alpha_3 \operatorname{tr}\left[\operatorname{sym}(\overline{U} - 1\!\!1)\right]^2, \\ &= -(2\alpha_1 + 6\alpha_3)\operatorname{tr}\left[\overline{U}\right] + \alpha_3 \operatorname{tr}\left[\overline{U}\right]^2 + \frac{(\alpha_1 - \mu_c)}{2}\operatorname{tr}\left[\overline{U}^2\right] + \frac{(\alpha_1 + \mu_c)}{2}\operatorname{tr}\left[\overline{U}^T\overline{U}\right] + \text{const.}\,, \end{aligned} \tag{2.9.151}$$

where $(2.2.6)_{2,3}$ is the form used in [Gra03, San99], respectively and we note the induced relations

$$\begin{aligned} D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}) &= (\alpha_1 + \mu_c)\,\overline{U} + (\alpha_1 - \mu_c)\,\overline{U}^T - 2\,\alpha_1\, 1\!\!1 + 2\,\alpha_3 \operatorname{tr}\left[\overline{U} - 1\!\!1\right] 1\!\!1 \\ D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^T &= (\alpha_1 + \mu_c)\,\overline{U}\,\overline{U}^T + (\alpha_1 - \mu_c)\,\overline{U}^T\overline{U}^T - 2\,\alpha_1\,\overline{U}^T + 2\,\alpha_3 \operatorname{tr}\left[\overline{U} - 1\!\!1\right]\overline{U}^T \\ \operatorname{skew}\left(D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^T\right) &= \operatorname{skew}\left((\alpha_1 - \mu_c)\,\overline{U}^T\overline{U}^T - 2\,\alpha_1\,\overline{U}^T + 2\,\alpha_3 \operatorname{tr}\left[\overline{U} - 1\!\!1\right]\overline{U}^T\right)\,, \end{aligned} \tag{2.9.152}$$

with material constants $\alpha_i, i = 1,2,3$ such that $\alpha_1,\, 3\alpha_3 - \alpha_1, \mu_c \geq 0$ from non negativity [Eri99].

2.9.4 Derivation of the geometrically exact micropolar balance equations

The derivation of $(2.2.12)_1$ is standard. For $(2.2.12)_2$ we consider simultaneously in each space point a one parameter group of rotations $\frac{\mathrm{d}}{\mathrm{dt}}\hat{R}(x,t) = A(x,t)\cdot\hat{R}(x,t)$, $\hat{R}(x,0) = \overline{R}(x)$, $A \in C_0^\infty(\Omega, \mathfrak{so}(3,\mathbb{R}))$. The stationarity condition is obtained from $\frac{\mathrm{d}}{\mathrm{dt}}_{|t=0} I(\varphi, \hat{R}(x,t)) = 0$. This yields three terms: the derivatives involving W_{mp} and $\langle M, \overline{R}\rangle$ are straightforward, using the definition of the one parameter group, and yield

$$\langle -\overline{R}\operatorname{skew}(D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T)\,\overline{R}^T, A\rangle, \quad \langle -\overline{R}\operatorname{skew}(\overline{R}^T M)\,\overline{R}^T, A\rangle\,, \tag{2.9.153}$$

respectively, where $\langle\cdot,\cdot\rangle$ means additionally integration w.r.t. x. For the term containing the curvature part, we note

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}_{|t=0}\int_\Omega W_{\mathrm{curv}}(\mathfrak{K}(x,t))\,\mathrm{dV} &= \sum_{i=1}^3 \langle \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3), \overline{R}^T\nabla(A\overline{R}.e_i) + (A\overline{R})^T\nabla(\overline{R}.e_i)\rangle_{\mathbb{M}^{3\times3}} \\ &= \sum_{i=1}^3 \langle \overline{R}\,\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3), \nabla(A\overline{R}.e_i)\rangle_{\mathbb{M}^{3\times3}} + \langle \overline{R}\,\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\overline{R}^T, A^T\rangle_{\mathbb{M}^{3\times3}} \\ &= \sum_{i=1}^3 -\langle \operatorname{Div}\left[\overline{R}\,\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\right], A\overline{R}.e_i\rangle_{\mathbb{R}^3} + \langle \overline{R}\left(\sum_{i=1}^3 \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\right)\overline{R}^T, A^T\rangle \\ &= -\langle \operatorname{Div}\left[\overline{R}\,\partial_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right], A\overline{R}\rangle_{\mathbb{M}^{3\times3}} + \langle \overline{R}\left(\sum_{i=1}^3 \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\,\overline{R}^T\right), A^T\rangle \\ &= -\langle \operatorname{Div}\left[\overline{R}\,\partial_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right]\overline{R}^T, A\rangle + \langle \overline{R}\left(\sum_{i=1}^3 \partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\right)\overline{R}^T, A^T\rangle \\ &= -\langle \overline{R}\,\overline{R}^T \operatorname{Div}\left[\overline{R}\,\partial_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right]\overline{R}^T, A\rangle - \langle \overline{R}\operatorname{skew}\left(\sum_{i=1}^3 \left(\partial_{\mathfrak{K}^i} W_{\mathrm{curv}}(\mathfrak{K}^1,\mathfrak{K}^2,\mathfrak{K}^3)\mathfrak{K}^{i,T}\right)\right)\overline{R}^T, A\rangle \\ &= -\langle \overline{R}\operatorname{skew}\left(\overline{R}^T \operatorname{Div}\left[\overline{R}\,\partial_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right]\right)\overline{R}^T, A\rangle - \langle \overline{R}\operatorname{skew}\left(D_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\mathfrak{K}^T\right)\overline{R}^T, A\rangle\,. \end{aligned} \tag{2.9.154}$$

Since $A \in C_0^\infty(\Omega, \mathfrak{so}(3,\mathbb{R}))$ is arbitrary, equation $(2.2.12)_2$ follows.

2.9.5 Scaling relations for Cosserat models

For completeness let us summarize the scaling relations appearing in a finite elastic Cosserat theory. Our goal is to relate the response of large and small samples of the same material and to asses the influence of the characteristic length L_c.

First, in our definition, the **characteristic length** L_c is a given material parameter, corresponding to the **smallest discernable distance** to be accounted for in the model. A simple consequence is that geometrical dimensions L of the bulk material must be larger than L_c, indeed for a continuum theory to apply L should be significantly larger than L_c.

Now let $\Omega_L = [0,L]\times[0,L]\times[0,L]$ be the cube with edge length L, representing the bulk material. Consider a deformation $\varphi_L : \xi \in \Omega_L \mapsto \mathbb{R}^3$ and microrotation $\overline{R}_L(\xi) : \Omega_L \mapsto \mathrm{SO}(3,\mathbb{R})$ as solution of the simplified minimization problem

$$\int_{\xi\in\Omega_L} \mu\,\|\overline{R}_L^T(\xi)F(\xi) - 1\!\!1\|^2 + \mu\,L_c^q\,\|\mathrm{D}_\xi \overline{R}_L(\xi)\|^q\,\mathrm{d}\xi \mapsto \ \min.\ \text{w.r.t.}\ (\varphi_L, \overline{R}_L) \tag{2.9.155}$$

The simple scaling transformation $\zeta : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $\zeta(x) = L\cdot x$ maps the unit cube $\Omega_1 = [0,1]\times[0,1]\times[0,1]$ into Ω_L. Defining the related deformation $\varphi : x \in \Omega_1 \mapsto \mathbb{R}^3$ and microrotation $\overline{R}(x) : \Omega_1 \mapsto \mathrm{SO}(3,\mathbb{R})$ as

$$\varphi(x) := \zeta^{-1}\left(\varphi_L(\zeta(x))\right), \quad \overline{R}(x) := \overline{R}_L(\zeta(x))\,, \tag{2.9.156}$$

shows

$$\mathrm{D_x}\overline{R}(x) = \mathrm{D}_\xi \overline{R}_L(\zeta(x)) \cdot \nabla_x \zeta(x) = \mathrm{D}_\xi \overline{R}_L(\xi) \cdot L$$
$$\nabla_x \varphi(x) = \frac{1}{L}\nabla_\xi \varphi_L(\zeta(x))\nabla_x\zeta(x) = \nabla_\xi \varphi_L(\xi)\,. \tag{2.9.157}$$

Hence, the minimization problem can be transformed

$$\int_{\xi\in\Omega_L} \mu\,\|\overline{R}_L^T(\xi)\nabla_\xi\varphi(\xi) - 1\!\!1\|^2 + \mu\,L_c^q\,\|\mathrm{D}_\xi\overline{R}_L(\xi)\|^q\,\mathrm{d}\xi = \int_{x\in\Omega_1} \mu\,\|\overline{R}^T(x)\nabla_x\varphi(x) - 1\!\!1\|^2 \det[\nabla_x\zeta(x)] + \mu\,L_c^q\,\|\frac{1}{L}\mathrm{D_x}\overline{R}(x)\|^q \det[\nabla_x\zeta(x)]\,\mathrm{dx}$$
$$= \int_{x\in\Omega_1} \mu\,\|\overline{R}^T(x)\nabla_x\varphi(x) - 1\!\!1\|^2\,L^3 + \mu\,L_c^q\,L^{3-q}\,\|\mathrm{D_x}\overline{R}(x)\|^q\,\mathrm{dx}\,, \tag{2.9.158}$$

and we may consider the problem defined on Ω_1:

$$\int_{x\in\Omega_1} \mu\,\|\overline{R}^T(x)\nabla_x\varphi(x) - 1\!\!1\|^2 + \mu\,L_c^q\,L^{3-q-3}\,\|\mathrm{D_x}\overline{R}(x)\|^q\,\mathrm{dx} \mapsto \min.\ \mathrm{w.r.t.}\ (\varphi,\overline{R}) \tag{2.9.159}$$

Comparison of different sample sizes is afforded by transformation to the unit cube repectively, e.g. we compare two samples of the same material with sizes $L_1 > L_2$. Transformation to the unit cube shows that the response of sample two is stiffer than the response of sample one.

It is plain to see that for L large compared to L_c, the influence of the rotations will be small and in the limit $\frac{L_c}{L} \to 0$ classical behaviour results. Otherwise, the larger $\frac{L_c}{L} < 1$, the more pronounced the Cosserat effects become and a small sample is relatively stiffer than a large one.

2.9.6 Finite elasto-plastic Cosserat theory for small elastic rotations

Since we are at present not in a position to mathematically treat the geometrically exact elasto-plastic Cosserat model (2.3.31) it seems expedient to introduce a first partial reduction of the model which will allow an adequate analysis in the near future. We assume only that elastic rotations $\overline{R}_e$ remain small. Hence by expansion $\overline{R}_e = 1\!\!1 + \overline{A}_e + \ldots$, $\overline{A}_e \in \mathfrak{so}(3)$ and $\mathrm{D_x}\overline{R}_e \approx \mathrm{D_x}\overline{A}_e + \ldots$. Furthermore we take $\mu_c = 0$, $\alpha_4 = 0$, $p = 1$ and dispose of external volume and surface couples.

This simplification results in the following nonlinear system of coupled partial differential and evolution equations for the finite deformation $\varphi : [0,T]\times\overline{\Omega} \mapsto \mathbb{R}^3$, the plastic deformation $F_p : [0,T]\times\overline{\Omega} \mapsto \mathrm{GL}^+(3,\mathbb{R})$ and the independent Cosserat elastic "microrotation" $\overline{R}_e : [0,T]\times\overline{\Omega} \mapsto \mathrm{GL}^+(3)$

$$\int_\Omega W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] + W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p] - \langle f,\varphi\rangle\det[F_p]\,\mathrm{dV} - \int_{\Gamma_S}\langle N,\varphi\rangle\,\|\operatorname{Cof}F_p.\vec{n}\|\,\mathrm{dS}$$
$$\mapsto \min.\ \mathrm{w.r.t.}\ (\varphi,\overline{R}_e)\ \text{at constant}\ F_p,$$
$$\overline{R}_{e_{|\Gamma}} = 1\!\!1 + \begin{cases}\operatorname{skew}(\nabla g_{\mathrm{d}}\cdot F_p^{-1})_{|\Gamma}\,, & \text{rigid}\\ \operatorname{skew}(\nabla\varphi\cdot F_p^{-1})_{|\Gamma}\,, & \text{consistent}\end{cases}\quad,\quad \varphi_{|\Gamma} = g_{\mathrm{d}}(t)$$
$$W_{\mathrm{mp}}(\overline{U}_e) = \mu\,\|\operatorname{sym}(\overline{U}_e - 1\!\!1)\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\overline{U}_e - 1\!\!1)\right]^2,\quad W_{\mathrm{curv}}(\mathfrak{K}_e) = \mu\,\frac{L_c^2}{12}\,\|\mathfrak{K}_e\|^2 \tag{2.9.160}$$
$$F_e = \nabla\varphi\cdot F_p^{-1},\quad \overline{U}_e = \overline{R}_e^T F_e,\quad \overline{R}_e = 1\!\!1 + \overline{A}_e\quad \mathfrak{K}_e = [\mathrm{D_x}\overline{A}_e(x)\cdot F_p^{-1}]$$
$$\frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right] \in -F_p^{-1}\cdot \mathfrak{f}(\Sigma_E),\quad \Sigma_E = \Sigma_{E,\mathrm{mp}} + \Sigma_{E,\mathrm{curv}}\,.$$

No assumptions on the magnitude of deformations or plastic deformation are introduced. We venture to say that local existence can be established along the lines of [Nef04].

2.9.7 Partially linearized finite elasto-plastic Cosserat theory

A further reduction is achieved if we assume additionally that plastic deformations F_p remain small.

To this end we proceed similar to (2.4.46) and write $\overline{R}_e = 1\!\!1 + \overline{A}_e + \ldots$, $\overline{A}_e \in \mathfrak{so}(3)$, $\|\overline{A}_e\|^2 \ll 1$, $F_p = 1\!\!1 + p + \ldots$, then the elastic micropolar stretch tensor $\overline{U}_e = \overline{R}_e^T F_e = \overline{R}_e^T F F_p^{-1}$ may be expanded as follows

$$\overline{U}_e = \overline{R}_e^T F F_p^{-1} = (1\!\!1 + \overline{A}_e + \ldots)^T(1\!\!1 + \nabla u)(1\!\!1 - p + \ldots) \approx 1\!\!1 + \nabla u - p - \overline{A}_e + \ldots\,, \tag{2.9.161}$$

such that to leading order (in (2.4.46) by contrast we keep one order more)

$$W_{\mathrm{mp}}(\overline{U}_e) \approx \mu\,\|\operatorname{sym}(\nabla u - p)\|^2 + \mu_c\|\operatorname{skew}(\nabla u - p - \overline{A}_e)\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\nabla u - p)\right]^2_; \tag{2.9.162}$$

and the elastic micropolar curvature is expanded as

$$\mathfrak{K}_e = \overline{R}_e^T\left[\mathrm{D_x}\overline{R}_e\cdot F_p^{-1}\right] = (1\!\!1 + \overline{A}_e + \ldots)^T\left[\mathrm{D_x}(1\!\!1 + \overline{A}_e + \ldots)(1\!\!1 - p + \ldots)\right] \approx \mathrm{D_x}\overline{A}_e(1\!\!1 - p) + \ldots\,. \tag{2.9.163}$$

Introducing these reductions consistently into (2.3.31) we obtain the following (seemingly more complicated) system of coupled partial differential and evolution equations for the finite macroscopic displacement $u : [0,T]\times\overline{\Omega} \mapsto \mathbb{R}^3$, the infinitesimal plastic deformation $p:[0,T]\times\overline{\Omega}\mapsto\mathbb{M}^{3\times3}$ and the independent infinitesimal Cosserat elastic microrotation $\overline{A}_e:[0,T]\times\overline{\Omega}\mapsto\mathfrak{so}(3)$

$$\begin{aligned}
&\int_\Omega W_{\rm mp}(\nabla u,p,\overline{A}_e)\,[1+\mathrm{tr}\,[\varepsilon_p]]+W_{\rm curv}(\mathrm{D_x}\overline{A}_e,p)[1+\mathrm{tr}\,[\varepsilon_p]\,]-\langle f,u\rangle[1+\mathrm{tr}\,[\varepsilon_p]]\,\mathrm{dV}\\
&\quad-\int_\Omega\langle M,\overline{A}_e\rangle[1+\mathrm{tr}\,[\varepsilon_p]\,]\,\mathrm{dV}-\int_{\Gamma_S}\langle N,u\rangle\,[1+\mathrm{tr}\,[\varepsilon_p]]\sqrt{1-2\langle\varepsilon_p.\vec{n},\vec{n}\rangle}\,\mathrm{dS}\\
&\qquad-\int_{\Gamma_C}\langle M_c,\overline{A}_e\rangle\,[1+\mathrm{tr}\,[\varepsilon_p]]\sqrt{1-2\langle\varepsilon_p.\vec{n},\vec{n}\rangle}\,\mathrm{dS}\mapsto\min.\quad\text{w.r.t. }(u,\overline{A}_e)\text{ at constant }p\,,\\
&W_{\rm mp}(\nabla u,p,\overline{A}_e)=\mu\|\,\mathrm{sym}(\nabla u-p)\|^2+\mu_c\,\|\,\mathrm{skew}(\nabla u-p-\overline{A}_e)\|^2+\frac{\lambda}{2}\,\mathrm{tr}\,[\mathrm{sym}(\nabla u-p)]^2\\
&W^{\rm small}_{\rm curv}(\mathrm{D_x}\overline{A}_e,p)=\mu\frac{L_c^2}{12}\|\mathrm{D_x}\overline{A}_e(1\!\!1-p)\|^2\\
&\frac{\mathrm{d}}{\mathrm{dt}}\,[p(t)]\in\partial\chi(T_E),\quad T_E=-\partial_p\big[\,\big[W_{\rm mp}(\nabla u,p,\overline{A}_e)+W^{\rm small}_{\rm curv}(\mathrm{D_x}\overline{A}_e,p)\big]\,[1+\mathrm{tr}\,[\varepsilon_p]\,]\big]\\
&T_E=T_{E,\rm mp}+T_{E,\rm curv}\,,\quad\varepsilon_p=\tfrac12(p^T+p)\,,\quad p(0)\in\mathbb{M}^{3\times3}\\
&T_{E,\rm mp}=DW_{\rm mp}(\nabla u,p,\overline{A}_e)\,[1+\mathrm{tr}\,[\varepsilon_p]]-W_{\rm mp}(\nabla u,p,\overline{A}_e)\,1\!\!1\\
&T_{E,\rm curv}=\mathrm{D_x}\overline{A}_e^T\,DW^{\rm small}_{\rm curv}(\mathrm{D_x}\overline{A}_e,p)\,[1+\mathrm{tr}\,[\varepsilon_p]]-W_{\rm curv}(\mathrm{D_x}\overline{A}_e,p)\,1\!\!1\\
&u_{|_\Gamma}(t,x)=g_{\rm d}(t,x)-x,\quad x\in\Gamma\,,\quad\overline{A}_{e|_\Gamma}(t,x)=\mathrm{skew}(\nabla g_{\rm d}(t,x)-p(t,x))_{|_\Gamma}\,.
\end{aligned}\tag{2.9.164}$$

Here T_E is the **reduced elastic Eshelby tensor**. Observe that the coupling in $T_{E,\rm curv}$ is of second order, otherwise the Cosserat contribution would not appear in the plastic flow part. This system is intrinsically thermodynamically admissible. Note that in the formal limit $\mu_c\to\infty$, the total infinitesimal continuum rotation splits additively into elastic infinitesimal microrotations $\overline{A}_e$ and **infinitesimal plastic spin** $\mathrm{skew}(p)$:

$$\mathrm{skew}(\nabla u)=\overline{A}_e+\mathrm{skew}(p)\,.\tag{2.9.165}$$

If we choose the elastic domain $\mathcal{E}:=\{\Sigma_E\,|\,\|\,\mathrm{dev}(\mathrm{sym}\,\Sigma_E)\|\le\sigma_{\rm y}\}$ [57], then the system further reduces to

$$\begin{aligned}
&\int_\Omega W_{\rm mp}(\nabla u,p,\overline{A}_e)+W_{\rm curv}(\mathrm{D_x}\overline{A}_e,p)-\langle f,u\rangle-\langle M,\overline{A}_e\rangle\,\mathrm{dV}\\
&\quad-\int_{\Gamma_S}\langle N,u\rangle\,\mathrm{dS}-\int_{\Gamma_C}\langle M_c,\overline{A}_e\rangle\,\mathrm{dS}\mapsto\min.\quad\text{w.r.t. }(u,\overline{A}_e)\text{ at constant }\varepsilon_p\,,\\
&W_{\rm mp}(\nabla u,p,\overline{A}_e)=\mu\,\|\varepsilon-\varepsilon_p\|^2+\mu_c\,\|\,\mathrm{skew}(\nabla u-\overline{A}_e)\|^2+\frac{\lambda}{2}\,\mathrm{tr}\,[\varepsilon]^2\\
&W^{\rm small}_{\rm curv}(\mathrm{D_x}\overline{A}_e,\varepsilon_p)=\mu\,\frac{L_c^2}{12}\,\|\mathrm{D_x}\overline{A}_e(1\!\!1-\varepsilon_p)\|^2\\
&\varepsilon_e=\varepsilon-\varepsilon_p,\quad\varepsilon(\nabla u(x))=\tfrac12(\nabla u^T+\nabla u),\quad\varepsilon_p=\tfrac12(p^T+p),\\
&\dot{\varepsilon_p}(t)\in\partial\chi(T_E),\quad T_E=-\partial_{\varepsilon_p}\,[W_{\rm mp}(\nabla u,\varepsilon_p,\overline{A}_e)+W_{\rm curv}(\mathrm{D_x}\overline{A}_e,\varepsilon_p)]\\
&T_E=T_{E,\rm mp}+T_{E,\rm curv},\\
&T_{E,\rm mp}=-\partial_pW_{\rm mp}(\nabla u,p,\overline{A}_e)=2\mu\,(\varepsilon-\varepsilon_p),\ T_{E,\rm curv}=\mathrm{D_x}\overline{A}_e^T\,DW^{\rm small}_{\rm curv}(\mathrm{D_x}\overline{A}_e,p)\\
&u_{|_\Gamma}(t,x)=g_{\rm d}(t,x)-x,\quad x\in\Gamma\,,\quad\mathrm{tr}\,[\varepsilon_p(0)]=0\,,\quad\varepsilon_p(0)\in\mathrm{Sym}(3)\\
&\overline{A}_{e|_\Gamma}(t,x)=\mathrm{skew}(\nabla g_{\rm d}(t,x)-\varepsilon_p)_{|_\Gamma}=\mathrm{skew}(\nabla g_{\rm d}(t,x))_{|_\Gamma}\,.
\end{aligned}\tag{2.9.166}$$

A simple calculation yields the relations $\|\mathrm{D_x}\overline{A}_e\|^2=2\|\nabla\,\mathrm{axl}(\overline{A}_e)\|^2$ and $\mathrm{D_x}\overline{A}_e^T\cdot\mathrm{D_x}\overline{A}_e(1\!\!1-\varepsilon_p)=\sum_{i=1}^3\mathrm{D_x}\overline{A}_e^{i,T}\cdot\mathrm{D_x}\overline{A}_e^{i,T}(1\!\!1-\varepsilon_p)=2\,[\nabla\,\mathrm{axl}(\overline{A}_e)]^T[\nabla\,\mathrm{axl}(\overline{A}_e)](1\!\!1-\varepsilon_p)$, where $\mathrm{axl}:\mathfrak{so}(3,\mathbb{R})\mapsto\mathbb{R}^3$ is the canonical identification of a skew-symmetric matrix with its axial vector. Therefore,

$$T_E=2\mu\,(\varepsilon-\varepsilon_p)+(2\mu+\lambda)\,2\,\frac{L_c^2}{12}\,[\nabla\,\mathrm{axl}(\overline{A}_e)]^T[\nabla\,\mathrm{axl}(\overline{A}_e)](1\!\!1-\varepsilon_p)\,,\tag{2.9.167}$$

and we appreciate the role, the elastic Cosserat contribution takes in the elasto-plastic model: in first order it resembles a **softening** mechanism. A similar softening effect can be observed in the model of [Bes74, (11.5)], where, however, the kinematical description is different from ours. In spite of the softening effect, we expect that the system (2.9.166) admits a global in-time solution with slightly improved regularity in the rate-independent ideal plasticity case.

If we finally neglect all second order terms, we obtain the infinitesimal, geometrically linear model (2.3.36).

[57] no **infinitesimal plastic spin**: $p=\varepsilon_p$ and isochoric plasticity: $\mathrm{tr}\,[\varepsilon_p]=0$ and assume $\langle\varepsilon_p.\vec{n},\vec{n}\rangle=0$ on $\Gamma_S\cup\Gamma_C$.

2.9.8 A remark on the elasto-plastic decomposition of the curvature tensor

There is some discussion in the literature as regards the suitable definition of an independent quantity of plastic curvature $\overline{\kappa}_p$. We did not advocate its use. Ehlers has already observed that $\overline{\kappa}_p$ cannot really be independent under a disguised consistency requirement. The argument runs as follows (only for the infinitesimal case for simplicity): the micropolar decomposition of the displacement gradient into microstrain $\overline{\varepsilon}$ and microrotation $\overline{A}$

$$\overline{\varepsilon} = \nabla u - \overline{A}\,, \tag{2.9.168}$$

shows that $\overline{\kappa} := \operatorname{Curl}\overline{A} = -\operatorname{Curl}\overline{\varepsilon}$. In infinitesimal micropolar elasto-plasticity the microstrain is additively decomposed into elastic and plastic parts: $\overline{\varepsilon} = \overline{\varepsilon}_e + \overline{\varepsilon}_p$. This yields

$$\overline{\varepsilon}_e + \overline{\varepsilon}_p = \overline{\varepsilon} = \nabla u - \overline{A}, \Leftrightarrow \quad \operatorname{Curl}\overline{\varepsilon}_e + \operatorname{Curl}\overline{\varepsilon}_p = -\operatorname{Curl}\overline{A} = -\overline{\kappa}\,. \tag{2.9.169}$$

If we assume in addition that the total curvature splits as well additively into elastic and plastic parts, $\overline{\kappa} = \overline{\kappa}_e + \overline{\kappa}_p$, we are naturally led to assume by consistency with (2.9.169), that $\overline{\kappa}_e = -\operatorname{Curl}\overline{\varepsilon}_e$ and $\overline{\kappa}_p = -\operatorname{Curl}\overline{\varepsilon}_p$.

The corresponding thermodynamical dissipation potential

$$\begin{aligned}
&\int_\Omega \mu\,\|\operatorname{sym}\overline{\varepsilon}_e\|^2 + \mu_c\,\|\operatorname{skew}(\overline{\varepsilon}_e)\|^2 + \mu\,L_c^2\,\|\overline{\kappa}_e\|^2\,\mathrm{dV} = \int_\Omega \mu\,\|\operatorname{sym}\nabla u - \overline{\varepsilon}_p\|^2 + \mu_c\,\|\operatorname{skew}(\nabla u - \overline{\varepsilon}_p - \overline{A})\|^2 + \mu\,L_c^2\,\|\overline{\kappa} - \overline{\kappa}_p\|^2\,\mathrm{dV}\\
&\quad = \int_\Omega \mu\,\|\operatorname{sym}\nabla u - \overline{\varepsilon}_p\|^2 + \mu_c\,\|\operatorname{skew}(\nabla u - \overline{\varepsilon}_p - \overline{A})\|^2 + \mu\,L_c^2\,\|\operatorname{Curl}\overline{A} - \operatorname{Curl}\overline{\varepsilon}_p\|^2\,\mathrm{dV}\,,
\end{aligned} \tag{2.9.170}$$

with only remaining independent dissipative variable $\overline{\varepsilon}_p$ (instead of two independent variables $(\overline{\varepsilon}_p, \overline{\kappa}_p)$) should lead to a **parabolic** flow rule for $\overline{\varepsilon}_p$ due to a **nonlocal** evaluation (instead of a traditional flow rule of ordinary differential equations for $(\overline{\varepsilon}_p, \overline{\kappa}_p)$. While the consistency requirement (2.9.169) has already been postulated in [EDV98], in their work they still use the flow rule coming from the ordinary differential approach.

2.9.9 Notes on parameter identification

We include this discussion because there seems to be some confusion on what significance certain material parameters appearing in a Cosserat context have. Here it suffices to consider only the infinitesimal case in equilibrium format

$$\begin{aligned}
0 &= \operatorname{Div}\sigma + f, \quad x \in \Omega\\
\sigma &= 2\mu\,\operatorname{sym}\nabla u + 2\,\mu_c\,(\operatorname{skew}(\nabla u) - \overline{A}) + \lambda\operatorname{tr}[\operatorname{sym}\nabla u]\cdot 1\!\!1\\
0 &= \mu\frac{L_c^2}{12}\,\Delta\operatorname{axl}(\overline{A}) + \mu_c\,\operatorname{axl}((\operatorname{skew}(\nabla u) - \overline{A})\\
u_{|\partial\Omega}(t,x) &= g_{\mathrm{d}}(t,x) - x, \quad x \in \partial\Omega\,, \quad \overline{A}_{|\partial\Omega} = \operatorname{skew}(\nabla g_{\mathrm{d}}(t,x))_{|\partial\Omega}\,.
\end{aligned} \tag{2.9.171}$$

Suppose homogeneous boundary conditions are prescribed: for $B \in \mathrm{GL}(3,\mathbb{R})$ a constant matrix, we set $g_{\mathrm{d}}(x) := B.x$. Thus we are able e.g. to describe uniform traction, uniform compression or simple shear. It is clear that $u(x) = B.x - x$ and $\overline{A}(x) = \operatorname{skew}(B)$ satisfy the boundary conditions and equilibrium equations. They are also the unique solutions. However, for these unique solutions, the Cosserat mechanisms are not activiated and we may determine μ, λ as classical moduli independent of the couple modulus μ_c. In any homogeneous situation, only classical mechanisms are involved. Turning this argument upside down, we conclude that classical infinitesimal elasticity is appropriate for homogeneous situations only.

In order to get some information on the value of μ_c and the length scale L_c we need to perform experiments leading to **inhomogeneous** response. One of the simplest cases is torsion of a cylinder. A sequence of torsion experiments allows to determine μ_c and L_c if analytical formula are available relating torque and twist and incorporating the appearing parameters. Such formulas exist, showing that torsion in a Cosserat material would be stiffer (and depending on the length of the specimen) than ought to be expected by calculations based on classical linear elasticity and the already determined classical coefficients.

Investigations to this end on many materials have been performed, with the deceiving result that μ_c should be set to zero, thus implying that the infinitesimal Cosserat model is not appropriate for a more realistic description than classical linear elasticity. However, this shows only that Cosserat effects, if they really exist, are second order effects, not discernable in a first linear approximation.

Since, however, the parameter μ_c appears as well in the geometrically exact description, the foregoing experiments have already shown conclusively that $\mu_c = 0$ is the correct value for the finite theory. This possibility of $\mu_c = 0$ together with a true finite Cosserat theory has been consistently overlooked by overemphasizing the linear model.

It remains to determine the length scale L_c. But now we cannot use the infinitesimal model and its solution formula in torsion since with $\mu_c = 0$ no Cosserat effects remain. We are either in need of a solution formula for the torsion problem of the nonlinear model or we have to calculate the torsional response directly numerically. However, one simplification is possible: we do not really need the full nonlinear system, instead an intermediate model incorporating second order effects would suffice. Below, we will give such a model.

2.9.10 Stability of the homogeneous solution

Consider the simplified finite Cosserat problem with consistent coupling condition

$$\begin{aligned}
&\int_\Omega \mu\,\|\operatorname{sym}(\overline{R}^T F) - 1\!\!1)\|^2 + \mu_c\,\|\operatorname{skew}(\overline{R}^T F)\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T F - 1\!\!1)\right]^2 + \mu\,L_c^2\,\|\mathrm{D_x}\overline{R}\|^2\,\mathrm{dV} \mapsto \min.\ \text{w.r.t. } (\varphi, \overline{R}),\\
&\qquad \overline{R}_{|\Gamma} = \operatorname{polar}(\nabla\varphi), \quad \varphi_{|\Gamma} = g_{\mathrm{d}}\,,
\end{aligned} \tag{2.9.172}$$

and impose homogeneous boundary conditions: $g_{\mathrm{d}}(x) = B.x$ with a constant matrix B. It is clear that the homogeneous solution $\varphi(x) = B.x$ and $\overline{R} = \operatorname{polar}(F) = \operatorname{polar}(B)$ solves the corresponding equilibrium equation and boundary conditions. Is it possible to conclude that this solution is also a (unique?) global minimizer of the energy? At least we surmise that the homogeneous solution is locally stable. No rigourous conclusion is possible at this stage of the investigation. Note that for the consistent coupling boundary condition we have the trivial estimate

$$\int_\Omega W(\nabla\varphi, \overline{R})\,\mathrm{dV} \le \int_\Omega W(B, \mathrm{polar}(B))\,\mathrm{dV} = W(B, \mathrm{polar}(B)) \cdot |\Omega|\,. \tag{2.9.173}$$

Moreover, the homogeneous solution will be energetically more favourable, the higher the value of $L_c > 0$ (or the smaller the specimen is). The height of a potential well around the homogeneous solution should be strictly related to $L_c > 0$. If $L_c = \infty$, then the homogeneous solution is the only possible one.

The Cosserat model allows therefore in principal for inhomogeneous minimizers in situations where homogeneous stationary solutions are possible.

2.9.11 A simplified elasto-plastic model for easy numerical implementation

Here we propose a model, based on our development, which should cover the essential behaviour of the geometrically exact model while being slightly simplified in order to arrive at a reasonable numerical implementation. It should as well serve the purpose of finding the value of the characteristic length L_c for zero Cosserat couple modulus $\mu_c = 0$.

Numerical implementations based on an infinitesimal system are already in use. Tentative calculations of geometrical exact equations have also been done. However, due to the nonlinear manifold structure of $\mathrm{SO}(3,\mathbb{R})$ the implementation is awkward and the performance of the finite codes is in general insufficient. In order to circumvent these problems right from the start we propose a **penalty** formulation for the treatment of the finite rotations. We augment the free energy with a penalty term $\frac{\aleph}{4}\,\|\overline{R}_e^T\overline{R}_e - 1\!\!1\|^2$ where $\aleph$ is not a material parameter but supposed to approach ∞ in order to adjust $\overline{R}$ to exact rotations.

The geometrical exact elasto-plastic Cosserat model was given by

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] + W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p] - \langle f, \varphi\rangle\det[F_p] - \langle M, \overline{R}_e\rangle\det[F_p]\,\mathrm{dV} - \int_{\Gamma_S}\langle N, \varphi\rangle\,\|\,\mathrm{Cof}\,F_p.\vec{n}\|\,\mathrm{dS}\\
&\qquad - \int_{\Gamma_C}\langle M_c, \overline{R}_e\rangle\,\|\,\mathrm{Cof}\,F_p.\vec{n}\|\,\mathrm{dS} \mapsto \min.\,\mathrm{w.r.t.}\ (\varphi, \overline{R}_e)\ \text{at constant}\ F_p,\\
&\overline{R}_{e_{|\Gamma}} = \mathrm{polar}(\nabla g_{\mathrm{d}}\cdot F_p^{-1})_{|\Gamma}, \quad \varphi_{|\Gamma} = g_{\mathrm{d}}(t)\\
&W_{\mathrm{mp}}(\overline{U}_e) = \mu\,\|\,\mathrm{sym}(\overline{U}_e - 1\!\!1)\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{U}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}_e - 1\!\!1)\right]^2\\
&W_{\mathrm{curv}}(\mathfrak{K}_e) = \mu\frac{L_c^{1+p}}{12}\,(1 + \alpha_4\,L_c^q\,\|\mathfrak{K}_e\|^q)\,\|\mathfrak{K}_e\|^{1+p}\\
&F_e = \nabla\varphi\cdot F_p^{-1}, \quad \overline{U}_e = \overline{R}_e^T F_e, \quad \mathfrak{K}_e = \overline{R}_e^T[\mathrm{D_x}\overline{R}_e(x)\cdot F_p^{-1}]\\
&\frac{\mathrm{d}}{\mathrm{dt}}\,[F_p^{-1}] \in -F_p^{-1}\cdot\mathfrak{f}(\Sigma_E), \quad \Sigma_E = \Sigma_{E,\mathrm{mp}} + \Sigma_{E,\mathrm{curv}}\\
&\Sigma_{E,\mathrm{mp}} = \overline{U}_e^T D_{\overline{U}_e}W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] - W_{\mathrm{mp}}(\overline{U}_e)\det[F_p]\,1\!\!1\\
&\Sigma_{E,\mathrm{curv}} = \mathfrak{K}_e^T D_{\mathfrak{K}_e}W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p] - W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p]\,1\!\!1\\
&F_p^{-1}(0) = F_{p_0}^{-1}, \quad F_{p_0} \in \mathrm{GL}^+(3,\mathbb{R})\,.
\end{aligned} \tag{2.9.174}$$

An immediate permitted simplification is obtained by setting $\alpha_4 = 0,\quad p = 1$ and discarding external volume and surface couples. This implies already that we can reduce the consideration of $\mathfrak{K}_e$ to $\mathfrak{K}_e = [\mathrm{D_x}\overline{R}_e(x)\cdot F_p^{-1}]$, the rotations do not appear explicitly in the curvature.

No we introduce the penalty term and relax the rotations in the sense that we only require $\overline{R}_e \in \mathbb{M}^{3\times 3}$:

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] + W_{\mathrm{curv}}(\mathfrak{K}_e)\det[F_p] + \frac{\aleph}{4}\,\|\overline{R}_e^T\overline{R}_e - 1\!\!1\|^2\det[F_p] - \langle f, \varphi\rangle\det[F_p]\,\mathrm{dV} - \int_{\Gamma_S}\langle N, \varphi\rangle\,\|\,\mathrm{Cof}\,F_p.\vec{n}\|\,\mathrm{dS}\\
&\qquad \mapsto \min.\,\mathrm{w.r.t.}\ (\varphi, \overline{R}_e)\ \text{at constant}\ F_p,\\
&\overline{R}_{e_{|\Gamma}} = \mathrm{polar}(\nabla g_{\mathrm{d}}\cdot F_p^{-1})_{|\Gamma}, \quad \varphi_{|\Gamma} = g_{\mathrm{d}}(t)\\
&W_{\mathrm{mp}}(\overline{U}_e) = \mu\,\|\,\mathrm{sym}(\overline{U}_e - 1\!\!1)\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{U}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}_e - 1\!\!1)\right]^2, \quad W_{\mathrm{curv}}(\mathfrak{K}_e) = \mu\frac{L_c^2}{12}\,\|\mathfrak{K}_e\|^2\\
&F_e = \nabla\varphi\cdot F_p^{-1}, \quad \overline{U}_e = \overline{R}_e^T F_e, \quad \mathfrak{K}_e = [\mathrm{D_x}\overline{R}_e(x)\cdot F_p^{-1}]\,.
\end{aligned} \tag{2.9.175}$$

While the formal structure is thus kept, the differences appear in the related Euler-Lagrange equations. The force balance equation (translational equilibrium) remains invariant. The balance of angular momentum (rotational equilibrium) is modified: taking free variations w.r.t. $\overline{R}_e \in \mathbb{M}^{3\times 3}$ yields the stationarity condition

$$\begin{aligned}
0 &= \langle D_{\overline{U}_e}W_{\mathrm{mp}}(\overline{U}_e)\,F_e^T, \delta\overline{R}_e^T\rangle\det[F_p] + 2\mu\frac{L_c^2}{12}\langle \mathrm{D_x}\overline{R}_e\cdot F_p^{-1}, \mathrm{D_x}(\delta\overline{R}_e)\cdot F_p^{-1}\rangle\det[F_p] + \frac{\aleph}{2}\langle\overline{R}_e^T\overline{R}_e - 1\!\!1, (\delta\overline{R}_e)^T\overline{R}_e + \overline{R}_e^T(\delta\overline{R}_e)\rangle\det[F_p]\\
0 &= \langle[D_{\overline{U}_e}W_{\mathrm{mp}}(\overline{U}_e)\,F_e^T]^T, \delta\overline{R}_e\rangle - 2\mu\frac{L_c^2}{12}\langle\mathrm{Div}[\mathrm{D_x}(\overline{R}_e)\cdot F_p^{-1}F_p^{-T}], \delta\overline{R}_e\rangle + \aleph\langle\overline{R}_e[\overline{R}_e^T\overline{R}_e - 1\!\!1], \delta\overline{R}_e\rangle\,.
\end{aligned} \tag{2.9.176}$$

Hence the strong form is given by

$$0 = 2\mu\,\frac{L_c^2}{12}\,\mathrm{Div}[\mathrm{D_x}(\overline{R}_e)\cdot F_p^{-1}F_p^{-T}] - [D_{\overline{U}_e}W_{\mathrm{mp}}(\overline{U}_e)\,F_e^T]^T - \aleph\overline{R}_e[\overline{R}_e^T\overline{R}_e - 1\!\!1]\,, \tag{2.9.177}$$

with $F_p^{-1}F_p^{-T}$ playing the role of a **plastic metric**. The complete penalized model reads

$$
\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}_e)\det[F_p] - \langle f,\varphi\rangle \det[F_p]\,\mathrm{dV} - \int_{\Gamma_S} \langle N,\varphi\rangle \,\|\operatorname{Cof} F_p.\vec{n}\|\,\mathrm{dS} \mapsto \min \text{ .w.r.t. } \varphi \text{ at constant } (\overline{R}_e, F_p),\\
&0 = 2\mu\,\frac{L_c^2}{12}\,\mathrm{Div}[\mathrm{D_x}(\overline{R}_e)\cdot F_p^{-1}F_p^{-T}] - D_{\overline{U}_e}W_{\mathrm{mp}}(\overline{U}_e)\,F_e^T - \aleph\overline{R}_e[\overline{R}_e^T\overline{R}_e - 1\!\!1]\\
&\overline{R}_{e_{|\Gamma}} = \mathrm{polar}(\nabla g_{\mathrm{d}}\cdot F_p^{-1})_{|\Gamma}, \quad \varphi_{|\Gamma} = g_{\mathrm{d}}(t)\\
&W_{\mathrm{mp}}(\overline{U}_e) = \mu\,\|\mathrm{sym}(\overline{U}_e - 1\!\!1)\|^2 + \mu_c\,\|\mathrm{skew}(\overline{U}_e)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U}_e - 1\!\!1)\right]^2, \quad W_{\mathrm{curv}}(\mathfrak{K}_e) = \mu\frac{L_c^2}{12}\,\|\mathfrak{K}_e\|^2\\
&F_e = \nabla\varphi\cdot F_p^{-1}, \quad \overline{U}_e = \overline{R}_e^T F_e, \quad \mathfrak{K}_e = [\mathrm{D_x}(\overline{R}_e)\cdot F_p^{-1}]\\
&D_{\overline{U}_e}W_{\mathrm{mp}}(\overline{U}_e) = 2\mu\,\mathrm{sym}(\overline{U}_e - 1\!\!1) + 2\mu_c\,\mathrm{skew}(\overline{U}_e) + \lambda\,\mathrm{tr}\left[\overline{U}_e - 1\!\!1\right]1\!\!1\\
&\frac{\mathrm{d}}{\mathrm{dt}}\,[F_p^{-1}] \in -F_p^{-1}\cdot \mathfrak{f}(\Sigma_E), \quad \Sigma_E = \Sigma_{E,\mathrm{mp}} + \Sigma_{E,\mathrm{curv}}\,.
\end{aligned}
\tag{2.9.178}
$$

The ensuing model is still **thermodynamical consistent** since the modifications affect only the elastic behaviour. A welcome feature of the penalized model is the fact that it remains **frame-indifferent**.

The significance of a computed solution can easily be checked by evaluating $\|\overline{R}_e^T\overline{R}_e - 1\!\!1\|$ and/or inserting the result into the exact Euler-Lagrange equations (2.2.12). It is also possible to make $\aleph$ a function of the residuum of the exact Euler-Lagrange equation.

If we augment the penalty term further with $\|\mathrm{skew}(F_e\overline{R}_e^T\|^2$, then in the limit $(\aleph\to\infty, L_c\to 0)$ we recover the classical result $\overline{R}_e = \mathrm{polar}(F_e)$ and we are close to the model investigated numerically in [NW03, M2].

Chapter 3

Two-dimensional Cosserat theory

This chapter is concerned with the consistent dimensional reduction of the previously introduced finite three-dimensional Cosserat micropolar elasticity model to the two-dimensional situation of thin plates and shells. The resulting membrane energy turns out to be a quadratic, elliptic, first order, non degenerate energy in contrast to classical approaches, the standard bending contribution is augmented with a term representing an additional stiffness of the Cosserat model and the corresponding system of balance equations remains of second order. The model includes size effects, transverse shear resistance, thickness stretch and drilling degrees of freedom. The thin shell limit is non-degenerate due to the additional Cosserat bending stiffness.

It is shown that the dimensionally reduced formulation is well-posed along the same line of argument which showed the well posedness of the three-dimensional model [Nef03c]. Decisive use is made of a dimensionally reduced version of an extended Korn's first inequality recently proved by the author [Nef02].

3.1 Introduction

3.1.1 Generalities on shells

The dimensional reduction of a given model is already an old and mature subject and it has seen many 'solutions'. The different approaches toward elastic shell theory proposed in the literature and relevant references thereof are, therefore, too numerous to list here. In any case our own proposal falls within the so called **derived approach**, i.e., reducing a given three-dimensional model via (physically) reasonable constitutive assumptions on the kinematics to a two-dimensional model[1] as opposed to either the **intrinsic approach** which views the shell from the onset as a two-dimensional surface and invokes concepts from differential geometry or the **asymptotic methods** which try to establish two-dimensional equations by formal expansion of the three-dimensional solution in power series in terms of a small parameter. The intrinsic approach is closely related to the **direct approach**[2] which takes the shell to be a two-dimensional directed medium in the sense of a **restricted Cosserat-surface** [CC09].[3] Two-dimensional equilibrium in appropriate new stress and strain variables is postulated ab-initio independent of three-dimensional considerations, cf. [Ant95, GNW65, ET58].

A detailed presentation of the classical shell theories can be found in [Nag72]. A thorough mathematical analysis of linear, infinitesimal shell theory, based on asymptotic methods is to be found in [Cia98a] and the extensive references therein, see also [Cia97, Cia99, Ant95, DS96, Dik82]. Excellent reviews and insightful discussions of the modelling and finite element implementation may be found in [SB92, San95, SB98, GSW89, GT92, BGS96, BR92] and in the series of papers [SF89, SFR89, SFR90, SRF90, SK92, SF92]. Recently, new C^1-conforming implementations for thin Kirchhoff-Love shells have been proposed in [COS00, CO01]. Properly invariant elastic plate theories are derived by formal asymptotic methods in [FRS93].

Let us sketch first the apparent areas of agreement in the development of the elastic case. The various shell models based on linearized three-dimensional elasticity proposed in the literature have been rigorously justified in those cases, where some normality assumption is introduced, either a priori or as a result of an

[1] This line of thought is expressed by W.T. Koiter [Koi69, p.93]:"Any two-dimensional theory of thin shells is necessarily of an approximate character. An exact two-dimensional theory of shells cannot exist, because the actual body we have to deal with, thin as it may be, is always three-dimensional. ... Since the theory we have to deal with is approximate in character, we feel that extreme rigour in its development is hardly desirable. ... Flexible bodies like thin shells require a flexible approach."

[2] The philosophy behind the direct approach is best framed by P.M. Naghdi[GN69, p.58]: "The theory of Cosserat is exact, but shell theory derived from the three-dimensional equations is approximate. It may be a matter of taste, but we prefer to regard an exact theory as more fundamental. The Cosserat theory of shells (Cosserat surface) is on a comparable footing with any exact three-dimensional continuum theory." This remark remains partly true today: while properly invariant derived shell models are now available, they do not necessarily guarantee invertibility.

[3] Restricted, since no material length scale usually enters the direct approach, only the relative thickness h appears in the model.

asymptotic analysis, see notably the extensive work of Ciarlet and his co-workers [Cia97, Cia99]. Membrane and bending equations are identified as leading order terms of asymptotic expansions of the three-dimensional solution. Convergence of the computed solution (and error estimates) to the 'exact' solution of linearized three-dimensional elasticity is established in all relevant cases if various scaling assumptions on the data are made.

The situation is slightly less clear as far as infinitesimal restricted Cosserat models (Reissner-Mindlin plate, Timoshenko beam etc.) are considered. Here the convergence as the thickness tends to zero of some director to the (linearized) normal of the surface poses additional difficulties, but can be overcome, see e.g. [Ebe99] for the plate bending problem. It is known, that the solution of the infinitesimal Reissner-Mindlin model for various values of the shear correction factor κ converges to the solution of the infinitesimal Kirchhoff-Love model for vanishing thickness.

Already in the infinitesimal case it becomes apparent that **a model, involving membrane and bending simultaneously cannot be obtained by formal asymptotic methods** but is a result of careful modelling. One such successful model, the Koiter model [Koi70] is simply the sum of the correctly identified membrane and bending contribution, properly scaled with the thickness. The mathematical analysis establishing the well-posedness of all these infinitesimal models is fairly well established and will not be our principal concern. Though analytically understood, the numerical implementation of these infinitesimal, linear shell models is still an area of very active research, mostly because of intricacies related to the singular character of the considered systems as the thickness tends to zero. In the engineering community, the infinitesimal Reissner-Mindlin model is usually preferred numerically to which witnesses the uncountable proposals of new implementation variants,[4] since one only needs to solve a second order problem with standard C^0-finite elements for an augmented field instead of a fourth order problem with difficult to handle C^1-finite elements in the Kirchhoff-Love model. Moreover, the Reissner-Mindlin model allows for **transverse shear**, which may occur at free or loaded edges of the three-dimensional plate. However, the infinitesimal Reissner-Mindlin FEM-implementation is notoriously ill-conditioned without further provision while the underlying mathematical problem is well-posed. Membrane and shear **locking**, roughly meaning that the calculated solution on coarse meshes only poorly approximates the exact solution has motivated the search for locking free implementations and has stipulated to some extent the development of nonconforming elements and of discontinuous Galerkin methods (cf. references in [LNSO04]) which in principal should not suffer from locking. In this respect we mention also the **hierarchical plate models** [Sch96] which are a direct outcome of the finite element methods applied to thin structures. The idea there is to discretely minimize the three-dimensional energy functional over some thickness-restricted ansatz-space, preferably a polynomial approximation in thickness direction.

In the finite, elastic case, mostly based on the Saint Venant-Kirchhoff (SVK) free energy, the formal asymptotic methods are still successful in that they identify again leading membrane and bending terms. As far as the occurring membrane contribution is concerned, it is W_{mp} in (3.7.83) which is given in [GKM96, FRS93]. However, methods based on variational Γ-convergence [DR95a] suggest a fundamentally different membrane term which leads to a nonresistance of the membrane shell in compression. It should be noted, that the widely accepted membrane term of (3.7.83) shows the characteristic apparent change of the Lamé-moduli for the two-dimensional structure. As far as the bending term is concerned, some agreement has been obtained that the term consistent with the 3D-SVK energy is a quadratic expression in the second fundamental form of the surface. Nevertheless, the coefficients of this quadratic form give still room for some discussion: the Hamiltonian based derivation in [GKM96] differs from the results obtained by formal asymptotic analysis in [FRS93, Cia97] precisely in whether there is the same apparent change of the elastic moduli as occurs in the membrane case. This difference is immaterial as regards the mathematical analysis and can be explained by the use of a linear kinematical ansatz in thickness direction in [GKM96] whereas a quadratic ansatz in a Hamiltonian framework would yield the same result as in [FRS93, Cia97] and W_{bend} in (3.7.83).

It must be noted, that proceeding by asymptotic analysis is based itself on certain a priori assumptions, namely that all appearing quantities indeed admit an expansion in terms of a small parameter and satisfy certain scaling assumptions. No rigorous justification of the formal asymptotic approach has been given so far for finite elasticity, precisely because of the lack of some encompassing theory which guarantees the well-posedness of the three-dimensional problem. The application of formal asymptotic methods has never led to basically new plate or shell models, it seems to be restricted to an a posteriori justification of existing models. By contrast, the equations obtained by a variational approach i.e. energy projection and those for a Cosserat surface are independent of scaling assumptions.

We wish to remark that in the finite regime, no 'unique' elastic three-dimensional model exists: we have always to make constitutive choices for the bulk behaviour which has consequences for the reduced theory. In

[4] There is a certain discrepancy between the effort put into the investigations of the **infinitesimal Reissner-Mindlin** model and its physical significance, given that the model is **not frame-indifferent** and for that matter, strictly speaking, irrelevant.

this case, making additional, physically sound, constitutive assumptions on the two-dimensional response itself, seems to be just another viable step in the modelling procedure. However, for infinitesimal strains we know the isotropic elastic bulk behaviour exactly[5] and subsequently it is reasonable to establish the convergence for vanishing relative thickness h to precisely one model without additional constitutive two-dimensional assumptions. This remark constitutes a strong justification for the asymptotic method in the infinitesimal case.

It has already been observed that the leading order term without additional provisions on the data is either a membrane or a bending term. But in applications, there are usually regions of a shell where membrane effects dominate while in others, bending is dominant. A fully three-dimensional resolution of a thin shell problem remains elusive notwithstanding the increased computer power. Hence, there is still a need to come up with a sound finite model, combining both effects in one system of equations, as does the Koiter model in the infinitesimal case.

Since we have in mind the future extension of the herein presented plate theories to multiplicative plasticity let us add that the picture is all the more complicated as far as elasto-plastic extensions are concerned, in part because of the (limited) state of the art of finite elasto-plasticity itself and in part because it is not straight forward to transform an existing 3D-model to its 2D-counterpart, see [BS99, BW91, SRF90] for representative examples. It is technically difficult to carry through the program of the formal asymptotic methods and in fact such a development seems not to have been undertaken in the finite case.

In order to get two-dimensional limit equations for plasticity despite these difficulties of some sort anyhow, additional mechanical assumptions on the stress distribution in the shell are usually introduced (e.g. **plane stress**, zero normal stress $S_1.e_3 = 0$ or less demanding: **zero normal tractions on the midsurface**, $S_{2,33} = 0$), moreover, the implementation of generally smooth, higher order shell elements is at variance with the lack of regularity either in (finite) plasticity or for very thin rigid shells. More problematic from a mathematical point of view, in many cases not an underlying self-consistent two-dimensional mathematical shell model is discretized in actual computations, but the shell like behaviour is enforced on the implementational element level only (this is the so called **degenerated solid approach**). There, evolution laws for plasticity are fully three-dimensionally integrated and elastic equilibrium is computed through numerical integration over the thickness. Only the two-dimensional kinematical constitutive ansatz for the total shell deformation reduces the problem. One has termed this method 2.5−dimensional. This applies to both elasticity and elasto-plasticity but, as already mentioned, the resulting problem is not consistent with any really two-dimensional definite model and a mathematical analysis for such a numerically motivated approach seems to be out of reach at present.

The Hamiltonian based, variational approach, which we will follow in disguise, has the distinctive advantage of being flexible enough to treat simultaneously finite elasticity, finite Cosserat models as well as finite elasto-plasticity in the framework of the multiplicative decomposition. This is to be contrasted with classical approaches for shells in curvilinear coordinates and indicial notation which must remain a mystery for all those not initiated.

The classical models proposed in the literature lead to effective numerical schemes only if the relative thickness h of the structure is still appreciable, i.e. classical bending terms are present and regularize the computation. However, there is an abundance of new applications where very thin structures are used, e.g. very thin metal layers on a substrate (in computer hardware, for the characteristic relative thickness $h \leq 5 \cdot 10^{-4}$). In these cases, classical bending energy, which comes with a factor of h^2 compared with the membrane energy contribution, cannot play a preponderant role for non-vanishing membrane energy. See also [BJ99] for an application to thin films. But the membrane terms e.g. in a finite, invariant Kirchhoff-Love plate or finite Reissner-Mindlin model are non-elliptic and the remaining (minimization) problem is not well-posed even if bending is included.

3.1.2 Outline and scope of this contribution

We therefore face several problems: first, there is no as yet generally accepted finite, properly invariant, elastic plate and shell model (and perhaps there cannot be); second, classical finite shell models are in general insufficient to account for very thin structures, the thin plate limit is degenerated; third, non-classical size effects, which cannot be neglected for very thin structures [CCC+03] are usually not accounted for; fourth, classical infinitesimal or finite shell models predict unrealistically high levels of smoothness, typically at least $C^{0,\alpha}(\omega)$ for the midsurface.

We propose therefore a new shell model for very thin almost rigid materials in addition to those already established which should remedy some of the aforementioned shortcomings with a view towards a subsequent

[5] If we assume that $\mu_c = 0$ in the finite three-dimensional Cosserat model, then the linearization coincides in fact with the classical infinitesimal model and the three-dimensional bulk has a unique infinitesimal response! The very possibility of $\mu_c = 0$ for a fully invariant three-dimensional finite Cosserat model has been considered impossible in the Cosserat community, since in effect, no infinitesimal, linear Cosserat bulk model would exist. While we keep completely track of $\mu_c > 0$ and $\mu_c = 0$ simultanuously, it is our belief that $\mu_c = 0$ is physically the correct choice.

stringent mathematical analysis and possible numerical implementation. We want to provide a model which is both theoretically and physically sound, such that the numerical implementation can concentrate on real convergence issues.

We view the obtained two-dimensional models as models in its own right: rather than trying to establish convergence results of the underlying three-dimensional model to its two-dimensional counterpart for vanishing thickness (which seems to be elusive given the appearing nonlinearities) we focus in a first attempt on the intrinsic mathematical problems inherent in the reduced models.

After introducing the underlying parent three-dimensional finite Cosserat model with **size effects** and **independent microrotations** and recalling the obtained existence results for this model, we proceed by considering a quadratic kinematical ansatz over the thickness where the director is automatically related to the rotations. Using generalized zero normal tractions on the transverse boundary, the two unknown leading coefficients in the quadratic ansatz can be determined in analytical form. The three-dimensional energy is then evaluated for the assumed form of plate deformation and analytically integrated over the thickness, this constitutes the energy projection. Boundary conditions are consistently reduced. The full minimization problem for the plate is gathered in section 4. The new model has six degrees of freedom (6 dof), including naturally one-drilling degree and allows for transverse shear. It is shown that the membrane part is uniformly Legendre-Hadamard elliptic at given rotations. This finishes the Cosserat modelling part.

Following, we derive a new Korn's first inequality for plates and elasto-plastic shells which is decisive for the mathematical treatment of the new models in a variational context. Depending on material constants and boundary conditions, different mathematical existence theorems are proposed. Generically, we obtain for the midsurface deformation $m \in H^{1,2}(\omega, \mathbb{R}^3)$, i.e. the midsurface must not necessarily be continuous. It is shown, that the limit of vanishing relative thickness $h \to 0$ in the new model is non-degenerate. The limits $\mu_c \to \infty$ and the zero internal length limit $L_c \to 0$, as well as the pure membrane limit $h \to 0$, $L_c \to 0$ and the pure bending for vanishing internal length, are also described. We propose as well a modification of the new plate model which ensures local invertibility of the reconstructed deformation gradient and allows for large stretch. This modification takes place on the two-dimensional level only which implies that there need not exist any underlying three-dimensional model. Nevertheless, the modified two-dimensional model is shown to be physically more plausible than the preceeding model.

For comparison, we next present a derivation of a rather classical finite, invariant Reissner-Mindlin model with one independent director and of the finite, invariant Kirchhoff-Love plate model. It is shown that both finite models exhibit a certain unphysical response. A modification of the Kirchhoff-Love model in view of expected small strain behaviour allows to establish the existence of minimizers. However, the obtained regularity is unrealistically high and the implementational cost is known to be very large thus limiting in effect the usefulness of the Kirchhoff-Love model. The pure bending problem based on either Reissner-Mindlin or Kirchhoff-Love is shown to admit minimizers and to coincide with the pure bending problem obtained from the new Cosserat model.

In the appendix we introduce the relevant notation, detail the treatment of external loads and present the observed scaling relations. Generalized convexity conditions are recalled and macroscopic shear failure for plates is defined, including a Baker-Ericksen inequality for plates.

In order to relate the new finite Cosserat plate model to more traditional approaches, we show, that a linearization of the new model basically results in the classical infinitesimal Reissner-Mindlin model (without extra size effects) and **shear correction factor** $\kappa = 1$.

3.2 The underlying finite three-dimensional Cosserat model in variational form

In [Nef03c] a finite, fully frame-invariant Cosserat model is introduced. The problem has been posed in a variational setting. The task is to find a pair $(\varphi, \overline{R}) \in \mathbb{R}^3 \times \mathrm{SO}(3,\mathbb{R})$ of deformation φ and **independent microrotation** $\overline{R}$ satisfying

$$\begin{aligned}
&\int_\Omega W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{curv}}(\mathfrak{K}) - \langle f, \varphi\rangle - \langle M, \overline{R}\rangle \,\mathrm{dV} - \int_{\Gamma_S} \langle N, \varphi\rangle \,\mathrm{dS} - \int_{\Gamma_C} \langle M_c, \overline{R}\rangle \,\mathrm{dS} \mapsto \min .\ \text{w.r.t.}\ (\varphi, \overline{R}),\\
&\quad \overline{U} = \overline{R}^T F, \quad F = \nabla\varphi, \quad \varphi_{|_\Gamma} = g_{\mathrm{d}}\\
&\quad \overline{R}_{|_\Gamma} = \begin{cases} \overline{R}_{\mathrm{d}}\,, & \text{rigid prescription}\\ \mathrm{polar}(\nabla\varphi)\,, & \text{consistent coupling} \Rightarrow S_2 := F^{-1} D_F W_{\mathrm{mp}}(\overline{U}) \in \mathrm{Sym} \text{ on } \Gamma \end{cases}\\
&W_{\mathrm{mp}}(\overline{U}) = \mu \,\|\,\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c \,\|\,\mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2
\end{aligned} \tag{3.2.1}$$

$$W_{\mathrm{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12} \left(1 + \alpha_4 L_c^q \|\mathfrak{K}\|^q\right) \left(\alpha_5 \|\operatorname{sym}\mathfrak{K}\|^2 + \alpha_6 \|\operatorname{skew}\mathfrak{K}\|^2 + \alpha_7 \operatorname{tr}[\mathfrak{K}]^2\right)^{\frac{1+p}{2}},$$
$$\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R} = \left(\overline{R}^T\nabla(\overline{R}.e_1), \overline{R}^T\nabla(\overline{R}.e_2), \overline{R}^T\nabla(\overline{R}.e_3)\right), \quad \text{third order } \textbf{curvature tensor}.$$

The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ depends on the deformation gradient $F = \nabla\varphi$ and microrotations $\overline{R}$ together with their space derivatives. In general, the **micropolar stretch tensor** $\overline{U}$ is not symmetric. Here $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}}, \overline{R}_{\mathrm{d}}$ for displacements and microrotations, respectively, are prescribed while $\Gamma_S \subset \partial\Omega$ is a part of the boundary, where traction boundary conditions N are applied with $\Gamma \cap \Gamma_S = \emptyset$. The external volume force is f and M takes on the role of external volume couples. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where external surface couples M_c are applied with $\Gamma \cap \Gamma_C = \emptyset$. The parameters μ, $\lambda > 0$ are the Lamé constants of classical elasticity, $\mu_c \geq 0$ is called the **Cosserat couple modulus** and $L_c > 0$ introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal. The internal length $L_c > 0$ is responsible for size effects in the sense that smaller samples are relatively stiffer than larger samples. If not stated otherwise, we assume that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$. **Consistent coupling** ensures that no non-classical effects are artificially introduced at the Dirichlet boundary.[6]

3.2.1 The different three-dimensional cases

We distinguish five completely different situations:

I: $\mu_{\mathbf{c}} > \mathbf{0}, \alpha_{\mathbf{4}} \geq \mathbf{0}, \mathbf{p} \geq \mathbf{1}, \mathbf{q} \geq \mathbf{0}$, unconditional elastic macro-stability, local first order Cosserat micropolar, unqualified existence, microscopic specimens, non-zero Cosserat couple modulus. Fracture excluded.

II: $\mu_{\mathbf{c}} = \mathbf{0}, \alpha_{\mathbf{4}} > \mathbf{0}, \mathbf{p} \geq \mathbf{1}, \mathbf{q} > \mathbf{1}$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Fracture excluded.

III: $\mu_{\mathbf{c}} = \infty, \alpha_{\mathbf{4}} \geq \mathbf{0}, \mathbf{p} \geq \mathbf{1}, \mathbf{q} \geq \mathbf{0}$, unconditional elastic macro-stability, the constrained gradient Cosserat micropolar problem (indeterminate couple stress model). Compatible Dirichlet boundary conditions: $\varphi_{|_\Gamma} = g_{\mathrm{d}}$, $\operatorname{polar}(\nabla\phi)_{|_\Gamma} = \operatorname{polar}(\nabla g_{\mathrm{d}})_{|_\Gamma}$.

IV: $\mu_{\mathbf{c}} = \mathbf{0}, \alpha_{\mathbf{4}} = \mathbf{0}, \mathbf{0} < \mathbf{p} \leq \mathbf{1}, \mathbf{q} = \mathbf{0}$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Since possibly $\varphi \notin W^{1,1}(\Omega, \mathbb{R}^3)$, due to lack of elastic coercivity, including **fracture** in multiaxial situations.

V: $\mu_{\mathbf{c}} = \mathbf{0}, \mathbf{L_c} = \mathbf{0}$, elastic pre-stability, finite elasticity with free rotations and microstructure. Weak solutions of a corresponding finite elasticity model are stationary points of this minimization problem. Allowing for **sharp interfaces**.

We refer to $0 < p < 1$, $q \geq 0$ as the **sub-critical case**, $p = 1$, $q \geq 0$ as the **critical case** and $p \geq 1$, $q > 1$ as the **super-critical case**. In [Nef03c] the first three cases are mathematically treated and case V is indeed shown to allow for sharp interfaces.

3.2.2 The coercive inequality in three-dimensions

The decisive analytical tool for the treatment of case II (super-critical) is the following non-trivial novel coercive inequality:

Theorem 3.2.1 (Extended 3D-Korn's first inequality)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with non vanishing 2-dimensional Lebesgue measure. Define $H_\circ^{1,2}(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$ and let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$. Moreover suppose that $\operatorname{Curl} F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\exists\, c^+ > 0\ \forall\, \phi \in H_\circ^{1,2}(\Omega, \Gamma): \quad \|\nabla\phi F_p^{-1}(x) + F_p^{-T}(x)\nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\, \|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

[6]If, instead, we assume for the stretch energy

$$W_{\mathrm{mp}}(\overline{U}) = \mu\,\|\operatorname{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\operatorname{skew}(\overline{U})\|^2 + \lambda\left(\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2\right) + \beta^+\|\operatorname{Cof}\overline{U} - 1\!\!1\|^2, \qquad (3.2.2)$$

then $W_{\mathrm{mp}}(\overline{U})$ is polyconvex w.r.t. F and local invertibility of the deformation φ can be guaranteed. However, basing the dimensional reduction on this modification, would lead to excessive formulas.

Proof. The proof has been presented in [Nef02]. Note that for $F_p = \nabla\Theta$ we would only have to deal with the classical Korn's inequality evaluated on the transformed domain $\Theta(\Omega)$. However, in general, F_p is **incompatible** giving rise to a **non-riemannian manifold** structure. Compare to [CG01] for an interpretation and the physical relevance of the quantity $\operatorname{Curl} F_p$. ■

Motivated by the investigations in [Nef02], it has been shown recently by my colleague W. Pompe [Pom03] that the extended Korn's inequality can be viewed as a special case of a general class of coercive inequalities for quadratic forms. He was able to show that indeed $F_p \in C(\overline{\Omega}, \mathrm{GL}(3,\mathbb{R}))$ is sufficient for (3.2.1) to hold without any condition on the compatibility.

However, taking the special structure of the extended Korn's inequality again into account, work in progress suggests that continuity is not really necessary: instead $F_p \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ and $\operatorname{Curl} F_p \in L^{3+\delta}(\Omega)$ should suffice, whereas $F_p \in L^\infty(\Omega, \mathrm{GL}(3,\mathbb{R}))$ alone is not sufficient, see the counterexample presented in [Pom03].

In view of the important role of the extended Korn's first inequality let us agree in saying that a bulk-material is **elastically pre-stable**, whenever

$$\begin{aligned} &\exists\, H \in \mathbb{M}^{3\times 3},\ H \neq 0: \quad D_F^2 W(x,F).(H,H) = 0 \\ &\exists\, c^+ > 0\ \exists\, G \in \mathrm{GL}^+(3,\mathbb{R})\ \forall\, H \in \mathbb{M}^{3\times 3}\ :\ D_F^2 W(x,F).(H,H) \geq c^+ \|G(x)^T H + H^T G(x)\|^2 . \end{aligned} \tag{3.2.3}$$

In this terminology, infinitesimal classical elasticity is pre-stable with $G = 1\!\!1$ due to the classical Korn's first inequality and the extended Korn's first inequality links the smoothness of G to the positive definiteness of the elastic tangent stiffness tensor.

3.2.3 Mathematical results for the three-dimensional problem

Using the extended Korn's inequality, in [Nef03c] the following has been shown:

Theorem 3.2.2 (Existence for 3D-finite elastic Cosserat model: case I.)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega,\mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S,\mathbb{R}^3)$ together with $M \in L^1(\Omega,\mathbb{M}^{3\times 3})$ and $M_c \in L^1(\Gamma_C,\mathbb{M}^{3\times 3})$. Then (3.2.1) with material constants conforming to case I admits at least one minimizing solution pair $(\varphi,\overline{R}) \in H^1(\Omega,\mathbb{R}^3) \times W^{1,1+p}(\Omega,\mathrm{SO}(3,\mathbb{R}))$.

Theorem 3.2.3 (Existence for 3D-finite elastic Cosserat model: case II.)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega,\mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\Omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $f \in L^2(\Omega,\mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S,\mathbb{R}^3)$ together with $M \in L^1(\Omega,\mathbb{M}^{3\times 3})$ and $M_c \in L^1(\Gamma_C,\mathbb{M}^{3\times 3})$. Then (3.2.1) with material constants conforming to case II admits at least one minimizing solution pair $(\varphi,\overline{R}) \in H^1(\Omega,\mathbb{R}^3) \times W^{1,1+p+q}(\Omega,\mathrm{SO}(3,\mathbb{R}))$.

3.3 Dimensional reduction of the Cosserat model

3.3.1 The three-dimensional problem on a thin domain

The basic task of any shell theory is a consistent reduction of some presumably 'exact' 3D-theory to 2D. The problem (3.2.1) will now be adapted to a shell like theory. Let us assume that we are given a three-dimensional **absolutely thin domain**

$$\Omega_h := \omega \times [-\frac{h}{2}, \frac{h}{2}], \quad \omega \subset \mathbb{R}^2 , \tag{3.3.4}$$

with **transverse boundary** $\partial\Omega_h^{\mathrm{trans}} = \omega \times \{-\frac{h}{2}, \frac{h}{2}\}$ and **lateral boundary** $\partial\Omega_h^{\mathrm{lat}} = \partial\omega \times [-\frac{h}{2}, \frac{h}{2}]$, where ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\omega$ and $h > 0$ is the thickness, and a deformation φ and **microrotation** $\overline{R}^{3d}$

$$\varphi: \quad \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 , \qquad \overline{R}^{3d} : \Omega_h \subset \mathbb{R}^3 \mapsto \mathrm{SO}(3,\mathbb{R}) , \tag{3.3.5}$$

solving the following minimization problem on Ω_h:

$$\begin{aligned}
&\int_{\Omega_h} W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{curv}}(\mathfrak{K}) - \langle f, \varphi\rangle \,\mathrm{dV} - \int_{\Omega_h^{\mathrm{trans}} \cup \{\gamma_s \times [-\frac{h}{2},\frac{h}{2}]\}} \langle N, \varphi\rangle \,\mathrm{dS} \mapsto \min . \text{ w.r.t. } (\varphi, \overline{R}),\\
&\overline{U} = \overline{R}^T F, \quad \varphi_{|_{\Gamma_0^h}} = g_{\mathrm{d}}, \quad \Gamma_0^h = \gamma_0 \times [-\frac{h}{2},\frac{h}{2}], \quad \gamma_0 \subset \partial\omega, \ \gamma_s \cap \gamma_0 = \emptyset\\
&\overline{R}_{|_{\Gamma_0^h}} = \mathrm{polar}(\nabla\varphi)\,, \quad \text{consistent coupling} \qquad (3.3.6)\\
&W_{\mathrm{mp}}(\overline{U}) = \mu\, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\, \|\mathrm{skew}(\overline{U})\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2\\
&W_{\mathrm{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12}\, (1 + \alpha_4\, L_c^q\, \|\mathfrak{K}\|^q) \left(\alpha_5\, \|\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_6\, \|\mathrm{skew}\,\mathfrak{K}\|^2 + \alpha_7\,\mathrm{tr}\left[\mathfrak{K}\right]^2\right)^{\frac{1+p}{2}}\,,\\
&\mathfrak{K} = \overline{R}^T \mathrm{D_x}\overline{R} = \left(\overline{R}^T\nabla(\overline{R}.e_1), \overline{R}^T\nabla(\overline{R}.e_2), \overline{R}^T\nabla(\overline{R}.e_3)\right), \quad \text{third order } \textbf{curvature tensor}\,.
\end{aligned}$$

We want to find a reasonable approximation $(\varphi_s, \overline{R}_s)$ of $(\varphi, \overline{R}^{3d})$ involving only two-dimensional quantities. The reduction is based on assumed kinematics and energy projection.

3.3.2 Enriched quadratic Cosserat kinematics

In the engineering shell community it is well known [Che80, Sch85, Pie85] that the ansatz over the thickness should at least be quadratic[7] in order to avoid the so called **Poisson thickness locking**[8] and to fully capture the three-dimensional kinematics without artificial modification of the material laws[9], see the detailed discussion of this point in [BR00] and compare with [BR92, BBR94, RR96, BR97, SB98].

For a Cosserat theory for small elastic strains[10] we assume therefore the **quadratic ansatz** in the thickness direction for the (reconstructed) finite deformation $\varphi_s : \mathbb{R}^3 \mapsto \mathbb{R}^3$ of the shell like structure

$$\varphi_s(x,y,z) = m(x,y) + \left(z\, \varrho_m(x,y) + \frac{z^2}{2}\varrho_b(x,y)\right) \cdot \overline{R}_{s,3}(x,y,0)\,, \qquad (3.3.7)$$

where $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ takes on the role of the deformation of the midsurfaces of the shell viewed as a parametrized surface, the (reconstructed) rotation $\overline{R}_s : \Omega \mapsto \mathrm{SO}(3,\mathbb{R})$ and with yet indeterminate functions $\varrho_m, \varrho_b : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}$ allowing for **thickness stretch** ($\varrho_m \neq 1$) and **transverse shear** ($\overline{R}_{s,3} \neq \vec{n}$).[11] The (reconstructed) rotations in the thin shell are assumed to be constant over the thickness

$$\overline{R}_s(x,y,z) = \overline{R}(x,y)\,. \qquad (3.3.8)$$

This is then a kind of plate formulation since for the moment the unstressed reference configuration ω was assumed to lie in the plane. This implies for the (reconstructed) deformation gradient of the shell (plate)

$$F_s = \nabla\varphi_s(x,y,z) = (\nabla m|\, \varrho_m\, \overline{R}_3) + z\cdot(\nabla(\varrho_m\, \overline{R}_3)|\varrho_b\, \overline{R}_3) + \frac{z^2}{2}(\nabla(\varrho_b\, \overline{R}_3)|0) = \tilde{A}_m + z\, \tilde{A}_r + \frac{z^2}{2}\tilde{B}_r\,. \qquad (3.3.9)$$

It should be noted that the augmented ansatz changes already the term which is linear in the transverse direction.[12] Invertibility of the shell deformation (as a physical requirement) entails

$$\forall\, z \in [-h/2, h/2] : \ \det[\nabla\varphi_s(x,y,z)] > 0 \Rightarrow \varrho_m(x,y) > 0\,, \qquad (3.3.10)$$

[7]This mimics the (1|1|2)-hierarchic plate models: linear in-plane displacement and quadratic transverse displacement, instead of (1|1|0)-plate models with constant transverse displacement. The dimensional reduction is achieved by energy projection on the enriched ansatz space. In this sense, we propose a $(\infty|\infty|2)$-model.

[8]Meaning that the bending stiffness of the reduced theory would tend to ∞ as the Poisson-number $\nu \to \frac{1}{2}$.

[9]Let us quote from [Sch85]: "Due to bending this change of length is generally **asymmetric** about (the midsurface) and leads to a shift of the original midsurfaces.... This asymmetry requires at least a **quadratic** representation of the (deformation in thickness direction)."

[10]Which captures already shells with large in plane rigidity and high transverse flexibility.

[11]This leads at first glance to a 8 'dof' theory: 3 components of the membrane deformation, 3 degrees of freedom for $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$, including naturally one **drilling degree** of freedom for in-plane rotations, 2 degrees of freedom over the thickness. However, the two thickness coefficients ϱ_m, ϱ_b will be eliminated, leaving us finally with a 6 'dof' model. Already in the classical elasticity context the beneficial influence of drill rotations for the numerical implementation has been investigated in the linear case in [HB89] and in the finite case in [SFH92].

[12]The corresponding stress field through the thickness $\overline{R}_s^T S_1(\nabla\varphi_s(x,y,z), \overline{R}_s).e_3$ is at least linear in the transverse variable z and not constant, as would be the case in a first order (linear) ansatz for the deformation.

and we must guarantee that $\varrho_m : \omega \mapsto \mathbb{R}^+$. The three-dimensional local part of the elastic Cosserat energy in (3.2.1) has the form

$$W(F,\overline{R}) = \frac{\mu}{4}\,\|\overline{R}^T F + F^T\overline{R} - 2 1\!\!1\|^2 + \frac{\mu_c}{4}\,\|\overline{R}^T F - F^T\overline{R}\|^2 + \frac{\lambda}{8}\,\mathrm{tr}\left[\overline{R}^T F + F^T\overline{R} - 2 1\!\!1\right]^2 . \tag{3.3.11}$$

The equilibrium equations of the three-dimensional Cosserat problem given in [Nef03c] show that on the transverse boundary (exact)

$$\begin{aligned} S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3 &= N^{\mathrm{trans}}(x,y,+h/2)\\ S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).(-e_3) &= N^{\mathrm{trans}}(x,y,-h/2)\,, \end{aligned} \tag{3.3.12}$$

where N^{trans} are the prescribed tractions N on the transverse boundary given globally in the basis (e_1,e_2,e_3). This implies (exact)

$$\begin{aligned} \overline{R}^{3d}(x,y,+h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3 &= \overline{R}^{3d}(x,y,+h/2))^T N^{\mathrm{trans}}(x,y,+h/2)\\ \overline{R}^{3d}(x,y,-h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).(-e_3) &= \overline{R}^{3d}(x,y,-h/2))^T N^{\mathrm{trans}}(x,y,-h/2)\,. \end{aligned} \tag{3.3.13}$$

Abbreviate

$$N^+ := N^{\mathrm{trans}}(x,y,+h/2), \quad N^- := N^{\mathrm{trans}}(x,y,-h/2)\,, \tag{3.3.14}$$

and define

$$N_{\mathrm{res}} := N^{\mathrm{trans}}(x,y,+h/2) + N^{\mathrm{trans}}(x,y,-h/2)\,,\ N_{\mathrm{diff}} := \frac{1}{2}\left[N^{\mathrm{trans}}(x,y,+h/2) - N^{\mathrm{trans}}(x,y,-h/2)\right]\,. \tag{3.3.15}$$

Then also (exact)

$$\begin{aligned} \langle\overline{R}^{3d}(x,y,+h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3,e_3\rangle &= \langle N^+,\overline{R}^{3d}(x,y,+h/2)).e_3\rangle\\ \langle\overline{R}^{3d}(x,y,-h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).e_3,e_3\rangle &= -\langle N^-,\overline{R}^{3d}(x,y,-h/2)).e_3\rangle\,. \end{aligned} \tag{3.3.16}$$

We determine ϱ_m, ϱ_b from the corresponding requirement in terms of the assumed kinematics $(\varphi_s,\overline{R}_s)$, yielding

$$\begin{aligned} \langle\overline{R}_s^T(x,y,\pm h/2)S_1(\nabla\varphi_s(x,y,\pm h/2),\overline{R}_s).e_3,e_3\rangle &= \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2),\overline{R}_s(x,y,\pm h/2).e_3\rangle \quad \Rightarrow\\ \langle\overline{R}^T S_1(\nabla\varphi_s(x,y,\pm h/2),\overline{R}).e_3,e_3\rangle &= \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle\,, \end{aligned} \tag{3.3.17}$$

which condition reduces to **zero normal tractions on the transverse free boundary** (in the absence of tractions N^{trans}) in the classical continuum limit of $\overline{R} = \mathrm{polar}(\nabla\varphi)$. Since

$$S_1(F,\overline{R}) = \overline{R}\left[\mu\left(F^T\overline{R} + \overline{R}^T F - 2 1\!\!1\right) + 2\mu_c\,\mathrm{skew}(\overline{R}^T F) + \frac{\lambda}{2}\,\mathrm{tr}\left[F^T\overline{R} + \overline{R}^T F - 2 1\!\!1\right] 1\!\!1\right]\,, \tag{3.3.18}$$

the requirement $\langle\overline{R}^T S_1(\nabla\varphi_s(x,y,z),\overline{R}).e_3,e_3\rangle = \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle$ turns into

$$\begin{aligned} \pm\,\langle N^{\mathrm{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle &= \mu\,(2(\varrho_m - 1) + 2z\,\varrho_b)\\ &+ \lambda\left(\langle\overline{R}^T(\nabla m|0),1\!\!1\rangle + \varrho_m + z\,\varrho_m\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle + z\,\varrho_b - 3 + \frac{z^2}{2}\,\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),1\!\!1\rangle\right)\,, \end{aligned} \tag{3.3.19}$$

independent of the Cosserat couple modulus μ_c. Let us evaluate the last equation for $z = \pm h/2$. This yields two **linear** equations in ϱ_m, ϱ_b

$$\begin{aligned} \langle N^+,\overline{R}.e_3\rangle &= \mu\,(2(\varrho_m - 1) + h\,\varrho_b)\\ &+ \lambda\left(\langle\overline{R}^T(\nabla m|0),1\!\!1\rangle + \varrho_m + h/2\varrho_m\langle\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle + h/2\,\varrho_b - 3 + \frac{h^2}{8}\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),1\!\!1\rangle\right)\\ -\langle N^-,\overline{R}.e_3\rangle &= \mu\,(2(\varrho_m - 1) - h\,\varrho_b)\\ &+ \lambda\left(\langle\overline{R}^T(\nabla m|0),1\!\!1\rangle + \varrho_m - h/2\varrho_m\langle\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle - h/2\,\varrho_b - 3 + \frac{h^2}{8}\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),1\!\!1\rangle\right)\,. \end{aligned} \tag{3.3.20}$$

The exact solution is given by

$$\begin{pmatrix} \varrho_m \\ \varrho_b \end{pmatrix} = \frac{1}{(2\mu+\lambda)^2\, h - \frac{\lambda^2 h^3}{8}\langle(\nabla\overline{R}_3|0)^T\overline{R}, 1\!\!1\rangle^2} \begin{pmatrix} (2\mu+\lambda)\, h & -\frac{\lambda h^2}{8}\langle(\nabla\overline{R}_3|0)^T\overline{R}, 1\!\!1\rangle \\ -\lambda\, h\langle(\nabla\overline{R}_3|0)^T\overline{R}, 1\!\!1\rangle & (2\mu+\lambda) \end{pmatrix}$$
$$\begin{pmatrix} \langle N_{\rm diff}, \overline{R}_3\rangle + (2\mu+\lambda) - \lambda\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right] \\ \langle N_{\rm res}, \overline{R}_3\rangle \end{pmatrix}, \tag{3.3.21}$$

which will be approximated through

$$\begin{pmatrix} \varrho_m \\ \varrho_b \end{pmatrix} \approx \frac{1}{(2\mu+\lambda)^2\, h} \begin{pmatrix} (2\mu+\lambda)\, h & -\frac{\lambda h^2}{8}\langle(\nabla\overline{R}_3|0), \overline{R}\rangle \\ -\lambda\, h\langle(\nabla\overline{R}_3|0), \overline{R}\rangle & (2\mu+\lambda) \end{pmatrix} \begin{pmatrix} \langle N_{\rm diff}, \overline{R}_3\rangle + (2\mu+\lambda) - \lambda\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right] \\ \langle N_{\rm res}, \overline{R}_3\rangle \end{pmatrix}. \tag{3.3.22}$$

Hence the leading terms[13] are:

$$\begin{aligned} \varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right] + \frac{\langle N_{\rm diff}, \overline{R}_3\rangle}{(2\mu+\lambda)} - \frac{\lambda\, h}{8(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0), \overline{R}\rangle\,\langle N_{\rm res}, \overline{R}_3\rangle \\ \varrho_b &= -\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0), \overline{R}\rangle + \frac{\langle N_{\rm res}, \overline{R}_3\rangle}{(2\mu+\lambda)\, h} - \frac{\lambda}{2(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0), \overline{R}\rangle\langle N_{\rm diff}, \overline{R}_3\rangle \\ &\quad \frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0), \overline{R}\rangle\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right]. \end{aligned} \tag{3.3.23}$$

The term $\frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0), \overline{R}\rangle\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right]$ represents a **nonlinear coupling** between midsurface in-plane strain and normal curvature, an artefact of the derivation not present in the underlying three-dimensional theory where only products of deformationgradient and rotations occur, we therefore neglect this term.[14] Moreover, for a **almost rigid** material with $\lambda \gg 1$ we have $\frac{\lambda}{(2\mu+\lambda)^2} \ll 1$, leading finally to the reduced expressions:

$$\begin{aligned} \varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right] + \frac{\langle N_{\rm diff}, \overline{R}_3\rangle}{(2\mu+\lambda)}, \\ \varrho_b &= -\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0), \overline{R}\rangle + \frac{\langle N_{\rm res}, \overline{R}_3\rangle}{(2\mu+\lambda)\, h}. \end{aligned} \tag{3.3.24}$$

The formula (3.3.24) shows the physically reasonable behaviour that to first order, **fibers will be elongated by opposite transverse tractions** and **in-plane stretch leads to thickness reduction**.

Having obtained the general form of the relevant coefficients ϱ_m, ϱ_b, it is expedient to base the expansion of the three-dimensional elastic Cosserat energy on a further simplified expression, namely

$$F_s = \nabla\varphi_s(x,y,z) \approx (\nabla m|\,\varrho_m\,\overline{R}_3) + z\cdot(\nabla\overline{R}_3|\varrho_b\,\overline{R}_3) = A_m + z\,A_r = \overline{F}_s\,, \quad A_m = \tilde{A}_m\,. \tag{3.3.25}$$

This modification has only consequences as far as the resulting bending contribution is concerned and is motivated by our

Remark 3.3.1 (Guiding principle of reduction)

(G1.) *The reduced model should at no place contain mixed products of normal curvature $\overline{R}^T(\nabla\overline{R}_3|0)$ and midsurface in-plane strain $\langle\overline{R}^T(\nabla m|0), -\rangle 2$, since such a coupling is not present in the underlying three-dimensional model.*

(G2.) *The reduced model should at no place contain space derivatives of the thickness stretch ϱ_m, since in the underlying three-dimensional Cosserat model curvature is only present through the third order curvature tensor $\mathfrak{K}$ related only to rotations $\overline{R}$.*

The use of (3.3.25) excludes (up to order h^3) exactly those terms which would violate our principle had we used (3.3.9) instead. A simple but tedious calculation reveals that

$$\begin{aligned} &\frac{\mu}{4}\|\overline{R}^T A_r + A_r^T\overline{R}\|^2 + \frac{\lambda}{8}\mathrm{tr}\left[\overline{R}^T A_r + A_r^T\overline{R}\right]^2 \\ &= \mu\|\,\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 + \frac{\langle N_{\rm res}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)\, h^2}. \end{aligned} \tag{3.3.26}$$

[13]Note that ϱ_m, ϱ_b have different units. ϱ_m is dimensionless, whereas $[\varrho_b] = \mathrm{m}^{-1}$.

[14]It would be possible to base all further considerations indeed on the exact solution of ϱ_m, ϱ_b and it seems that the resulting two-dimensional model would allow an existence proof. However, the much more involved expressions are not easily interpreted and do not reduce to the classical expressions upon linearization.

Exactly the same computations as for the bending term allows us to conclude that

$$\begin{aligned}
&\frac{\mu}{4}\|\overline{R}^T A_r + A_r^T\overline{R} - 2\mathbb{1}\|^2 + \frac{\lambda}{8}\mathrm{tr}\left[\overline{R}^T A_r + A_r^T\overline{R}\right]^2 \\
&= \mu\|\,\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\right]^2 + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)}\,.
\end{aligned} \tag{3.3.27}$$

3.3.3 Dimensionally reduced energy: energy approach

Now we perform the analytical integration over the thickness in terms of the reduced kinematics. We insert the result $\overline{F}_s$ (3.3.25) and $\overline{R}_s$ instead of F and $\overline{R}^{3d}$ into (3.3.6). Since

$$\begin{aligned}
\|\,\mathrm{sym}(\overline{R}_s^T\overline{F}_s) - \mathbb{1}\|^2 &= \frac{1}{4}\|A_m^T\overline{R} + \overline{R}^T A_m + z\,A_r^T\overline{R} + z\,\overline{R}^T A_r - 2\mathbb{1}\|^2 \\
&= \frac{1}{4}\|A_m^T\overline{R} + \overline{R}^T A_m - 2\mathbb{1}\|^2 + z\,\langle A_m^T\overline{R} + \overline{R}^T A_m - 2\mathbb{1}, A_r^T\overline{R}\rangle + \frac{z^2}{4}\|A_r^T\overline{R} + \overline{R}^T A_r\|^2\,.
\end{aligned} \tag{3.3.28}$$

and a similar expression for the trace and skew part, we obtain by explicitly integrating over the (absolutely thin plate like referential) domain $\Omega_h = \omega\times[-\frac{h}{2},\frac{h}{2}]$

$$\begin{aligned}
\int_\omega\int_{-\frac{h}{2}}^{\frac{h}{2}} W_{\mathrm{mp}}(\overline{F}_s,\overline{R}_s)\,\mathrm{dV} &= \int_\omega h\left(\frac{\mu}{4}\|A_m^T\overline{R} + \overline{R}^T A_m - 2\mathbb{1}\|^2 + \frac{\mu_c}{4}\|A_m^T\overline{R} - \overline{R}^T A_m\|^2 + \frac{\lambda}{8}\mathrm{tr}\left[A_m^T\overline{R} + \overline{R}^T A_m - 2\mathbb{1}\right]^2\right)\mathrm{d}\omega \\
&\quad + 0 + \int_\omega \frac{h^3}{12}\left(\frac{\mu}{4}\|A_r^T\overline{R} + \overline{R}^T A_r\|^2 + \frac{\mu_c}{4}\|A_r^T\overline{R} - \overline{R}^T A_r\|^2 + \frac{\lambda}{8}\mathrm{tr}\left[A_r^T\overline{R} + \overline{R}^T A_r\right]^2\right)\mathrm{d}\omega \\
&= \int_\omega h\left(\mu\|\,\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{R}^T(\nabla m|\overline{R}_3))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\right]^2\right. \\
&\quad \left. + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)}\right)\mathrm{d}\omega + \qquad (3.3.29) \\
&\quad \int_\omega \frac{h^3}{12}\left(\mu\|\,\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 + \frac{\langle N_{\mathrm{res}}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)\,h^2}\right)\mathrm{d}\omega \\
&= \int_\omega h\left(\mu\|\,\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{R}^T(\nabla m|\overline{R}_3))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - \mathbb{1}\right]^2\right. \\
&\quad \left. + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)} + \frac{\langle N_{\mathrm{res}}, \overline{R}_3\rangle^2}{24\,(2\mu+\lambda)}\right)\mathrm{d}\omega \\
&\quad + \int_\omega \frac{h^3}{12}\left(\mu\|\,\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\right)\mathrm{d}\omega\,,
\end{aligned}$$

and we may call $(A_m^T\overline{R} + \overline{R}^T A_m - 2\mathbb{1})$ the **membrane part** and $(A_r^T\overline{R} + \overline{R}^T A_r)$ the **bending part**. The influence of

$$h\left(\frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle^2}{2(2\mu+\lambda)} + \frac{\langle N_{\mathrm{res}}, \overline{R}_3\rangle^2}{24\,(2\mu+\lambda)}\right)\,, \tag{3.3.30}$$

in the reduced energy is of higher order than the comparable influence of the assumed resultant loading, cf. (3.9.102). Moreover, for large tractions, the influence of this term in the energy would coerce the component $\overline{R}_3$ to adjust orthogonal to tractions N instead of presumably parallel. Since $2\mu+\lambda \gg 1$ for a rigid material it is therefore suggested to neglect this contribution as well. This is all the more necessary, since (3.3.30) would be a **non-frame-indifferent** contribution to the plate elastic energy.

3.3.4 Reduction of the curvature

Similarly the Cosserat curvature term is integrated over the thickness. Consider

$$\mathfrak{K}_s = \overline{R}_s^T\mathrm{D_x}\overline{R}_s = \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\,, \tag{3.3.31}$$

the **reduced** third order **curvature tensor**. Integration over the domain $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$ yields

$$\int_\omega \int_{-\frac{h}{2}}^{\frac{h}{2}} W_{\mathrm{curv}}(\mathfrak{K}_s)\,\mathrm{dV} = \int_\omega \mu \frac{h\, L_c^{1+p}}{12} \,(1 + \alpha_4\, L_c^q\, \|\mathfrak{K}_s\|^q)\, \left(\alpha_5\, \|\operatorname{sym}\mathfrak{K}_s\|^2 + \alpha_6\, \|\operatorname{skew}\mathfrak{K}_s\|^2 + \alpha_7 \operatorname{tr}[\mathfrak{K}_s]^2\right)^{\frac{1+p}{2}} \mathrm{d}\omega\,. \tag{3.3.32}$$

3.3.5 Reduction/deduction of the boundary conditions

Taking the Dirichlet boundary conditions for φ into account and the kinematical ansatz, we have

$$\varphi_s(x,y,z) = m(x,y) + \left(z\,\varrho_m(x,y) + \frac{z^2}{2}\varrho_b(x,y)\right) \cdot \overline{R}_{s,3}(x,y,0)\,, \qquad \varphi_s(x,y,z)_{|_{\Gamma_0}} = g_{\mathrm{d}}(x,y,z)\,. \tag{3.3.33}$$

Evaluating for $\pm h/2$ yields two vector equations:

$$\begin{aligned} g_{\mathrm{d}}(x,y,+h/2) &= m(x,y) + \left(h/2\,\varrho_m(x,y) + \frac{h^2}{8}\varrho_b(x,y)\right) \cdot \overline{R}_{s,3}(x,y,0) \\ g_{\mathrm{d}}(x,y,-h/2) &= m(x,y) + \left(-h/2\,\varrho_m(x,y) + \frac{h^2}{8}\varrho_b(x,y)\right) \cdot \overline{R}_{s,3}(x,y,0)\,. \end{aligned} \tag{3.3.34}$$

Adding and subtracting shows

$$\begin{aligned} g_{\mathrm{d}}(x,y,+h/2) + g_{\mathrm{d}}(x,y,-h/2) &= 2m(x,y) + \frac{h^2}{4}\varrho_b(x,y) \cdot \overline{R}_{s,3}(x,y,0) \\ g_{\mathrm{d}}(x,y,+h/2) - g_{\mathrm{d}}(x,y,-h/2) &= h\,\varrho_m(x,y)\,\overline{R}_{s,3}(x,y,0) \Rightarrow \nabla g_{\mathrm{d}}(x,y,0).e_3 = \varrho_m(x,y)\,\overline{R}_{s,3}(x,y,0) + o(h)\,. \end{aligned} \tag{3.3.35}$$

This implies

$$m(x,y) = \frac{1}{2}\left(g_{\mathrm{d}}(x,y,+h/2) + g_{\mathrm{d}}(x,y,-h/2)\right) \approx g_{\mathrm{d}}(x,y,0)\,. \tag{3.3.36}$$

In order get a boundary condition for the rotation we use the best available information of the three-dimensional theory: consider the three-dimensional **consistent coupling** boundary condition $\Gamma_0^h \subset \partial\Omega$:

$$\overline{R}^{3d}(x,y,z) = \operatorname{polar}(\nabla\varphi(x,y,z)) = \operatorname{polar}((\partial_x\varphi(x,y,z)|\partial_y\varphi(x,y,z)|\partial_z\varphi(x,y,z)))\,. \tag{3.3.37}$$

Since g_{d} is given on Γ_0^h, it holds that

$$\begin{aligned} &g_{\mathrm{d}}(x,y,+h/2) = \varphi(x,y,+h/2) \\ &g_{\mathrm{d}}(x,y,-h/2) = \varphi(x,y,-h/2) \Rightarrow \partial_z\varphi(x,y,0) = \nabla g_{\mathrm{d}}(x,y,0).e_3 + o(h)\,. \end{aligned} \tag{3.3.38}$$

Hence

$$\begin{aligned} \overline{R}^{3d}(x,y,0) &= \operatorname{polar}(\nabla\varphi(x,y,0)) = \operatorname{polar}((\partial_x\varphi(x,y,0)|\partial_y\varphi(x,y,0)|\partial_z\varphi(x,y,0))) \\ &= \operatorname{polar}((\partial_x\varphi(x,y,0)|\partial_y\varphi(x,y,0)|\nabla g_{\mathrm{d}}(x,y,0).e_3))\,, \end{aligned} \tag{3.3.39}$$

which, in view of the assumed kinematics necessitates the **consistent coupling for plates**

$$\overline{R}_{|_{\gamma_0}}(x,y) = \operatorname{polar}((\nabla m(x,y)|\nabla g_{\mathrm{d}}(x,y,0).e_3))\,. \tag{3.3.40}$$

This condition disposes us from the need to motivate rather artificially any boundary conditions for the rotations. Observe that this last boundary condition **does not imply** that the **rigid plate prescription**

$$\overline{R}_{3|_{\gamma_0}} = \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \tag{3.3.41}$$

holds, which would correspond to a form of clamping[15] and which can be seen as a consequence of (3.3.35). Note, however, that (3.3.40) implies (3.3.41) in the limit of small-strain: i.e. if $(\nabla m|\nabla g_{\mathrm{d}}.e_3)_{|_{\gamma_0}} \in \mathrm{SO}(3,\mathbb{R})$. In this sense, (3.3.41) is a small strain approximation of (3.3.40).

[15] We reserve the notion **clamped**, meaning that $\vec{n}_m = \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\nabla g_{\mathrm{d}}(x,y,0).e_3}$ on γ_0 to traditional fourth order Kirchhoff-Love models (3.7.83).

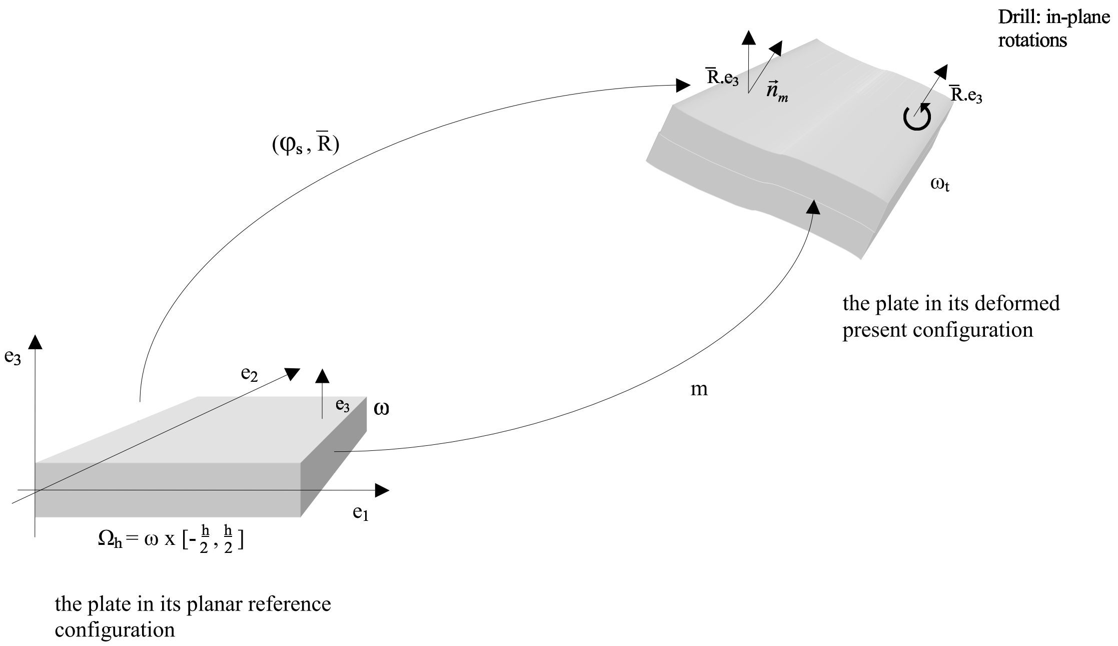

Figure 3.1: The assumed Cosserat plate kinematics incorporating transverse shear ($\overline{R}_3 \neq \vec{n}$), thickness stretch ($\varrho_m \neq 1$) and drill-rotations. Reconstructed three-dimensional deformation φ_s, midsurface deformation m, microrotation $\overline{R}$.

3.4 The new finite Cosserat thin plate model with size effects

Gathering our results we have obtained the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ solving on ω:

$$
\begin{aligned}
I &= \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12}\, W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min .\ \text{w.r.t. } (m, \overline{R}), \\
\overline{U} &= \overline{R}^T \hat{F}, \quad \hat{F} = (\nabla m|\overline{R}_3), \quad F_s = (\nabla m|\varrho_m \overline{R}_3) \quad \text{reconstructed deformation gradient} \\
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right] + \overbrace{\frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle}{(2\mu+\lambda)}}^{\text{non-invariant}} = \underbrace{1 - \frac{\lambda}{2\mu+\lambda}\mathrm{tr}\left[\overline{U} - 1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle}{(2\mu+\lambda)}}_{\text{first order thickness stretch}} \\
m_{|\gamma_0} &= g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported (fixed)} \qquad (3.4.42) \\
\overline{R}_{|\gamma_0} &= \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3))_{|\gamma_0}\,, \quad \text{reduced consistent coupling} \\
\overline{R}_{3_{|\gamma_0}} &= \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \qquad \text{alternatively: rigid prescription} \\
W_{\mathrm{mp}}(\overline{U}) &= \mu\, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\, \|\mathrm{skew}(\overline{U})\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\, \mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2 \\
&= \mu\, \|\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\|^2 + \mu_c\, \|\mathrm{skew}((\overline{R}_1|\overline{R}_2)^T\nabla m)\|^2 \\
&\quad + \underbrace{\frac{\kappa(\mu+\mu_c)}{2}\left(\langle\overline{R}_3, m_x\rangle^2 + \langle\overline{R}_3, m_y\rangle^2\right)}_{\text{transverse shear energy}} + \frac{\mu\lambda}{2\mu+\lambda}\, \mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\right]^2 \\
W_{\mathrm{curv}}(\mathfrak{K}_s) &= \mu\frac{L_c^{1+p}}{12}\, (1 + \alpha_4\, L_c^q\, \|\mathfrak{K}_s\|^q) \left(\alpha_5\, \|\mathrm{sym}\, \mathfrak{K}_s\|^2 + \alpha_6\, \|\mathrm{skew}\, \mathfrak{K}_s\|^2 + \alpha_7\, \mathrm{tr}\left[\mathfrak{K}_s\right]^2\right)^{\frac{1+p}{2}}\,, \\
\mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\,, \quad \text{reduced third order } \textbf{curvature tensor} \\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\, \|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \mu_c\, \|\mathrm{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\, \mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2 \\
\mathfrak{K}_b &= \overline{R}^T(\nabla\overline{R}_3|0) = \mathfrak{K}_s^3\,, \quad \text{second order, } \textbf{non-symmetric bending tensor}\,.
\end{aligned}
$$

The (relative) thickness of the plate (shell) is $h > 0$. The total elastically stored energy due to **membrane**, **curvature** and **bending**

$$
W = h\, W_{\mathrm{mp}} + h\, W_{\mathrm{curv}} + \frac{h^3}{12} W_{\mathrm{bend}}\,, \tag{3.4.43}
$$

depends on the midsurface deformation gradient ∇m and microrotations $\overline{R}$ together with their space derivatives only through $\overline{U}$ and $\mathfrak{K}_s$. The **micropolar stretch tensor** $\overline{U}$ of the plate is in general non-symmetric. Here $\omega \subset \mathbb{R}^2$ is a domain with boundary $\partial\omega$ and $\gamma_0 \subset \partial\omega$ is that part of the boundary, where Dirichlet conditions $g_{\mathrm{d}}, \overline{R}_{3,\mathrm{d}}$ for displacements and microrotations, respectively, are prescribed. The reduced external loading functional $\Pi(m, \overline{R}_3)$ is a linear form in $(m, \overline{R}_3)$ defined in (3.9.102) in terms of the underlying three-dimensional loads. The parameters μ, $\lambda > 0$ are the Lamé constants of classical elasticity, $\mu_c \geq 0$ is called the **Cosserat couple modulus** and $L_c > 0$ introduces an **internal length** which is **characteristic** for the material, e.g. related to the grain size in a polycrystal and which is responsible for the size effects. If not stated otherwise, we assume that $\alpha_5 > 0, \alpha_6 > 0, \alpha_7 \geq 0$. We have included the so called **shear correction factor** κ $(0 < \kappa \leq 1)$ to keep in line with infinitesimal models, in our derivation however, we obtain $\kappa = 1$. The model is fully **frame-indifferent**, meaning that

$$
\forall\, Q \in \mathrm{SO}(3,\mathbb{R}): \quad W(Q\hat{F}, Q\overline{R}) = W(\hat{F}, \overline{R})\,. \tag{3.4.44}
$$

The non-invariant term ϱ_m is only needed to reconstruct the 3D-deformation, which of course depends on the non-invariant loading.[16] Strain and curvature parts are additively decoupled, as in the underlying parent model (3.3.6).

[16] Of course, if the external tractions are rotated as well, we obtain invariance: $\langle Q.N_{\mathrm{diff}}, Q.\overline{R}_3\rangle = \langle N_{\mathrm{diff}}, \overline{R}_3\rangle$.

3.4.1 The different cases for the Cosserat plate

As in the three-dimensional case, we may distinguish five different situations: (different values of p, q compared with the three-dimensional case)

I: $\mu_c > 0$, $\alpha_4 \geq 0$, $p \geq 1$, $q \geq 0$, unconditional elastic macro-stability, local first order Cosserat micropolar, unqualified existence, microscopic specimens, non-zero Cosserat couple modulus. Fracture excluded.

II: $\mu_c = 0$, $\alpha_4 = 0$, $p > 1$, $q = 0$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Fracture excluded.

III: $\mu_c = \infty$, $\alpha_4 \geq 0$, $p \geq 1$, $q \geq 0$, unconditional elastic macro-stability, the constrained gradient Cosserat micropolar plate problem (indeterminate couple stress plate model (3.4.60)). Compatible Dirichlet boundary conditions: $m_{|\gamma_0} = g_{\mathrm{d}}$, $\mathrm{polar}((\nabla m|\varrho_m \vec{n}_m))_{|\gamma_0} = \mathrm{polar}(\nabla g_{\mathrm{d}})_{|\gamma_0}$.

IV: $\mu_c = 0$, $\alpha_4 = 0$, $0 < p \leq 1$, $q = 0$, elastic pre-stability, nonlocal second order Cosserat micropolar, macroscopic specimens, in a sense close to classical elasticity, zero Cosserat couple modulus. Since possibly $m \notin W^{1,1}(\omega, \mathbb{R}^3)$, due to lack of elastic coercivity, including **fracture** in multiaxial situations.

V: $\mu_c = 0, L_c = 0$, elastic pre-stability, finite elasticity with free rotations and microstructure. Weak solutions of corresponding finite elasticity are stationary points of this minimization problem. Allowing for **sharp interfaces**.

We refer to $0 < p < 1$, $q \geq 0$ as the **sub-critical case**, $p = 1$, $q \geq 0$ as the **critical case** and $p \geq 1$, $q > 1$ as the **super-critical case**. We will mathematically treat the first three cases.

3.4.2 Constitutive consequences of the value for the Cosserat couple modulus

Looking at the membrane energy W_{mp} with $\mu_c > 0$ we see that the implication of this choice at a first glance is an inoccuous rise in the macroscopic elastic membrane strain energy $W_{\mathrm{mp}}(\overline{U})$ of the plate if $\overline{R} \neq \mathrm{polar}(\nabla m|\overline{R}_3))$. The choice $\mu_c > 0$ acts like a local 'elastic spring' between both continuum rotations and microrotations.

Let us consider the mathematical implications of $\mu_c = 0$ and $0 < \mu_c \leq \mu$, respectively, for the membrane, in more detail. We compute the second derivative of the membrane strain energy $W_{\mathrm{mp}}(\overline{R}^T \hat{F})$ at fixed $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$ w.r.t. $\nabla m \in \mathbb{M}^{2\times 3}$. For $H \in \mathbb{M}^{2\times 3}$ we have

$$D^2_{\nabla m} W_{\mathrm{mp}}(\overline{R}^T \hat{F}).(H, H) \geq D^2_{\nabla m} \left(\mu \|\,\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\|^2 + \mu_c \|\,\mathrm{skew}(\overline{R}^T(\nabla m|\overline{R}))\|^2\right).(H, H) \quad (3.4.45)$$

$$= 2\mu \|\,\mathrm{sym}(\overline{R}^T(H|0))\|^2 + 2\mu_c \|\,\mathrm{skew}(\overline{R}^T(H|0))\|^2 = \begin{cases} \geq 2\mu_c\, \|\overline{R}^T(H|0))\|^2 = 2\mu_c\, \|(H|0))\|^2 & \text{if } \mu_c > 0 \\ = 2\mu\, \|\,\mathrm{sym}(\overline{R}^T(H|0))\|^2 & \text{if } \mu_c = 0 \end{cases}.$$

Hence the choice $\mu_c > 0$ leads to **uniform convexity** of $W_{\mathrm{mp}}(\overline{R}^T \hat{F})$ w.r.t. ∇m and **unconditional elastic stability** on the macroscopic level: regardless of what distribution of microrotations $\overline{R}(x)$ is given, the macroscopic equation of balance of linear momentum is uniquely solvable and this equation is insensible to any deterioration of the spatial features of the microstructure. Uniform convexity is difficult to accept from a constitutive point of view since it is impossible for a geometrically exact description in the framework of a classical macroscopic continuum but clear from the above discussion: the additional elastic spring between micro- and continuum rotation extremely rigidifies the material and completely changes the type of the mathematical boundary value problem compared with the classical finite theory.[17]

Fortunately, such a far reaching unsatisfactory conclusion does not hold for $\mu_c = 0$. Choose $\xi \in \mathbb{R}^3$ and $\eta = (\eta_1, \eta_2, 0)^T$. Then consider $(H|0) = \xi \otimes \eta \in \mathbb{M}^{3\times 3}$ and

$$D^2_{\nabla m} W_{\mathrm{mp}}(\overline{R}^T \hat{F}).(\xi \otimes \eta, \xi \otimes \eta) = \mu\Big(\|\overline{R}^T \xi \otimes \eta\|^2 + \langle \overline{R}^T \xi \otimes \eta, \eta \otimes \overline{R}^T \xi\rangle\Big) = \mu\left(\|\overline{R}^T \xi \otimes \eta\|^2 + \langle \overline{R}^T \xi, \eta\rangle^2\right),$$

which shows the physically much more appealing inequality

$$D^2_{\nabla m} W_{\mathrm{mp}}(\overline{R}^T \hat{F}).(\xi \otimes \eta, \xi \otimes \eta) \geq \mu\, \|\xi\|^2_{\mathbb{R}^3} \cdot \|\eta\|^2_{\mathbb{R}^2}, \quad (3.4.46)$$

expressing nothing but uniform **Legendre-Hadamard ellipticity** of the membrane acoustic-tensor with ellipticity constant μ **independent** of $\overline{R}$. The Legendre-Hadamard condition has the most convincing physical basis [Ant95, p.461] in that it implies the reality of wave speeds and the Baker-Ericksen inequalities (stress increases with strain, [MH83, p.19]). The choice $\mu_c = 0$ is consistent with the three-dimensional strain energy density proposed in [Nef03a, (P3)] and [NW03, M1] if the appearing independent viscoelastic rotations there are identified with the independent elastic Cosserat microrotations here.

[17] In the analytical section we will see that $\mu_c > 0$ implies that $m \in W^{1,1}(\omega, \mathbb{R}^3)$ irrespective of $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$, thus **excluding fracture**.

3.4.3 The coercitive inequality in two-dimensions

In this section we show how to use the three-dimensional extended Korn's first inequality Theorem 3.2.1 in our reduced two-dimensional context of plates and shells in order to improve Legendre-Hadamard ellipticity to uniform positivity. In order to show that the elastic membrane energy is uniformly convex we look at the second differential of $W_{\mathrm{mp}}(\overline{R}^T\hat{F})$ with respect to m

$$D^2_{\nabla m}W_{\mathrm{mp}}(\overline{R}^T\hat{F}).(\nabla\phi,\nabla\phi) \geq \frac{\mu}{2}\|(\nabla\phi|0)^T\overline{R}+\overline{R}^T(\nabla\phi|0)\|^2 \,. \tag{3.4.47}$$

Set for simplicity $\mu = 2$ and consider the slightly more general quadratic form (appropriate for elastic shells and elasto-plastic shells)

$$\begin{aligned}\|F_p^{-T}(\nabla\phi|0)^T\overline{R}_e+\overline{R}_e^T(\nabla m|0)F_p^{-1}\|^2 &= \|\overline{R}_e\left(F_p^{-T}(\nabla\phi|0)^T\overline{R}_e+\overline{R}_e^T(\nabla\phi|0)F_p^{-1}\right)\overline{R}_e^T\|^2\\ &= \|(\overline{R}_eF_p)^{-T}(\nabla\phi|0)^T+(\nabla\phi|0)(\overline{R}_eF_p)^{-1}\|^2\,,\end{aligned} \tag{3.4.48}$$

where $\phi:\omega\subset\mathbb{R}^2\mapsto\mathbb{R}^3$ and $\phi_{|_{\gamma_0}}=0$ for $\gamma_0\subset\partial\omega$. Extend now ϕ by $\bar{\phi}:\mathbb{R}^3\mapsto\mathbb{R}^3$ through

$$\bar{\phi}(x,y,z):=\phi(x,y)\Rightarrow\bar{\phi}(x,y,z)_{\gamma_0\times[-\frac{h}{2},\frac{h}{2}]}=0. \tag{3.4.49}$$

This extension implies

$$\nabla_{(x,y,z)}\bar{\phi}(x,y,z)=(\nabla_{(x,y)}\phi|0). \tag{3.4.50}$$

For $\bar{\phi}$ it is possible to use the 3D-extended Korn's first inequality Theorem 3.2.1. To this end consider $\Omega_h = \omega\times[-\frac{h}{2},\frac{h}{2}]$ and the lateral Dirichlet boundary $\Gamma_0^h=\gamma_0\times[-\frac{h}{2},\frac{h}{2}]\subset\partial\Omega_h$. Then Γ_0^h has non-vanishing 2-dimensional Lebesgue measure. Set by abuse of notation $F_p=(\overline{R}_eF_p)$ for the moment. With smooth enough, invertible F_p it holds on applying Theorem 3.2.1 that

$$\begin{aligned}&\int_{\omega\times[-\frac{h}{2},\frac{h}{2}]}\|\nabla\bar{\phi}^TF_p^{-1}+F_p^{-T}\nabla\bar{\phi}\|^2\,\mathrm{dV}\geq c^+_{3D}\cdot\int_{\omega\times[-\frac{h}{2},\frac{h}{2}]}\|\bar{\phi}\|^2+\|\nabla\bar{\phi}\|^2\,\mathrm{dV}\Rightarrow\\ &\int_\omega\int_{-\frac{h}{2}}^{\frac{h}{2}}\|\nabla\bar{\phi}^TF_p^{-1}+F_p^{-T}\nabla\bar{\phi}\|^2\,\mathrm{d}\omega\,\mathrm{dz}\geq c^+_{3D}\cdot\int_\omega\int_{-\frac{h}{2}}^{\frac{h}{2}}\|\bar{\phi}\|^2+\|\nabla\bar{\phi}\|^2\,\mathrm{d}\omega\,\mathrm{dz}\,.\end{aligned} \tag{3.4.51}$$

Since $\bar{\phi}$ is independent of z we get, however,

$$\int_\omega\|\nabla\bar{\phi}^TF_p^{-1}+F_p^{-T}\nabla\bar{\phi}\|^2\,\mathrm{d}\omega\geq c^+_{3D}\cdot\int_\omega\|\bar{\phi}\|^2+\|\nabla\bar{\phi}\|^2\,\mathrm{d}\omega\,, \tag{3.4.52}$$

or back in terms of ϕ

$$\int_\omega\|(\nabla\phi|0)^TF_p^{-1}+F_p^{-T}(\nabla\phi|0)\|^2\,\mathrm{d}\omega\geq c^+_{3D}\cdot\int_\omega\|\phi\|^2+\|(\nabla\phi|0)\|^2\;\mathrm{d}\omega. \tag{3.4.53}$$

Observe that the constant c^+_{3D} is **independent of the thickness** h which might be surprising at first glance. This observation allows one to bound $m\in H^{1,2}_\circ(\omega,\mathbb{R}^3;\gamma_0)$ independent of the relative thickness h only in terms of the membrane energy $\int_\omega W(\nabla m,\overline{R})\;\mathrm{d}\omega$ if $\overline{R}\in\mathrm{SO}(3,\mathbb{R})$ is smooth enough. Thus we have finally proved

Theorem 3.4.1 (Extended Korn´s first inequality for rigid shells)
Let $\omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain and let $\gamma_0\subset\partial\omega$ be a smooth part of the boundary with non vanishing 1-dimensional Lebesgue measure. Define $H^{1,2}_\circ(\omega,\mathbb{R}^3;\gamma_0):=\{\phi\in H^{1,2}(\omega),\ \phi:\omega\mapsto\mathbb{R}^3\mid\phi_{|_{\gamma_0}}=0\}$ and let $F_p,F_p^{-1}\in C^1(\overline{\omega},\mathrm{GL}(3,\mathbb{R}))$. Moreover suppose that $\mathrm{Curl}\,F_p\in C^1(\overline{\omega},\mathbb{M}^{3\times3})$. Then

$$\exists\,c^+>0\quad\forall\,\phi\in H^{1,2}_\circ(\omega,\mathbb{R}^3;\gamma_0):\ \|(\nabla\phi|0)F_p^{-1}(x)+F_p^{-T}(x)(\nabla\phi|0)^T\|^2_{L^2(\omega)}\geq c^+\ \|\phi\|^2_{H^{1,2}(\omega)}\,. \qquad\blacksquare$$

Based on the strengthening proposed in [Pom03] we get immediately the following

Corollary 3.4.2 (Improved Korn's inequality for rigid shells)
Let $\omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let $\gamma_0 \subset \partial\omega$ be a part of the boundary with non vanishing 1-dimensional Lebesgue measure. Define $H_\circ^{1,2}(\omega, \mathbb{R}^3; \gamma_0) := \{\phi \in H^{1,2}(\omega),\ \phi : \omega \mapsto \mathbb{R}^3 \mid \phi_{|_{\gamma_0}} = 0\}$ and let $F_p \in W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$. Then

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H_\circ^{1,2}(\omega, \mathbb{R}^3; \gamma_0) :\ \|(\nabla\phi|0)F_p^{-1}(x) + F_p^{-T}(x)(\nabla\phi|0)^T\|^2_{L^2(\omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\omega)}\,, \tag{3.4.54}$$

and the constant is bounded away from zero for F_p bounded in $W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$.

Proof. The Sobolev embedding shows that $F_p \in W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$ may be identified with a continuous function. A contradiction argument as in [Nef04] shows that the constant is bounded away from zero since $W^{1,2+\delta}(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$ is compactly embedded in $C(\overline{\omega}, \mathrm{GL}(3,\mathbb{R}))$. ■

However, taking the special structure of the extended Korn's inequality into account, work in progress suggests that even continuity is not really necessary: instead $F_p \in L^\infty(\omega, \mathrm{GL}(3,\mathbb{R}))$ and $\operatorname{Curl} F_p \in L^{N+\delta}(\omega)$ with $N = \dim(\omega)$ should suffice, whereas $F_p \in L^\infty(\omega, \mathrm{GL}(3,\mathbb{R}))$ alone is not sufficient, see the counterexample presented in [Pom03].

3.4.4 Mathematical analysis of the two-dimensional problem

The following results are the first existence theorems for geometrically exact[18] derived elastic Cosserat plate models known to the author:[19]

Theorem 3.4.3 (Existence for 2D-finite elastic Cosserat model: case I.)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_\mathrm{d} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_\mathrm{d} \in W^{1,1+p}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $\overline{f} \in L^2(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (3.9.102). Then (3.4.42) with material constants conforming to case I admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p}(\omega, \mathrm{SO}(3,\mathbb{R}))$.

Proof. We apply the direct methods of variations. First, the requirement on the data shows that

$$\Pi(m, \overline{R}_3) \leq C \cdot \left(\|m\|_{L^2(\omega)} + \|\overline{R}_3\|_\infty\right) = C\ \left(\|m\|_{L^2(\omega)} + 1\right)\,. \tag{3.4.55}$$

With the prescription of $(g_\mathrm{d}, \overline{R}_\mathrm{d})$ it is clear that $I < \infty$ for some pair $(m, \overline{R})$. Observe first that the micropolar curvature term $\mathfrak{K}_s$ controls $\overline{R} \in W^{1,1+p}(\omega, \mathrm{SO}(3,\mathbb{R}))$, since $\|\mathfrak{K}_s\| = \|\overline{R}^T \mathrm{D_x}\overline{R}\| = \|\mathrm{D_x}\overline{R}\|$, pointwise and $\alpha_5, \alpha_6 > 0$. Moreover, $\mathrm{SO}(3,\mathbb{R})$ is weakly closed in the topology of $W^{1,1+p}(\omega)$. We omit to show that I is bounded below: this will turn out not to be necessary. We may choose decreasing (infimizing) sequences of pairs $(m^k, \overline{R}^k)$. The curvature contribution together with the appropriate boundary conditions and Poincarés inequality yields boundedness of $\overline{R}^k \subset W^{1,1+p}(\omega, \mathrm{SO}(3,\mathbb{R}))$. We may extract a subsequence again denoted by $\overline{R}^k$ converging strongly in $L^{1+p}(\omega)$ to an element $\tilde{\overline{R}} \in W^{1,1+p}(\omega, \mathrm{SO}(3,\mathbb{R}))$ since $p > 0$ by assumption. Because $\mu_c > 0$, it is immediate that $(\nabla m_k|\overline{R}^k) = \hat{F}^k$ is **bounded** in $L^2(\omega, \mathbb{M}^{3\times3})$, **independent** of $\overline{R}^k$ on account of

$$W_\mathrm{mp}(\overline{R}^{k,T}\hat{F}^k) \geq \mu_c\,\|\overline{R}^{k,T}\hat{F}^k - 1\!\!1\|^2 = \mu_c\,\left(\|\hat{F}^k\|^2 - 2\langle\hat{F}^k, \overline{R}^k\rangle + 3\right) \geq \mu_c\,\left(\|\hat{F}^k\|^2 - 2\sqrt{3}\|\hat{F}^k\| + 3\right)\,, \tag{3.4.56}$$

and

$$\begin{aligned}
\infty > \int_\omega h\,W_\mathrm{mp}(\overline{U}_k) + h\,W_\mathrm{curv}(\mathfrak{K}_{s,k}) + \frac{h^3}{12}\,W_\mathrm{bend}(\mathfrak{K}_b)\,\mathrm{d}\omega - \Pi(m_k, \overline{R}_3^k) &\geq \int_\omega h\,W_\mathrm{mp}(\overline{U}_k) - \Pi(m_k, \overline{R}_3^k)\,\mathrm{d}\omega \\
\geq \int_\omega h\,W_\mathrm{mp}(\overline{U}_k)\,\mathrm{d}\omega - C\left(\|m_k\|_{L^2(\omega)} + 1\right) \qquad\qquad (3.4.57)\\
\geq \mu_c\,h\,\|\hat{F}^k\|^2_{L^2(\omega)} - 2\sqrt{3}\,\mu_c\,h\,\|\hat{F}^k\|_{L^2(\omega)} - C\|m_k\|_{H^{1,2}(\omega)} + 3\mu_c\,h - C \\
\geq \mu_c\,h\,\|\nabla m_k\|^2_{L^2(\omega)} - 2\sqrt{3}\,\mu_c\,h\,\|\nabla m_k\|_{L^2(\omega)} - C\,\|m_k\|_{H^{1,2}(\omega)} + 3\mu_c\,h - C \\
\geq \mu_c\,h\,\|\nabla m_k\|^2_{L^2(\omega)} - C\,\|m_k\|_{H^{1,2}(\omega)} - C \geq \mu_c\,c_\mathrm{P}^+\,h\,\|v_k\|^2_{H^{1,2}(\omega)} - C_1\,\|v_k\|_{H^{1,2}(\omega)} + C_2\,,
\end{aligned}$$

[18] same as frame-indifferent

[19] The proposed finite results determine the macroscopic midsurface plate deformation $m \in H^1(\omega, \mathbb{R}^3)$ and not more. This means that discontinuous macroscopic deformations by cavities or the formation of holes are not excluded (possible mode I failure). If $\mu_c > 0$ fracture is effectively ruled out, which is unrealistic. All results remain true for arbitrary shear correction factor $\kappa > 0$.

where we made use of the appropriate boundary conditions for $m_k = x + v_k(x)$, and applied Poincarés inequality to u_k since it has zero boundary values on γ_0. This yields the boundedness of v_k, thus m_k is bounded in $H^1(\omega, \mathbb{R}^3)$. Hence we may extract a subsequence, not relabelled, such that $m_k \rightharpoonup \tilde{m} \in H^1(\omega, \mathbb{R}^3)$. Furthermore, we may always obtain a subsequence of $(m_k, \overline{R}^k)$ such that $\overline{U}_k = \overline{R}^{k,T} \hat{F}^k$ converges weakly in $L^2(\omega)$ to an element $\tilde{\overline{U}}$ on account of the boundedness of the stretch energy and $\mu_c > 0$.

For $p \geq 1$ we have as well that $\overline{R}^k$ converges indeed strongly in $L^2(\omega)$ to an element $\tilde{\overline{R}} \in H^{1,2}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Thus $\overline{R}^{k,T} \hat{F}^k$ converges weakly to $\tilde{\overline{R}}^T \hat{F}$ in $L^1(\omega)$. The weak limit in $L^1(\omega)$ must coincide with the weak limit of $\overline{U}_k$ in $L^2(\omega)$. Hence, $\tilde{\overline{U}} = \tilde{\overline{R}}^T (\nabla \tilde{m} | \tilde{\overline{R}}_3)$.

Since the total energy is convex in $(\overline{U}, \mathfrak{K}_s, \mathfrak{K}_b)$ and $(\hat{F}, D\overline{R})$, we get

$$\begin{aligned}
I(\tilde{m}, \tilde{\overline{R}}) &= \int_\omega h\, W_{\mathrm{mp}}(\tilde{\overline{U}}) + h\, W_{\mathrm{curv}}(\tilde{\mathfrak{K}}_s) + \frac{h^3}{12} W_{\mathrm{bend}}(\tilde{\mathfrak{K}}_b)\, \mathrm{d}\omega - \Pi(\tilde{m}, \tilde{\overline{R}}_3) \\
&\leq \liminf_{k\to\infty} \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k) + h\, W_{\mathrm{curv}}(\mathfrak{K}_{s,k}) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_{b,k})\, \mathrm{d}\omega - \Pi(m_k, \overline{R}_3^k) = \lim_{k\to\infty} I(m_k, \overline{R}^k)\,,
\end{aligned} \tag{3.4.58}$$

which implies that the limit pair is a minimizer. Note that the limit microrotations $\tilde{\overline{R}}$ may fail to be continuous if $p \leq 2$ (non-existence or limit case of Sobolev embedding). Moreover, uniqueness cannot be ascertained, since $\mathrm{SO}(3,\mathbb{R})$ is a nonlinear manifold (and the considered problem is indeed nonlinear), such that convex combinations of rotations are not rotations in general. Since the functional I is differentiable the minimizing pair is a stationary point and therefore a solution of the corresponding field equations. Note again that the limit microrotations are trivial in $L^\infty(\omega)$ but may fail to be continuously distributed in space. That under these unfavourable circumstances a minimizing solution may nevertheless be found is entirely due to $\mu_c > 0$ and $p \geq 1$. ■

We continue with the (more realistic) super-critical case with zero loads for simplicity.

Theorem 3.4.4 (Existence for 2D-finite elastic Cosserat model: case II.)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Then (3.4.42) with material constants conforming to case II admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$.

Proof. We repeat the argument of case I. However, the boundedness of infimizing sequences is not immediately clear. Boundedness of the rotations $\overline{R}^k$ holds true in the space $W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$ with $1+p+q > N = 3$, hence we may extract a subsequence, not relabelled, such that $\overline{R}^k$ converges strongly to $\hat{\overline{R}} \in C^0(\overline{\omega}, \mathrm{SO}(3,\mathbb{R}))$ in the topology of $C^0(\overline{\omega}, \mathrm{SO}(3,\mathbb{R}))$ on account of the Sobolev-embedding theorem. Along such strongly convergent sequence of rotations, the corresponding sequence of midsurface-deformations m^k is also bounded in $H^1(\omega, \mathbb{R}^3)$. However, this is not due to a basically simple pointwise estimate as in case I, but only true after integration over the domain: at face value we only control certain mixed symmetric expressions in the reconstructed deformation gradient. More precisely, we have

$$\begin{aligned}
\infty &> \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k) + h\, W_{\mathrm{curv}}(\mathfrak{K}_{s,k}) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega \geq \int_\omega h\, W_{\mathrm{mp}}(\overline{U}_k)\, \mathrm{d}\omega \\
&\geq \int_\omega h\, \frac{\mu}{4} \|\overline{R}_k^T (\nabla m_k | \overline{R}_3) + (\nabla m_k | \overline{R}_3)^T \overline{R}_k - 2\mathbb{1}\|^2\, \mathrm{d}\omega \\
&\geq \int_\omega h\, \frac{\mu}{4} \|\overline{R}_k^T (\nabla m_k | 0) + (\nabla m_k | 0)^T \overline{R}_k\|^2\, \mathrm{d}\omega - C_1 \|m_k\|_{H^{1,2}(\omega)} + C_2 \\
&= \int_\omega h\, \frac{\mu}{8} \|(\overline{R}_k - \hat{R} + \hat{R})^T (\nabla v_k | 0) + (\nabla v_k | 0)^T (\overline{R}_k - \hat{R} + \hat{R})\|^2\, \mathrm{d}\omega - C_1 \|v_k\|_{H^{1,2}(\omega)} + C_2 \\
&\geq \oint h\, \frac{\mu}{8} \underbrace{\|\hat{R}^T (\nabla v_k | 0) + (\nabla v_k | 0)^T \hat{R}\|^2}_{\text{combinations of derivatives}}\, \mathrm{d}\omega - C_3 \|\hat{R} - \overline{R}_k\|_\infty \|v_k\|^2_{H^{1,2}(\omega)} \\
&\qquad - (C_1 + 2\|\hat{R} - \overline{R}_k\|_\infty) \|v_k\|_{H^{1,2}(\omega)} + C_2 \\
&\geq (\frac{\mu}{8} c_K^+ - C_3 \|\hat{R} - \overline{R}_k\|_\infty) \|v_k\|^2_{H^{1,2}(\omega)} - (C_1 + 2\|\hat{R} - \overline{R}_k\|_\infty) \|v_k\|_{H^{1,2}(\omega)} + C_2\,,
\end{aligned} \tag{3.4.59}$$

where we made use of the appropriate boundary conditions for $m^k = x + u_k$ and applied the extended Korn's inequality (3.2.1) in the improved version of [Pom03] yielding the positive constant c_K^+ for the continuous microrotation $\hat{R}$. Since $\|\hat{R} - \overline{R}_k\|_\infty \to 0$ we conclude the boundedness of v_k in $H^1(\omega)$. Hence, m_k is bounded as well in $H^1(\omega)$. Now we obtain that $\overline{U_k} \rightharpoonup \tilde{\overline{U}} = \hat{\overline{R}}^T \nabla \tilde{m}$ by construction with the notations as in case I.

The remainder proceeds as in case I. This finishes the argument. The limit microrotations $\hat{\overline{R}}$ are indeed found to be continuous. However, for mixed boundary conditions, the midsurface deformation m cannot be shown to be smooth for lack of elliptic regularity. ■

3.4.5 The limit problem for infinite Cosserat couple modulus $\mu_c \to \infty$: the Biot-plate

As in the three-dimensional case, a constrained plate model is obtained by setting formally $\mu_c = \infty$ in (3.4.42). This implies that $\overline{U} = \overline{R}^T(\nabla m|\overline{R}_3) \in \mathrm{Sym}$, which entails $\overline{R}_3 = \vec{n}_m$ and the constraint rotation $\overline{R} = \mathrm{polar}(\nabla m|\vec{n}_m)$. Moreover, $\mathfrak{K}_b \in \mathrm{Sym}$ is enforced. Independent variation is only possible w.r.t. m and (3.4.42) turns into the **constrained** minimization problem on ω:

$$
\begin{aligned}
I &= \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega - \Pi(m, \vec{n}_m) \mapsto \min .\ \text{w.r.t. } m, \\
\overline{U} &= \overline{R}^T \hat{F} = \sqrt{(\nabla m|\vec{n})^T(\nabla m|\vec{n})} \in \mathrm{Sym}, \quad \hat{F} = (\nabla m|\vec{n}_m), \quad F_s = (\nabla m|\varrho_m \vec{n}_m), \quad \overline{R} = \mathrm{polar}(\nabla m|\vec{n}) \\
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda} \mathrm{tr}\left[\overline{U} - 1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \vec{n}\rangle}{(2\mu+\lambda)}, \quad \text{first order thickness stretch} \\
m_{|\gamma_0} &= g_{\mathrm{d}}(x,y,0), \qquad \text{simply supported} \qquad\qquad (3.4.60)\\
\mathrm{polar}(\nabla m|\vec{n}) &= \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3))_{|\gamma_0}, \quad \text{reduced consistent coupling} \\
\vec{n}_{|\gamma_0} &= \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}, \quad \text{alternatively: classical rigid condition} \\
W_{\mathrm{mp}}(\overline{U}) &= \mu\,\|\overline{U} - 1\!\!1\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\overline{U} - 1\!\!1\right]^2 \\
&= \mu\,\|\sqrt{I_m} - 1\!\!1_2\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\sqrt{I_m} - 1\!\!1_2\right]^2, I_m\text{: } \textbf{first fundamental form} \text{ of the surface} \\
W_{\mathrm{curv}}(\mathfrak{K}_s) &= \mu \frac{L_c^{1+p}}{12}\left(1 + \alpha_4 L_c^q \|\mathfrak{K}_s\|^q\right)\left(\alpha_5\,\|\mathrm{sym}\,\mathfrak{K}_s\|^2 + \alpha_6\,\|\mathrm{skew}\,\mathfrak{K}_s\|^2 + \alpha_7\,\mathrm{tr}\left[\mathfrak{K}_s\right]^2\right)^{\frac{1+p}{2}}, \\
\mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right), \quad \text{reduced third order } \textbf{curvature tensor} \\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\,\|\mathfrak{K}_b\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathfrak{K}_b\right]^2 = \mu\,\|\nabla\vec{n}\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\overline{R}^T(\nabla\vec{n}|0)\right]^2 \\
\mathfrak{K}_b &= \mathrm{polar}((\nabla m|\vec{n}))^T(\nabla\vec{n}|0), \quad \text{second order, } \textbf{weighted, bending tensor} \\
\mathfrak{K}_b &\in \mathrm{Sym} \Leftrightarrow \overline{U}^{-1}\widehat{II}_m \in \mathrm{Sym}, \quad \text{symmetry constraint} \Leftrightarrow \langle \overline{R}_{1,y}, \vec{n}\rangle = \langle \overline{R}_{2,x}, \vec{n}\rangle \quad \text{for smooth fields} \\
\widehat{II}_m &: \quad \text{extended } \textbf{second fundamental form} \text{ of the surface}.
\end{aligned}
$$

Let us therefore define the set of admissible deformations

$$
\begin{aligned}
\mathcal{A} := \{ & m \in H^{1,2}(\omega, \mathbb{R}^3), \overline{R} \in W^{1,1+p}(\omega, \mathrm{SO}(3,\mathbb{R}))) \,| \\
& \overline{R}^T(\nabla m|\overline{R}_3) \in L^2(\omega, \mathrm{Sym}), \quad \overline{R}_{3|\gamma_0} = \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|} \}.
\end{aligned} \tag{3.4.61}
$$

This set is not empty: pure bending situations $(\nabla m|\vec{n}) \in \mathrm{SO}(3,\mathbb{R})$ and deformations, where $\overline{U} \in \mathrm{diag}$ and $II_m \in \mathrm{diag}$ are contained in $\mathcal{A}$.

Theorem 3.4.5 (Existence for 2D-constrained Cosserat plate model: case III)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and $\mathrm{polar}(\nabla g_{\mathrm{d}}) \in W^{1,1+p}(\omega, \mathrm{SO}(3,\mathbb{R}))$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $\overline{f} \in L^2(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (3.9.102). If $I < \infty$ over $\mathcal{A}$ then problem (3.4.60) with $p \geq 1$ admits at least one minimizing solution $m \in H^{1,2}(\omega, \mathbb{R}^3)$.

Proof. The proof mimics case I since the sequence of infimizing rotations $\overline{R}_k$ is constrained to the orthogonal part polar$(\hat{F}_k)$ of the corresponding sequence of deformation gradients F_k. Due to the extra Cosserat curvature control, the rotations $\overline{R}_k$ can be chosen such that they converge weakly in $H^1(\omega, \mathrm{SO}(3,\mathbb{R}))$. Taking into account weak convergence of F_k and strong convergence of $\overline{R}_k$, the composition $\overline{U}_k$ converges weakly in $L^2(\omega)$. However, $\overline{U}_k \in \mathrm{Sym}$ and this condition is weakly closed. Hence, the limit belongs to $\mathcal{A}$. ■

Remark 3.4.6
Complete higher regularity of m in the constrained Cosserat model, i.e. $m \in H^{2,2}(\omega, \mathbb{R}^3)$ cannot be ascertained in general since we only control certain second derivatives of m in the curvature term. One might wonder therefore, whether the additional C^1-continuity in treating the fourth order indeterminate couple stress problem numerically is worth the effort.

3.4.6 The limit problem for vanishing relative thickness $h \to 0$

While it does not make much sense to let $h \to 0$ at fixed in-plane elongation $L > 0$, since there is an absolute lower bound on the thickness in terms of the internal length L_c, we may consider a sequence of plates, whose absolute thickness is fixed, but whose in-plane elongation L is increased. This implies that the relative thickness h tends to zero. In a formal sense then, the thin plate limit problem is obtained by neglecting the h^3- bending tensor contribution and giving up the possibility/necessity to prescribe microrotations $\overline{R}_3$ at the Dirichlet boundary γ_0. In view of the expected limit behaviour of skew$(\overline{U}) = 0 \Rightarrow \overline{R}_3 = \vec{n}_m$ we consider $\mu_c = 0$ only. The two-dimensional limit problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the thin plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ solves formally the following minimization problem on ω:

$$\int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s)\,\mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min .\ \text{w.r.t. } (m, \overline{R}),$$

$$\begin{aligned}
m_{|_{\gamma_0}} &= g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \\
\overline{R}_{|_{\gamma_0}} &= \mathrm{polar}((\nabla m|\varrho_m\, \overline{R}_3))_{|_{\gamma_0}}\,, \quad \text{reduced consistent coupling} \Rightarrow \overline{R}_{3_{|_{\gamma_0}}} = \vec{n}_m, \quad \text{free} \\
W_{\mathrm{mp}}(\overline{U}) &= \mu\, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2 \\
W_{\mathrm{curv}}(\mathfrak{K}_s) &= \mu\frac{L_c^{1+p}}{12}\,(1 + \alpha_4\, L_c^q\, \|\mathfrak{K}_s\|^q)\left(\alpha_5\, \|\mathrm{sym}\,\mathfrak{K}_s\|^2 + \alpha_6\, \|\mathrm{skew}\,\mathfrak{K}_s\|^2 + \alpha_7\,\mathrm{tr}\,[\mathfrak{K}_s]^2\right)^{\frac{1+p}{2}}, \\
\mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right), \quad \text{reduced third order } \textbf{curvature tensor}\,.
\end{aligned} \tag{3.4.62}$$

Theorem 3.4.7 (Existence for 2D-finite Cosserat limit model for vanishing relative thickness)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $\overline{f} \in L^2(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (3.9.102). Then (3.4.62) admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$.

Proof. Exactly the same proof as for case II applies since the decisive control is afforded by W_{curv} and not W_{bend}. ■

Conjecture 3.4.8 (Γ-limit)
The Γ-limit for $h \to 0$ of suitably rescaled energies in (3.3.6) and $\mu_c \geq 0$ is given by the variational problem (3.4.62) with $\mu_c = 0$. ■

3.4.7 The limit problem for vanishing internal length $L_c \to 0$

This limit is practically encountered if very large, relatively thin plates are considered. The difference to the case $h \to 0$ from above is clear: we consider a sequence of ever larger plates with the **same relative thickness**. A scaling argument (3.9.2) shows easily that the respectively transformed L_c on a unit domain ω will tend to zero. We obtain formally the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ solving on ω:

$$\begin{aligned}
I &= \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min . \text{ w.r.t. } (m, \overline{R}),\\
\overline{U} &= \overline{R}^T \hat{F}, \quad \hat{F} = (\nabla m|\overline{R}_3), \quad F_s = (\nabla m|\varrho_m \overline{R}_3)\\
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda} \mathrm{tr}\left[\overline{U} - 1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle}{(2\mu+\lambda)}, \quad \text{first order thickness stretch}\\
m_{|\gamma_0} &= g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported}\\
\overline{R}_{|\gamma_0} &= \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3)_{|\gamma_0}\,, \quad \text{reduced consistent coupling}\\
\overline{R}_{3_{|\gamma_0}} &= \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \quad \text{rigid prescription}\\
W_{\mathrm{mp}}(\overline{U}) &= \mu\, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\, \|\mathrm{skew}(\overline{U})\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2\\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\, \|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \mu_c\, \|\mathrm{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2\\
\mathfrak{K}_b &= \overline{R}^T (\nabla \overline{R}_3|0)\,, \quad \text{second order, } \textbf{non-symmetric bending tensor}\,.
\end{aligned} \tag{3.4.63}$$

For $\mu_c = 0$ this is case V of our classification. In this form, the problem is not completely determined since the remaining bending term only controls the 'director' $\overline{R}_3$ but leaves **in plane rotations free**. However, anticipating that $\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1$ is small (appropriate for almost rigid materials), a modification of the bending term is suggested: we modify

$$\mathfrak{K}_b = \overline{R}^T (\nabla \overline{R}_3|0) \notin \mathrm{Sym} \Rightarrow \begin{pmatrix} -\|R_{1,x}\| & -\|R_{1,y}\| & 0\\ -\|R_{2,x}\| & -\|R_{2,y}\| & 0\\ 0 & 0 & 0 \end{pmatrix} \notin \mathrm{Sym}\,. \tag{3.4.64}$$

Remark 3.4.9 (Motivation)
The motivation of this modification for relatively thin Cosserat shells is as follows: either the **membrane energy is non-zero**, *in which case it dominates and the* **bending contribution can be neglected** *or the* **membrane energy is zero** *($\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1 = 0$) in which case the* **non-symmetric bending tensor** *of* (3.4.63) **coincides** *with the symmetric expression of* (3.4.64), *see Lemma 3.10.8.*

A formulation based on this modification supports an existence theorem if $\mu_c > 0$, notwithstanding the inherent nonlinearity along the same lines as in Theorem 3.4.3. The more interesting case of $\mu_c = 0$ must remain open at present, since the limit rotations in $H^{1,2}(\omega, \mathrm{SO}(3,\mathbb{R})))$ must not necessarily be smooth.

3.4.8 The limit problem for vanishing L_c: the pure bending case.

Assume that the boundary conditions for the Cosserat plate support multiple finite bending modes, i.e. $\nabla m^T \nabla m = 1\!\!1_2$ and the membrane energy $W_{\mathrm{mp}}(\overline{U})$ is zero. What can we say about the corresponding degenerated minimization problem? The variational problem for the Cosserat bending plate is then to find a deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ solving on ω:

$$\begin{aligned}
I &= \int_\omega \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega \mapsto \min . \text{ w.r.t. } (m, \overline{R}) \text{ such that } \nabla m^T \nabla m = 1\!\!1_2 \text{ and } W_{\mathrm{mp}}(\overline{U}) = 0,\\
\overline{U} &= \overline{R}^T \hat{F}, \quad \hat{F} = (\nabla m|\overline{R}_3) \quad m_{|\gamma_0} = g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported}\\
\overline{R}_{|\gamma_0} &= \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3)_{|\gamma_0}\,, \quad \text{reduced consistent coupling}\\
W_{\mathrm{mp}}(\overline{U}) &= \mu\, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\, \|\mathrm{skew}(\overline{U})\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2\\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\, \|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \mu_c\, \|\mathrm{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2\\
\mathfrak{K}_b &= \overline{R}^T (\nabla \overline{R}_3|0)\,, \quad \text{second order, } \textbf{non-symmetric bending tensor}\,.
\end{aligned} \tag{3.4.65}$$

It is easily seen, that $\nabla m^T \nabla m = 1\!\!1_2$ and $W_{\mathrm{mp}}(\overline{U}) = 0$ already constrains the microrotations to $\overline{R} = (\nabla m|\vec{n}_m) \in \mathrm{SO}(3,\mathbb{R})$ for $\mu_c > 0$[20] and $\mu_c = 0$.[21] This implies $\mathfrak{K}_b = \overline{R}^T(\nabla \overline{R}_3|0) = (\nabla m|\vec{n}_m)^T(\nabla \vec{n}|0)$ and the in general non-symmetric bending tensor $\mathfrak{K}_b$ coincides with the symmetric second fundamental form of the midsurface m. The resulting minimization **coincides** with the bending problem based on the Kirchhoff-Love theory (3.7.88) and admits therefore a solution $m \in H^2(\omega,\mathbb{R}^3)$, from which we recover $\overline{R} = (\nabla m|\vec{n}) \in H^{1,2}(\omega,\mathrm{SO}(3,\mathbb{R}))$.

3.4.9 The limit problem for vanishing L_c and vanishing h: the pure membrane.

The problem for vanishing relative thickness h and without consideration of the internal length L_c leads to the **pure membrane dominated** limit problem[22] for the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the microrotation of the thin plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$ on ω:

$$\int_\omega h\, W_{\mathrm{mp}}(\overline{U})\, \mathrm{d}\omega - \Pi(m,\overline{R}_3) \mapsto \min.\ \text{w.r.t. } (m,\overline{R}), \quad m_{|\gamma_0} = g_{\mathrm{d}}(x,y,0), \tag{3.4.66}$$

which is equivalent to

$$\begin{aligned}
&\int_\omega h\, W_{\mathrm{mp}}(\overline{U})\, \mathrm{d}\omega - \Pi(m,\overline{R}_3) \mapsto \min.\ \text{w.r.t. } m \text{ at given } \overline{R} \in \mathrm{SO}(3,\mathbb{R}).\,, \quad m_{|\gamma_0} = g_{\mathrm{d}}(x,y,0), \qquad (3.4.67)\\
&\overline{U} = \overline{R}^T(\nabla m|\overline{R}_3)\\
0 = &\underbrace{\mathrm{skew}\left(\overline{R}^T(\overline{M}|0|0)\right) + h\ \mathrm{skew}\left(\overline{R}^T D_{\overline{R}} W_{\mathrm{mp}}(\nabla m,\overline{R})\right)}_{\text{2D-balance of angular momentum}}, \quad \text{local, algebraic condition } \Leftrightarrow\\
0 = &\underbrace{\mathrm{skew}\left(\overline{R}^T(\overline{M}|0|0)\right) + h\ \mathrm{skew}\left(D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\overline{U}^T\right)}_{\text{thickness integrated 3D-balance of angular momentum}} - h\ \mathrm{skew}\left((0|0|D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}).e_3)\right),
\end{aligned}$$

see (3.9.102) for the definition of $\overline{M}$. The local condition comes from locally minimizing w.r.t. $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$, it is balance of angular momentum for the plate in disguise.[23] Note that at given $\overline{R}$, the membrane minimization problem w.r.t. m is still uniformly Legendre-Hadamard elliptic. However, coercivity w.r.t. m depends crucially on the smoothness of $\overline{R}$ if $\mu_c = 0$. There is no reason to expect $\overline{R}$ to be smoothly distributed. Existence to this problem is open: we expect therefore sharp interfaces.

In the absence of external loads, the remaining symmetry condition

$$\mathrm{skew}\left(D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\overline{U}^T\right) = \mathrm{skew}\left((0|0|D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}).e_3)\right) \tag{3.4.69}$$

is satisfied, if $\overline{U} \in \mathrm{Sym}$, which itself implies $\overline{R} = \mathrm{polar}(\nabla m|\vec{n})$. Nevertheless, considered as a local condition, the remaining symmetry condition does not automatically imply the symmetry of $\overline{U}$, see the discussion of a similar problem in the three-dimensional case in [Nef03c]. Such a discrepancy does not occur in the infinitesimal Reissner-Mindlin model (3.9.127).

[20] In this case, we could dispose of the requirement $\nabla m^T \nabla m = 1\!\!1_2$.

[21] $\mathrm{sym}(\overline{U} - 1\!\!1) = 0$ implies immediately $\overline{R}_3 = \vec{n}$. Write $\overline{R}_i = a_i^1 m_x + a_i^2 m_y$, $i = 1,2$. Using $\langle m_x, m_y\rangle = 0$ the result follows. Whether one can do without $\nabla m^T \nabla m = 1\!\!1_2$ in case $\mu_c = 0$ is open, since $\mathrm{sym}(\overline{U} - 1\!\!1) = 0$ for $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ and $(\nabla m|\vec{n}) \in \mathrm{GL}^+(3,\mathbb{R})$ considered **without gradient constraint** on m has nontrivial solutions.

[22] Observe that the Cosserat model does not automatically endow a thin plate limit with additional stiffness, since it is physically not possible to let $h \to 0$ and keep the in-plane elongations L constant.

[23] To see the equivalence of the two local statements in (3.4.67), consider variation of $\overline{R}$ along a one-parameter group of rotations $\frac{\mathrm{d}}{\mathrm{dt}}\overline{R} = A(t)\cdot\overline{R}$, $A \in \mathfrak{so}(3,\mathbb{R})$ and evaluate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} W_{\mathrm{mp}}(\overline{R}(\nabla m|\overline{R}_3)) &= \langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}), (\delta\overline{R})^T(\nabla m|\overline{R}_3) + \overline{R}^T(0|0|(\delta\overline{R})_3)\rangle = \langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}), (A\overline{R})^T(\nabla m|\overline{R}_3) + \overline{R}^T(0|0|(A\overline{R})_3)\rangle\\
&= -\langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\overline{U}^T, \overline{R}^T A\overline{R}\rangle + \langle \overline{R} D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}), (0|0|A\overline{R}.e_3)\rangle = -\langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\overline{U}^T, \overline{R}^T A\overline{R}\rangle + \langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}).e_3, \overline{R}^T A\overline{R}.e_3\rangle\\
&= -\langle D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\overline{U}^T, \overline{R}^T A\overline{R}\rangle + \langle (0|0|D_{\overline{U}} W_{\mathrm{mp}}(\overline{U}).e_3), \overline{R}^T A\overline{R}\rangle\,.
\end{aligned} \tag{3.4.68}$$

3.5 A modified finite Cosserat thin plate for large stretch and local invertibility

While the preceeding models have been derived from a three-dimensional model which itself is appropriate only for small strain and large rotations, let us present a modified model,[24] which in principle allows for arbitrary large stretch and which automatically preserves local invertibility if the reconstructed deformation is smooth. It is clear that such an extension is by no means unique. The model reads

$$\begin{aligned}
I &= \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min. \text{ w.r.t. } (m, \overline{R}),\\
\overline{U} &= \overline{R}^T \hat{F}, \quad \hat{F} = (\nabla m|\overline{R}_3), \quad F_s = (\nabla m|\varrho_m \overline{R}_3)\\
\varrho_m &= \frac{1}{1 + \frac{\lambda}{2\mu+\lambda}(\det[\overline{U}] - 1)} + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle}{(2\mu+\lambda)}\\
m_{|_{\gamma_0}} &= g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad\qquad (3.5.70)\\
\overline{R}_{|_{\gamma_0}} &= \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3))_{|_{\gamma_0}}\,, \quad \text{reduced consistent coupling}\\
\overline{R}_{3_{|_{\gamma_0}}} &= \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \quad \text{alternatively: rigid prescription}\\
W_{\mathrm{mp}}(\overline{U}) &= \mu\, \|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\frac{1}{2}\left((\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2\right)\\
W_{\mathrm{curv}}(\mathfrak{K}_s) &= \mu \frac{L_c^{1+p}}{12}\,(1 + \alpha_4\, L_c^q\, \|\mathfrak{K}_s\|^q)\left(\alpha_5\, \|\mathrm{sym}\, \mathfrak{K}_s\|^2 + \alpha_6\, \|\mathrm{skew}\, \mathfrak{K}_s\|^2 + \alpha_7\, \mathrm{tr}\,[\mathfrak{K}_s]^2\right)^{\frac{1+p}{2}}\,,\\
\mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\,, \quad \text{reduced third order } \textbf{curvature tensor}\\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\, \|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\,[\mathrm{sym}(\mathfrak{K}_b)]^2\\
\mathfrak{K}_b &= \overline{R}^T(\nabla\overline{R}_3|0) = \mathfrak{K}_s^3\,, \quad \text{second order, } \textbf{non-symmetric bending tensor}\,.
\end{aligned}$$

Let us summarize the salient features of this model. First, $W_{\mathrm{mp}}(\overline{U}) \to \infty$ if $\det[\overline{U}] \to 0$. Thus, if minimizers exist, then $\det[\overline{U}] > 0$ a.e. and the minimizing surface is locally regular. The modified energy contribution is **polyconvex** w.r.t ∇m and thus Legendre-Hadamard elliptic. If $\overline{R}_3 = \vec{n}$, then

$$\det[\overline{U}] = \|\,\mathrm{Cof}\,(\nabla m|0)\|, \quad \|\,\mathrm{Cof}\,(\nabla m|0)\|^2 = \|m_x \times m_y\|^2 = \|m_x\|^2\|m_y\|^2 - \langle m_x, m_y\rangle^2 = \det[I_m]\,, \quad (3.5.71)$$

a pure, intrinsic measure of the surface stretch. If $W_{\mathrm{mp}}(\overline{U}) = 0$ then $\overline{U} = 1\!\!1$ although $\mu_c = 0$. The thickness stretch ϱ_m has such a form, that at finite energy one has $0 < \varrho_m < \infty$ without restriction on the kinematics and transverse fibers will be elongated upon action of opposite tractions. Moreover, $\varrho_m \equiv 1$ for $\lambda = 0$ (extreme compressibility, $\nu = 0$) and $\varrho_m = \frac{1}{\det[\overline{U}]}$ for $\lambda = \infty$ (incompressibility, $\nu = \frac{1}{2}$) such that $\det[F_s] = \det[(\nabla m|\varrho_m\, \overline{R}_3)] \equiv 1$, i.e. exact incompressibility for the reconstructed deformation.

The formulation (3.5.70) has the same linearized behaviour as the initial model (3.4.42).[25] We can prove the following result

Theorem 3.5.1 (Existence for 2D-finite elastic Cosserat model with large stretch and invertibility) *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\omega, \mathbb{R}^3)$ and $\overline{R}_{\mathrm{d}} \in W^{1,1+p}(\omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, let $\overline{f} \in L^2(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (3.9.102). Then (3.5.70) with material constants conforming to case II admits at least one minimizing solution pair $(m, \overline{R}) \in H^1(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \mathrm{SO}(3,\mathbb{R}))$ with $\det[(\nabla m|\overline{R}_3)] > 0$ a.e.*

Proof. The proof mimics the arguments of the preceeding existence results for case II. We only need to observe in addition, that the modified membrane energy is in fact polyconvex at given $\overline{R}$ w.r.t. ∇m. The modified term provides us with the information that $\det[(\nabla m_k|\overline{R}_3^k)]$ is uniformly bounded in $L^2(\omega)$ for minimizing sequences. Hence we may always choose a minimizing sequence, such that $\det[(\nabla m_k|\overline{R}_3^k)] \rightharpoonup \zeta \in L^2(\omega)$. We have as well $\overline{R}^k \to \overline{R} \in C^0(\omega, \mathrm{SO}(3,\mathbb{R}))$. Moreover, $\nabla m_k \rightharpoonup \nabla m \in L^2(\omega, \mathbb{M}^{2\times 3})$. Thus, $\det[(\nabla m_k|\overline{R}_3^k)] \to \det[(\nabla m|\overline{R}_3)]$

[24]It is clear that a modification to large stretch does not concern the bending term since bending only plays a role for small stretch.

[25]Because $\left((\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2\right) = 2\,\mathrm{tr}\,[\overline{U} - 1\!\!1]^2 + O(\|\overline{U} - 1\!\!1\|^3)$.

strongly in the sense of distributions [Bal77, Th. 3.4]. This implies $\zeta = \det[(\nabla m|\overline{R}_3)]$. The remainder is standard. ■

It is therefore believed that (3.5.70) represents an improvement over (3.4.42), although (3.5.70) itself is not strictly obtained from a parent model.[26]

3.6 The finite, invariant Reissner-Mindlin plate

To contrast the previous models, let us directly derive a new nonlinear, finite, properly invariant Reissner-Mindlin plate starting from the three-dimensional SVK elasticity model. Again, we assume a quadratic ansatz in the thickness direction for the (reconstructed) finite deformation $\varphi_s : \mathbb{R}^3 \mapsto \mathbb{R}^3$ of the shell like structure

$$\varphi_s(x,y,z) = m(x,y) + \left(z\,\varrho_m(x,y) + \frac{z^2}{2}\varrho_b(x,y)\right)\cdot \vec{d}(x,y)\,, \tag{3.6.72}$$

where $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ takes on the role of the deformation of the midsurfaces of the shell viewed as a parametrized surface and $\vec{d} : \omega \subset \mathbb{R}^2 \mapsto \mathbb{S}^2$ is a **unit director** field; the functions $\varrho_m,\ \varrho_b : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}$ allow for **thickness stretch** ($\varrho_m \neq 1$) and **transverse shear** ($\vec{d} \neq \vec{n}$). [27] This implies for the (reconstructed) deformation gradient of the shell (plate)

$$\begin{aligned} F_s = \nabla\varphi_s(x,y,z) &= (\nabla m|\,\varrho_m\,\vec{d}) + z\cdot(\nabla(\varrho_m\,\vec{d})|\varrho_b\,\vec{d}) + \frac{z^2}{2}(\nabla(\varrho_b\,\vec{d})|0) = \tilde{A}_m + z\,\tilde{A}_r + \frac{z^2}{2}\tilde{B}_r \\ &\approx (\nabla m|\,\varrho_m\,\vec{d}) + z\cdot((\nabla\vec{d})|\varrho_b\,\vec{d}) = A_m + z\,A_r\,. \end{aligned} \tag{3.6.73}$$

The underlying three-dimensional Saint Venant Kirchhoff energy has the form

$$W_{\mathrm{SVK}}(F) = \frac{\mu}{4}\,\|F^TF - 1\!\!1\|^2 + \frac{\lambda}{8}\,\mathrm{tr}\left[F^TF - 1\!\!1\right]^2. \tag{3.6.74}$$

The equations of the three-dimensional finite elasticity problem show that on the transverse boundary (exact)

$$\begin{aligned} S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2)).e_3 &= N^{\mathrm{trans}}(x,y,+h/2) \\ S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2)).(-e_3) &= N^{\mathrm{trans}}(x,y,-h/2)\,, \end{aligned} \tag{3.6.75}$$

where N^{trans} are the prescribed tractions N on the transverse boundary.[28]

Following the steps which led to (3.3.17) we have (exact)

$$\langle F^{-1}(x,y,\pm h/2)S_1(\nabla\varphi(x,y,\pm h/2)).e_3, e_3\rangle = \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2), F^{-T}(x,y,\pm h/2).e_3\rangle\,, \tag{3.6.76}$$

which condition reduces to **zero normal tractions on the transverse free boundary**:

$$S_{2,33}(\nabla\varphi(x,y,\pm h/2)) = 0\,, \tag{3.6.77}$$

in the absence of tractions N^{trans}. In view of the **assumed rigidity** ($\mu \gg 1$) we expect that $\nabla\varphi^T\nabla\varphi - 1\!\!1 \ll 1$ such that $\nabla\varphi^{-T} \approx \nabla\varphi$ and we determine $\varrho_m,\ \varrho_b$ from the corresponding modified requirement in terms of the assumed kinematics for φ_s, yielding

$$\begin{aligned} \langle F_s^{-1}(x,y,\pm h/2)S_1(\nabla\varphi_s(x,y,\pm h/2)).e_3, e_3\rangle &= \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2), \overbrace{F_s(x,y,\pm h/2).e_3}^{\text{modified}}\rangle \\ &= \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2), (\varrho_m + z\,\varrho_b)\,\vec{d}\rangle\,. \end{aligned} \tag{3.6.78}$$

Since $S_1 = F\left[\mu(F^TF - 1\!\!1) + \frac{\lambda}{2}\mathrm{tr}\left[F^TF - 1\!\!1\right]1\!\!1\right]$, we obtain the two **nonlinear** equations

$$\langle\left[\mu(F_s^TF_s - 1\!\!1) + \frac{\lambda}{2}\mathrm{tr}\left[F_s^TF_s - 1\!\!1\right]1\!\!1\right].e_3, e_3\rangle = \pm\langle N^{\mathrm{trans}}(x,y,\pm h/2), (\varrho_m + z\,\varrho_b)\,\vec{d}\rangle\,. \tag{3.6.79}$$

[26] There is a general danger of direct theories to postulate two-dimensional models from scratch without recours to any underlying parent model : while general two-dimensional balance principles are easily applied, it is not clear how to incorporate any three-dimensional information.

[27] This leads finally to a 5 'dof' theory: 3 components of the membrane deformation and 2 degrees of freedom for the unit director field, the coefficients ϱ_m, ϱ_b will again be eliminated.

[28] Using the approximated F_s in (3.6.73) leads to an affine linear reconstruction of the transverse shear stress $\langle S_2(F_s).e_3, e_i\rangle$, $i = 1, 2$.

There is no simple way to solve these equations exactly. To leading order in h we obtain for ϱ_m

$$\varrho_m = +\frac{\langle N_{\text{diff}}, \vec{d}\rangle}{(2\mu+\lambda)} \pm \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}[\|\nabla m\|^2 - 2] + \frac{\langle N_{\text{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}}\,, \tag{3.6.80}$$

and for ϱ_b

$$\varrho_b = -\frac{\lambda}{2\mu+\lambda}\langle(\nabla m|\vec{d}), (\nabla\vec{d}|0)\rangle + \frac{1}{(2\mu+\lambda)\,h}\langle N_{\text{res}}, \vec{d}\rangle + \frac{1}{\varrho_m\,(2\mu+\lambda)}\langle(\nabla m|\varrho_m\,\vec{d}), (\varrho_{m,x}\vec{d}|\varrho_{m,y}\vec{d}|0)\rangle\,.$$

Since we do not want to consider space variations in the thickness-stretch ϱ_m we take finally

$$\begin{aligned}
\varrho_m &= \frac{\langle N_{\text{diff}}, \vec{d}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\,[\|\nabla m\|^2 - 2] + \frac{\langle N_{\text{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}}\\
\varrho_b &= -\frac{\lambda}{2\mu+\lambda}\langle(\nabla m|\vec{d}), (\nabla\vec{d}|0)\rangle + \frac{\langle N_{\text{res}}, \vec{d}\rangle}{(2\mu+\lambda)\,h}\,.
\end{aligned} \tag{3.6.81}$$

Note that if we identify $\vec{d} = \overline{R}_3$ then ϱ_b in the last formula coincides with the expression for ϱ_b found in (3.3.24) while ϱ_m is still different.

Following conceptually the same computation which starts after (3.3.25) we obtain after thickness integration the following minimization problem for the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the unit director field $\vec{d} : \omega \subset \mathbb{R}^2 \mapsto \mathbb{S}^2$ on ω:

$$\begin{aligned}
&\int_\omega h\,W_{\text{mp}}(\overline{C}) + \frac{h^3}{12}\,W_{\text{bend}}(\mathfrak{K}_b)\,\mathrm{d}\omega - \Pi(m,\vec{d}) \mapsto \min .\ \text{w.r.t.}\ (m,\vec{d})\\
&\overline{C} = \hat{F}^T\hat{F}, \quad \hat{F} = (\nabla m|\vec{d}), \quad F_s = (\nabla m|\varrho_m\vec{d})\\
&\varrho_m = \frac{\langle N_{\text{diff}}, \vec{d}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\,[\|\nabla m\|^2 - 2] + \frac{\langle N_{\text{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}}\\
&\quad = \frac{\langle N_{\text{diff}}, \vec{d}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\text{tr}\left[\overline{C} - 1\!\!1\right] + \frac{\langle N_{\text{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}}\,, \quad \text{first order thickness stretch}\\
&m_{|\gamma_0} = g_{\text{d}}(x,y,0)\,, \qquad \text{simply supported}, \quad \vec{d}_{|\gamma_0} = \frac{\nabla g_{\text{d}}(x,y,0).e_3}{\|\nabla g_{\text{d}}(x,y,0).e_3\|}\,, \quad \text{rigid prescription}\\
&W_{\text{mp}}(\overline{C}) = \frac{\mu}{4}\,\|\overline{C} - 1\!\!1\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\,\text{tr}\left[\overline{C} - 1\!\!1\right]^2\\
&\quad = \frac{\mu}{4}\,\|\underbrace{\nabla m^T\nabla m - 1\!\!1_2}_{\text{intrinsic energy}}\|^2 + \underbrace{\frac{\kappa\,\mu}{2}\left(\langle m_x, \vec{d}\rangle^2 + \langle m_y, \vec{d}\rangle^2\right)}_{\text{transverse shear energy}} + \frac{2\mu\lambda}{8(2\mu+\lambda)}\,\text{tr}\left[\nabla m^T\nabla m - 1\!\!1_2\right]^2\\
&W_{\text{bend}}(\mathfrak{K}_b) = \mu\,\|\,\text{sym}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\text{tr}\left[\text{sym}(\mathfrak{K}_b)\right]^2, \quad \mathfrak{K}_b = (\nabla m|\vec{d})^T(\nabla\vec{d}|0)\,.
\end{aligned} \tag{3.6.82}$$

It is immediate, that the obtained model is **frame-indifferent** in the sense that $\forall\ Q \in \text{SO}(3,\mathbb{R})\ :\ W(Q(\nabla m|\vec{d}) = W((Q\nabla m|Q.\vec{d}) = W((\nabla Q.m|Q.\vec{d}) = W(\nabla m|\vec{d})$. The membrane part is **coercive** in $H^{1,4}(\omega,\mathbb{R}^3)$. However, the membrane part neither satisfies the Baker-Ericksen inequalities nor is it Legendre-Hadamard elliptic. It is not obvious, which type of control can be expected in the bending contribution. Drill rotations are absent, but the model allows for transverse shear (again, $\kappa = 1$ is the shear correction factor). Invertibility of the reconstructed deformation is not ensured. Nothing seems to be known on existence. No extra size effects enter the description. While ϱ_m shows the physically correct behaviour that small opposite transverse tractions will elongate fibers, for non-infinitesimal transverse tractions which 'presurrize' the plate, the fibers would as well be elongated instead of shrinked. Linearization of this model results in the classical infinitesimal Reissner-Mindlin Model (3.9.127) and restricting the director $\vec{d}$ to the unit normal of the surface simplifies the model into the following finite Kirchhoff-Love plate. In this sense, the model has some merits.

3.7 The finite, invariant Kirchhoff-Love plate

3.7.1 Variational formulation

Either by formal asymptotic analysis (and adding together the leading membrane and bending part) or a proper kinematical ansatz[29] or else by restricting the director $\vec{d}$ in (3.6.82) to the unit normal of the midsurface, a finite, properly invariant[30] Kirchhoff-Love plate problem in variational form can be written in the form of a minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ on ω:

$$
\begin{aligned}
&\int_\omega h\, W_{\mathrm{mp}}(\overline{C}) + \frac{h^3}{12}\, W_{\mathrm{bend}}(\mathfrak{K}_b)\, \mathrm{d}\omega - \Pi(m, \vec{n}_m) \mapsto \min.\ \text{w.r.t. } m \\
&\qquad \overline{C} = \hat{F}^T \hat{F}, \quad \hat{F} = (\nabla m|\vec{n}_m), \quad F_s = (\nabla m|\varrho_m \vec{n}_m) \\
&\qquad \varrho_m = \frac{\langle N_{\mathrm{diff}}, \vec{n}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)} \operatorname{tr}\left[\overline{C} - 1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \vec{n}\rangle^2}{(2\mu+\lambda)^2}}\,, \quad \text{first order thickness stretch} \\
&\qquad m_{|\gamma_0} = g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported}, \quad \vec{n}_{m_{|\gamma_0}} = \frac{\nabla g_{\mathrm{d}}(x,y,0).e_3}{\|\nabla g_{\mathrm{d}}(x,y,0).e_3\|}\,, \quad \text{clamped} \\
&\qquad W_{\mathrm{mp}}(\overline{C}) = \frac{\mu}{4}\, \|\overline{C} - 1\!\!1\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr}\left[\overline{C} - 1\!\!1\right]^2 = \frac{\mu}{4}\, \|\nabla m^T \nabla m - 1\!\!1_2\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)} \operatorname{tr}\left[\nabla m^T \nabla m - 1\!\!1_2\right]^2 \\
&\qquad\qquad = \frac{\mu}{4}\, \|I_m - 1\!\!1_2\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)} \operatorname{tr}\left[I_m - 1\!\!1_2\right]^2, \quad I_m\text{: } \textbf{first fundamental form} \text{ of the surface} \\
&\qquad W_{\mathrm{bend}}(\mathfrak{K}_b) = \mu\, \|\operatorname{sym}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr}\left[\operatorname{sym}(\mathfrak{K}_b)\right]^2 = \mu\, \|II_m\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr}\left[II_m\right]^2 \\
&\qquad \mathfrak{K}_b = (\nabla m|\vec{n}_m)^T (\nabla \vec{n}_m|0) = II_m \in \mathrm{Sym}\,, \quad II_m\text{: } \textbf{second fundamental form} \text{ of the surface } m\,.
\end{aligned}
\tag{3.7.83}
$$

The reconstructed deformation $\varphi_s(x,y,z) = m(x,y) + \left(z\varrho_m + \frac{z^2}{2}\varrho_b\right)\vec{n}_m$ yields the plane stress condition $S_1(\nabla\varphi_s(x,y,0).e_3 = 0$, which is only consistent with three-dimensional equilibrium if there are no normal tractions at the transverse boundary. [31]

It is easily seen that the resultant membrane strain energy $W_{\mathrm{mp}}(\overline{C})$ is neither quasiconvex nor Legendre-Hadamard elliptic. Moreover, the resultant membrane strain energy does not satisfy the Baker-Ericksen inequalities in contrast to the Biot-plate model (3.4.60)! The significance of this statement can be seen as follows. Take $\omega = [-1,1] \times [-1,1]$ and consider zero external loads and boundary conditions for m on $\partial\omega$ which uniformly shrink the plate: $m_{|\partial\omega}(x,y) = \overline{B}.(x,y)^T$, $\overline{B} \in \mathrm{GL}^+(2,\mathbb{R})$. Now take a sequence of minimizing deformations m_k with $\vec{n}_m^k = e_3 = \text{const}$, i.e. $m_3^k(x,y) \equiv 0$. The sequence m_k is naturally bounded in $H^{1,4}(\omega)$. Hence a subsequence converges weakly: $m_k \rightharpoonup \tilde{m} \in H^{1,4}(\omega)$. The minimizing sequence can be chosen such that $\nabla\tilde{m} = \overline{B}$. However $I(m_k) \to 0$ but $I(\tilde{m}) > 0$. Thus the homogeneously shrinked plate is not energy-minimal,

---

[29] Or other constitutive requirements [LS98, p.476]. Indeed there is no general agreement as to what really constitutes an isotropic Kirchhoff-Love plate theory [LS98, p.xiii] and [Kil65]. One encompassing independent statement to obtain Kirchhoff-Love in an engineering context may read: **i. normals remain straight and normal to the midsurface (but may be extended), ii. plane stress, iii. the elastic plate energy is additively decoupled in membrane and curvature parts.** Formal energy projection would also yield indefinite mixed products like $\langle I_m - 1\!\!1, II_m\rangle$.

[30] not to be confused with the nonlinear, non frame-indifferent, Kirchhoff-Love plate model given in [Cia97, p. 318] and mathematically justified in [Mon03].

[31] In fact, the condition $\vec{d} = \vec{n}$ can also be motivated by eliminating locally the free, extensible director $\varrho_m \cdot \vec{d}$ from the finite Reissner-Mindlin model through taking

$$
\begin{aligned}
&(\vec{d}, \varrho_m) := \operatorname{argmin}_{\varrho_m \in \mathbb{R}^+, \vec{d} \in \mathbb{S}^2, (\nabla m|\varrho_m \vec{d}) \in \mathrm{GL}^+(3,\mathbb{R})} W_{\mathrm{mp}}(\overline{C})\,, \qquad \overline{C} = (\nabla m|\varrho_m \vec{d})^T (\nabla m|\varrho_m \vec{d}) \quad \Rightarrow \\
&\quad \vec{d} = \vec{n}_m\,, \qquad \varrho_m^2 = 1 - \frac{\lambda}{(2\mu+\lambda)} \left[\|\nabla m\|^2 - 2\right]\,.
\end{aligned}
\tag{3.7.84}
$$

In doing so, no available three-dimensional information has been used. If instead, one defines a reduced membrane energy $W_0 : \mathbb{M}^{2\times 3} \to \mathbb{R}$ without recourse to a specific kinematical ansatz as in [DR95b, p.573] and **without invertibility constraint**

$$
\begin{aligned}
&W_0(\nabla m) := \inf_{\eta \in \mathbb{R}^3} W_{\mathrm{mp}}((\nabla m|\eta)^T (\nabla m|\eta)) = \inf_{\widehat{\varrho}_m \in \mathbb{R}, \vec{d} \in \mathbb{S}^2} W_{\mathrm{mp}}((\nabla m|\widehat{\varrho}_m \vec{d})^T (\nabla m|\widehat{\varrho}_m \vec{d})) \Leftrightarrow \\
&\langle D_F W_{\mathrm{mp}}((\nabla m|\widehat{\varrho}_m \vec{d})^T (\nabla m|\widehat{\varrho}_m \vec{d})), (0|0|\delta)\rangle = 0 \quad \forall\, \delta \in \mathbb{R}^3 \Leftrightarrow \quad S_1((\nabla m|\widehat{\varrho}_m \vec{d})).e_3 = 0, \quad \textbf{plane stress} \Rightarrow \quad \vec{d} = \vec{n}\,, \\
&\widehat{\varrho}_m = \begin{cases} \varrho_m & 1 - \frac{\lambda}{(2\mu+\lambda)} \left[\|\nabla m\|^2 - 2\right] \geq 0\,, \quad (\nabla m|\widehat{\varrho}_m \vec{n}) \in \mathrm{GL}^+(3,\mathbb{R}) \\ 0 & 1 - \frac{\lambda}{(2\mu+\lambda)} \left[\|\nabla m\|^2 - 2\right] < 0\,, \quad (\nabla m|\widehat{\varrho}_m \vec{n}) \notin \mathrm{GL}^+(3,\mathbb{R}) \end{cases}, \quad W_0(\nabla m) = W_{\mathrm{mp}}((\nabla m|\widehat{\varrho}_m \vec{n})^T (\nabla m|\widehat{\varrho}_m \vec{n}))\,,
\end{aligned}
\tag{3.7.85}
$$

then zero normal tractions $S_{2,33}(\nabla m|\widehat{\varrho}_m \vec{n}) = 0$ are not satisfied for $1 - \frac{\lambda}{(2\mu+\lambda)} \left[\|\nabla m\|^2 - 2\right] < 0$, which shows the unphysical behaviour, cf. [DR96, DR95c, DR00].

which it clearly should be, given the stabilization inherent through $\vec{n}_m^k = e_3$. This deficiency must be seen as unphysical and will be called **in-plane failure**.

Thus it is motivated why it is not known whether minimization based on (3.7.83) does admit a solution for arbitrary data. Even the inclusion of the classical bending term might not be enough: the control of only certain second derivatives of m does not suffice to treat the highly nonlinear problem by a compactness argument and to pass to the limit by strong convergence in the non-elliptic membrane part. The above example suggests that the in-plane failure is somehow related to the absence of drill-rotations.

Moreover, the very feasibility of a Kirchhoff-Love ansatz with thickness stretch places a **restriction on the kinematics** in the sense that it must be guaranteed for the membrane deformation that

$$1 - \frac{\lambda}{(2\mu+\lambda)}\operatorname{tr}\left[\overline{C} - 1\!\!1\right] \geq 0 \Rightarrow \|\nabla m\|^2 \leq 3 + \frac{2\mu}{\lambda} \Leftrightarrow \operatorname{tr}\left[\overline{C} - 1\!\!1\right] < 1 + \frac{2\mu}{\lambda}, \tag{3.7.86}$$

in the absence of tractions. This condition figures in [Cia97, p.355] among others, under which the quasiconvex hull of the membrane energy $W_{\mathrm{mp}}(\overline{C})$ in (3.7.83) coincides with the energy itself. In our derivation, condition (3.7.86) is, as a mathematical consequence of a **physical requirement** from the three-dimensional problem (3.6.78), most natural. It has also appeared in [FRS93, p.180] where it is believed to be '...unduly restrictive' due to the shortcomings of the SVK energy. While the shortcomings of the SVK energy are well known, similar restrictions occur most natural also for our Cosserat model, there in the form $\varrho_m > 0$, dictated by invertibility of the reconstructed shell deformation. The physical significance of the computed solution is thus tied to $\varrho_m > 0$, which in turn expresses as well the physical (not mathematical) requirement $S_{2,33}(x, y, \pm h/2) = 0$. Looking for solutions with $\varrho_m = 0$ is, physically speaking, not realistic.[32]

Remark 3.7.1
The problem of the non-ellipticity in the case of the Kichhoff-Love theory has been dealt with in Le Dret and Raoult [DR95b, DR00]. They perform the thin shell limit analysis based on the St.Venant-Kirchhoff density. As a result, they get that the limit energy deformation (of the 3D-model) is independent of the transverse variable and minimizes a limit energy computed as the Γ-limit [Mas92] of the 3D-(St.Venant-Kirchhoff) energy. The limit stored energy is again that of a nonlinearly elastic 'membrane' shell, in the sense that it contains only first derivatives of the unknown deformation m of the midsurface. However, it turns out that the limit energy offers no resistance to compression, a feature that is appropriate only for 'soft' elastic materials like a deflated baloon or a sail but in our oppinion inacceptable for 'almost rigid' materials like metals or paper, the topic whe are interested in since the rigidity translates directly into the small strain assumption.[33] The non resistance to compression in the above analysis is related to the use of the quasiconvex hull [34] QW_0 of the reduced St.Venant Kirchhoff energy W_0 in (3.7.85), which, surprisingly enough, can be given in closed form [DR95c, HP96] and which shows to be in general positive but zero in the compression range [35] since St.Venant Kirchhoff typically looses ellipticity there. These remarks indicate that results based on Γ-convergence and global minimization are not in all cases the appropriate direction to take, certainly not for almost rigid materials.

However, given all these deficiencies of the finite Kirchhoff-Love model, anticipating that $\nabla m^T \nabla m - 1\!\!1_2$ is small (appropriate for almost rigid materials) as in (3.4.64), a modification of the bending term is suggested: we modify

$$\mathfrak{K}_b = (\nabla m|\vec{n}_m)^T(\nabla \vec{n}_m|0) = II_m \in \mathrm{Sym} \Rightarrow \begin{pmatrix} -\|m_{xx}\| & -\|m_{xy}\| & 0 \\ -\|m_{yx}\| & -\|m_{yy}\| & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Sym}\,. \tag{3.7.87}$$

Remark 3.7.2 (Motivation)
The motivation of this modification for thin shells is as follows: either the **membrane energy is non-zero**, *in which case it dominates and the* **bending contribution can be neglected** *or the* **membrane energy is zero** *in which case the* **bending term** *of* (3.7.83) **coincides** *with that of* (3.7.87), *see Lemma 3.10.7 and compare to [GKM96].*

---

[32] One might be inclined to think that the apparent problem of non-ellipticity of the membrane expression is only related to the use of the non-elliptic parent SVK-energy. This is not the case. Proceeding by energy projection from a **polyconvex** Neo-Hooke energy, the resulting membrane energy is again non-elliptic. This is well known feature, [DR95b, p.560,iii].

[33] They remark [DR95b, p.550]: "...then the corresponding nonlinear membranes offer no resistance to crumpling. This is an empirical fact, witnessed by anyone who ever played with a deflated ballon."

[34] "... the fact that this function ($W_{\mathrm{mp}}(\overline{C})$) is not quasiconvex already implied that it had to be relaxed in order to give rise to a well posed problem."[DR95b, p.575].

[35] Strictly speaking, the use of the quasiconvex hull leads to a so called **tension field theory** [Ste90]. Steigmann himself [Ste90, p.143] notes "A question then arises concerning the validity of tension filed theory as an approximation to a theory of shells with bending stiffness that is small in some sense. Evidently, the deformation is not well described, though the theory delivers solutions that approximate the average of the deformation observed in a real membrane containing many wrinkles. We conjecture that the stress is accurately predicted, however."

A formulation based on this modification supports an existence theorem.

Theorem 3.7.3 (Existence for finite almost rigid KL-plate)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^2(\omega, \mathbb{R}^3)$. Moreover, let $\overline{f} \in L^2(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (3.9.102). *Then* (3.7.83) *with the modification* (3.7.87) *admits at least one minimizing solution $m \in H^2(\omega, \mathbb{R}^3)$.*

Proof. We apply the direct methods of variations. The functional I is bounded above and below. We may choose an infimizing sequence m^k. Due to the boundary conditions and Poincaré's inequality the sequence is bounded in $H^2(\omega)$. The compact embedding $H^2(\omega) \subset W^{1,4}(\omega)$ shows that we may choose a weakly convergent subsequence, not relabelled, such that strongly $\nabla m^k \to \nabla m \in L^4(\omega)$. The weak limit is a minimizer since the bending term is convex in the second derivatives and the nonlinear, non-quasiconvex membrane term is handled by strong convergence. The modified bending term imparts as well additional control for in-plane deformations. ∎

Such a theorem might not be of much practical value because it is precisely the level of smoothness we want to avoid and it must be noted that the proposed modification of the bending term is **not consistent** with the classical Kirchhoff-Love theory upon linearization!

3.7.2 The pure finite bending Kirchhoff-Love problem

Assume that the boundary conditions for the plate support multiple finite bending modes, i.e. the membrane energy is zero, hence $I_m = 1\!\!1$. What can we say about the corresponding degenerated minimization problem based on the remaining term involving only curvature? The variational problem for the clamped plate reads then

$$\begin{aligned}\inf\{ \int_\omega \frac{h^3}{12} \left[\mu \, \|II_m\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \operatorname{tr}\left[II_m\right]^2\right] \mathrm{d}\omega, \; m \in H^2(\omega, \mathbb{R}^3) : \nabla m^T \nabla m = 1\!\!1_2, \\ m_{|\gamma_0} = g_{\rm d}(x,y,0), \, \partial_\nu m_{|\gamma_0} = \partial_\nu g_{{\rm d}\,|\gamma_0}, \, g_{\rm d} \in H^2(\omega, \mathbb{R}^3) : (\nabla g_{\rm d} | \vec{n}_{g_{\rm d}}) \in \mathrm{SO}(3)\} \, .\end{aligned} \tag{3.7.88}$$

Here ∂_ν are normal derivatives at the boundary. The proposed system coincides with that previously derived by [GKM96, p.44] apart from a modified material parameter $\lambda \mapsto \frac{2\mu\lambda}{2\mu+\lambda}$. Note that under pure bending of a plate, we have for the Gauss curvature $K = 0$ and using (3.10.139) we get, by adding zero, equivalently

$$\begin{aligned}&\frac{h^3}{12}\left(\mu \, \|II_m\|^2 + \frac{\lambda}{2}\frac{2\mu}{(2\mu+\lambda)} \operatorname{tr}\left[II_m\right]^2 + 2\mu \underbrace{\frac{\operatorname{tr}\left[II_m\right]^2 - \|II_m\|^2}{2\det[I_m]}}_{K=0}\right) = \frac{h^3}{12}\left(\mu + \frac{\lambda}{2}\frac{2\mu}{(2\mu+\lambda)}\right)\operatorname{tr}\left[II_m\right]^2 \\ &= \frac{h^3}{12} 2\mu \, \frac{\mu+\lambda}{2\mu+\lambda} \operatorname{tr}\left[II_m\right]^2 = \frac{h^3}{12}\frac{1}{2}\frac{E}{1-\nu^2}\operatorname{tr}\left[II_m\right]^2 ,\end{aligned} \tag{3.7.89}$$

on using (3.10.187). Inserting the linearized quantity $\operatorname{tr}\left[II_m\right]^2 \approx \|\Delta v_3\|^2 + \ldots$ we obtain

$$\underbrace{\frac{h^3}{12}\frac{1}{2}\frac{E}{1-\nu^2}}_{\text{flexural rigidity}} \|\Delta v_3\|^2 , \tag{3.7.90}$$

the classical infinitesimal plate bending energy leading to the biharmonic equation. It is possible to show that the finite minimization problem admits at least one solution.

Theorem 3.7.4 (Existence for pure bending finite KL-plate)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\rm d} \in H^2(\omega, \mathbb{R}^3)$. Then (3.7.88) *admits at least one minimizing solution $m \in H^2(\omega, \mathbb{R}^3)$.*

Proof. The proof is based on the crucial observation that on the space of admissible functions, the energy coincides with the quadratic expression

$$\int_\omega \frac{h^3}{12} \left[\mu \, \|\nabla \vec{n}\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \left(\|m_{xx}\| + \|m_{yy}\|\right)^2\right] \mathrm{d}\omega \, . \tag{3.7.91}$$

Standard arguments of the direct method of variations finish the proof. A detailed presentation was given in [Cia97, p.347]. ∎

Again, the level of smoothness is discomforting.

Corollary 3.7.5 (Existence for pure bending finite RM-plate)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume that the boundary data $g_{\mathrm{d}} \in H^2(\omega, \mathbb{R}^3)$ supports bending modes. Then the pure finite Reissner-Mindlin bending problem based on (3.6.82) admits at least one minimizing solution $m \in H^2(\omega, \mathbb{R}^3)$ and $\vec{d} = \vec{n}_m \in H^1(\omega, \mathbb{S}^2)$.

Remark 3.7.6 (Pure bending problems)
Note that all presented pure finite bending problems **coincide** *for the new Cosserat model, the new finite Reissner-Mindlin model and the finite Kirchhoff-Love model. The last two results show that the classical finite bending terms provide enough control in pure bending for models, in which the membrane part would have been non-elliptic. However, the classical bending terms are insufficient to stabilize joint membrane and bending situations.*

3.8 Discussion

Starting from a fully invariant three-dimensional physically linear Cosserat theory with independent rotations and size effects, we have obtained a family of fully invariant, finite Cosserat plates by means of assumed kinematics and energy projection. The models include in a natural way drilling degrees of freedom and size effects (smaller samples with the same relative thickness are relatively stiffer than corresponding larger samples). Since the assumed kinematics is quadratic through the thickness, one avoids the so called Poisson thickness locking. In contrast to traditional reduced theories, the membrane part is uniformly elliptic and allows a non-degenerate passage to zero relative thickness. The balance equations for the midsurface are not only uniformly Legendre-Hadamard elliptic, but linear at given rotations.

For vanishing Cosserat couple modulus $\mu_c = 0$, the formulation is shown to be downwards compatible with traditional infinitesimal linear Reissner-Mindlin theories and shear-correction factor $\kappa = 1$.

A detailed mathematical analysis of the resulting two-dimensional models is proposed which closely follows the three-dimensional ideas. It is based on a correspondingly dimensionally reduced version of a new extended Korn's first inequality. We have achieved a surprising unification of two- and three-dimensional concepts.

From a mechanical point of view, compared to more traditional, non-elliptic finite Reissner-Mindlin and Kirchhoff-Love models, it seems to be the beneficial influence of the drill-rotations in conjunction with the internal length $L_c > 0$ which stabilizes the new Cosserat thin plate model.

Certain limit cases related to Sobolev-embedding theorems must remain open for the moment, notably the case IV including possible fracture of the plate. They leave a wide field of challenging new mathematical problems.

A modification of the new Cosserat plate model is also proposed, which ensures invertibility of the reconstructed deformation gradient and which allows as well for minimizers. This model shows the most reasonable physical behaviour, but is not easily seen to be obtained by direct descend from three-dimensions.

While we have large freedom of specifying boundary conditions for the rotations at the Dirichlet boundary, we prefer a generalization of the three-dimensional consistent coupling condition, which includes as a special case prescriptions corresponding to clamping.

A major conceptual advantage of the new proposed model is the appearance of rotations already in the three-dimensional **parent** model. There is no need to artificially introduce independent directors of the plate.

In a subsequent contribution, it will be shown that the proposed method can be easily extended to shells and multiplicative elasto-plasticity with the possibility of exactly the same mathematical analysis in the elastic case.

From a numerical point of view, the new Cosserat plate model offers the highly welcome perspective to use only C^0-conforming finite elements. When interpolating the midsurface deformation one order higher than the rotations, shear locking should be avoided.

It remains to completely justify the apparently sound, new finite Cosserat thin plate model by means of either a convergence proof for vanishing relative thickness to the underlying three-dimensional parent Cosserat model or by showing, that a suitably rescaled three-dimensional problem Γ-converges to one of the two-dimensional limit problems.

Let us summarize and relate some basic features of the obtained new plate models. We abbreviate LH: Legendre-Hadamard elliptic, BE: Baker-Ericksen inequalities, dof: degrees of freedom, invariance: fully frame-indifferent, v: midsurface displacement, $\vec{d}$: unit director, $\vec{n}$: unit normal of the midsurface, θ: infinitesimal director, invertibility: local invertibility of the reconstructed deformation in the sense of a strictly positive determinant of the deformation gradient almost everywhere, pure bending: the problem obtained by restricting considerations to locally length preserving deformations (inextensional).

It can be seen, that **linearization does not always commute with dimensional reduction**. From a modelling point of view it is clear, however, that linearization is the last step to be performed. The unifying role of setting $\mu_c = 0$ is also appreciated.

new finite 3D-Cosserat parent model (3.3.6), invariance (+), invertibility (+/-), LH(+), BE(+), size effects (+), existence (+), uniqueness (-), higher regularity (?), indep. rotations (+), symmetric stress (-), dof (6)

$\mu_c = 0$, linearized

classical infinit. 3D linear elasticity, existence (+), uniqueness (+), higher regularity (+), symmetric stress (+), dof (3)

linearized

classical finite 3D SVK-elasticity, invariance (+), invertibility (-), LH (-), BE(-), existence (?), uniqueness (-), higher regularity (?), size effects (-), symmetric stress (+), dof (3)

dimensional reduction: assumed kinematics and energy projection

dimensional reduction: energy projection or asymptotic methods

dimensional reduction: formal asymptotic methods or energy projection

new finite 2D-Cosserat plate model (3.4.42), invariance (+), invertibility (-), LH(+), BE(+), size effects (+), existence (+), uniqueness (-), higher regularity (?), error estimates (?), thin plate limit (+), transverse shear (+), drill rotations (+), symmetric stress (-), pure bending (+), dof (6)

$\mu_c = 0$, linearized, $L_c^{2+\delta}$

classical infinit. 2D KL-plate (3.9.130), invariance (-) , existence (+), uniqueness (+), higher regularity (+), error estimates (+), size effects (-), drill rotations (-), symmetric stress (+), dof (3)

linearized

finite 2D KL-plate model (3.7.83), invariance (+), invertibility (-), LH(-), BE(-), existence (?), uniqueness (-), higher regularity (?), size effects (-), thin plate limit (-), transverse shear (-), drill rotations (-), symmetric stress (+), pure bending (+), dof (3)

physically motivated modification:

new finite 2D-Cosserat plate model (3.5.70), invariance (+), invertibility (+), LH(+), BE(+), polyconvex (+), size effects (+), existence (+), uniqueness (-), higher regularity (?), error estimates (?), thin plate limit (+), transverse shear (+), drill rotations (+), symmetric stress (-), pure bending (+), dof (6)

solution of RM_{lin}, KL_{lin} converges as $h \to 0$ to solution of $3D.SVK_{\text{lin}}$

constrain $\theta = (v_{3,x}, v_{3,y})^T$

constrain $\vec{d} = \vec{n}$

$\mu_c = 0$, linearized, $L_c^{2+\delta}$

classical infinit. 2D RM-plate (3.9.126), shear correction $\kappa = 1$, invariance (-), transverse shear (+), existence (+), uniqueness (+), higher regularity (+), error estimates (+), size effects (-), thin plate limit (-), drill rotations (-), symmetric stress (-), dof (5)

linearized

new finite 2D-RM plate model (3.6.82), invariance (+), invertibility (-), LH(-), BE(-), transverse shear (+), existence (?), uniqueness (-), higher regularity (?), error estimates (?), size effects (-), thin plate limit (-), drill rotations (-), symmetric stress (-), pure bending (+), dof (5)

new finite 2D-Biot plate model (3.4.60), invariance (+), invertibility (-), LH(-), BE(+), transverse shear (-), existence (+), uniqueness (-), higher regularity (?), error estimates (?), size effects (+), thin plate limit (+), drill rotations (-), symmetric stress (+), pure bending (+), dof (3), linearized: $L_c^{2+\delta} \to$ classical KL

3.9 Appendix A

3.9.1 The treatment of external loads

Dead load body forces for the thin plate

In the three-dimensional theory the dead load body forces $f(x,y,z) \in \mathbb{R}^3$ were simply included by appending the potential with the term

$$\int_{\Omega_h} f(x,y,z)\cdot\varphi(x,y,z)\,\mathrm{dV}\,. \tag{3.9.92}$$

Inserting the ansatz for the reconstructed deformation φ_s results in

$$\begin{aligned}\int_{\Omega_h} f(x,y,z)\cdot\varphi_s(x,y,z)\,\mathrm{dV} &\approx \int_{\Omega_h} f(x,y,z)\cdot\Big[m(x,y)+z\,\varrho_m\overline{R}_3+\frac{z^2}{2}\varrho_b\,\overline{R}_3\Big]\,\mathrm{dV}\\ &= \int_\omega h\,\hat{f}(x,y)\cdot m(x,y)\,\mathrm{d}\omega+\int_\omega\Big(\int_{-h/2}^{h/2} z\,f(x,y,z)\,\mathrm{dz}\Big)\varrho_m\overline{R}_3\,\mathrm{d}\omega+\int_\omega\Big(\int_{-h/2}^{h/2}\frac{z^2}{2}\,f(x,y,z)\,\mathrm{dz}\Big)\varrho_b\overline{R}_3\,\mathrm{d}\omega\end{aligned} \tag{3.9.93}$$

Let us define

$$\hat{f}_0(x,y):=\int_{-h/2}^{h/2} f(x,y,z)\,\mathrm{dz}\,,\quad \hat{f}_1(x,y):=\int_{-h/2}^{h/2} z\,f(x,y,z)\,\mathrm{dz}\,,\quad \hat{f}_2(x,y):=\int_{-h/2}^{h/2}\frac{z^2}{2}\,f(x,y,z)\,\mathrm{dz}\,, \tag{3.9.94}$$

such that $\hat{f}_0$, $\hat{f}_1$, $\hat{f}_2$ are the zero, first, second moment of f in thickness direction. Moreover

$$\begin{aligned}\int_{-h/2}^{h/2}\frac{z^2}{2}\,f(x,y,z)\,\mathrm{dz} &= \int_{-h/2}^{h/2}\frac{z^2}{2}\,(f(x,y,0)+z\,\partial_z f(x,y,0)+\ldots)\,\mathrm{dz}=\frac{h^3}{24}f(x,y,0)+O(h^5)\\ \int_{-h/2}^{h/2} z\,f(x,y,z)\,\mathrm{dz} &= \int_{-h/2}^{h/2} z\,(f(x,y,0)+z\,\partial_z f(x,y,0)+\ldots)\,\mathrm{dz}=0+\frac{h^3}{12}\partial_z f(x,y,0)+O(h^5)\,.\end{aligned} \tag{3.9.95}$$

Therefore

$$\int_{\Omega_h} f(x,y,z)\cdot\varphi_s(x,y,z)\,\mathrm{dV}\approx\int_\omega\hat{f}_0(x,y)\cdot m(x,y)\,\mathrm{d}\omega+\int_\omega\hat{f}_1(x,y)\varrho_m\overline{R}_3\,\mathrm{d}\omega+\int_\omega\hat{f}_2(x,y)\varrho_b\overline{R}_3\,\mathrm{d}\omega \tag{3.9.96}$$

Traction boundary conditions for the thin plate

In the three-dimensional theory the traction boundary forces $N(x,y,z)\in\mathbb{R}^3$ were simply included by appending the potential with the term

$$\int_{\partial\Omega_h^{\mathrm{trans}}\cup\{\gamma_s\times[-\frac{h}{2},\frac{h}{2}]\}} N(x,y,z)\cdot\varphi(x,y,z)\,\mathrm{dS}\,. \tag{3.9.97}$$

Inserting our ansatz for the reconstructed deformation φ_s results in

$$\begin{aligned}\int_{\partial\Omega_h^{\mathrm{trans}}\cup\{\gamma_s\times[-\frac{h}{2},\frac{h}{2}]\}} N(x,y,z)\cdot\varphi_s(x,y,z)\;\mathrm{dS}\;&\approx\int_{\omega\times\{-\frac{h}{2},\frac{h}{2}\}} N(x,y,z)\cdot\Big[m(x,y)+z\varrho_m\overline{R}_3+\frac{z^2}{2}\varrho_b\overline{R}_3\Big]\;\mathrm{dS}\\ &+\int_{\gamma_s\times[-\frac{h}{2},\frac{h}{2}]} N(x,y,z)\cdot\Big[m(x,y)+z\varrho_m\overline{R}_3+\frac{z^2}{2}\varrho_b\overline{R}_3\Big]\;\mathrm{dS}.\end{aligned}$$

Let us define

$$\hat{N}_{\mathrm{lat},0}(x,y):=\int_{-h/2}^{h/2} N(x,y,z)\,\mathrm{dz}\,,\quad \hat{N}_{\mathrm{lat},1}(x,y):=\int_{-h/2}^{h/2} z\,N(x,y,z)\,\mathrm{dz}\,,\quad \hat{N}_{\mathrm{lat},2}(x,y):=\int_{-h/2}^{h/2}\frac{z^2}{2}\,N(x,y,z)\,\mathrm{dz}\,, \tag{3.9.98}$$

such that $\hat{N}_{\mathrm{lat},0}$, $\hat{N}_{\mathrm{lat},1}$, $\hat{N}_{\mathrm{lat},2}$ are the zero, first, second moment of the tractions N at the lateral boundary in thickness direction. Hence

$$\begin{aligned}\int_{\partial\Omega_h} N(x,y,z)\cdot\varphi(x,y,z)\,\mathrm{dS} &\approx \int_\omega[N(x,y,\frac{h}{2})+N(x,y,-\frac{h}{2})]\cdot m(x,y)\,\mathrm{d}\omega+\int_\omega\frac{h}{2}[N(x,y,\frac{h}{2})-N(x,y,-\frac{h}{2})]\varrho_m\overline{R}_3\,\mathrm{d}\omega\\ &+\int_\omega[\frac{h^2}{8}N^+ +\frac{h^2}{8}N^-]\varrho_b\overline{R}_3\,\mathrm{d}\omega+\int_{\gamma_s}\hat{N}_{\mathrm{lat}}(x,y)\cdot m(x,y)\,\mathrm{ds}+\int_{\gamma_s}\hat{N}_{\mathrm{lat},1}(x,y)\,\varrho_m\overline{R}_3\,\mathrm{ds}+\int_{\gamma_s}\hat{N}_{\mathrm{lat},2}(x,y)\,\varrho_b\overline{R}_3\,\mathrm{ds}\\ &=\int_\omega N_{\mathrm{res}}(x,y)\cdot m(x,y)\;\mathrm{d}\omega+\int_\omega h\,N_{\mathrm{diff}}(x,y)\varrho_m\overline{R}_3\,\mathrm{d}\omega+\int_\omega\frac{h^2}{8}N_{\mathrm{res}}\varrho_b\overline{R}_3\,\mathrm{d}\omega\\ &+\int_{\gamma_s}\hat{N}_{\mathrm{lat},0}(x,y)\cdot m(x,y)\,\mathrm{ds}+\int_{\gamma_s}\hat{N}_{\mathrm{lat},1}(x,y)\,\varrho_m\overline{R}_3\,\mathrm{ds}+\int_{\gamma_s}\hat{N}_{\mathrm{lat},2}(x,y)\,\varrho_b\overline{R}_3\,\mathrm{ds}\,,\end{aligned} \tag{3.9.99}$$

with

$$N_{\mathrm{res}}:=[N(x,y,\frac{h}{2})+N(x,y,-\frac{h}{2})]\,,\quad N_{\mathrm{diff}}:=\frac{1}{2}[N(x,y,\frac{h}{2})-N(x,y,-\frac{h}{2})]\,. \tag{3.9.100}$$

The external loading functional

Let us gather all influences of the external loading terms. It would be possible to account for all appearing influences, however, in view of a reasonable simplification we consider only those terms, which would have appeared, if we had made the restricted linear ansatz without thickness stretch $\varphi_s = m + z\,\overline{R}_3$. To leading order we have the

$$\begin{aligned}
\overline{f} &= \hat{f}_0 + N_{\mathrm{res}}\,, && \textbf{resultant body force} \\
\overline{M} &= \hat{f}_1 + h\,N_{\mathrm{diff}}\,, && \textbf{resultant body couple} \\
\overline{N} &= \hat{N}_{\mathrm{lat},0}\,, && \textbf{resultant surface traction} \\
\overline{M}_c &= \hat{N}_{\mathrm{lat},1}\,, && \textbf{resultant surface couple}\,.
\end{aligned} \tag{3.9.101}$$

The **resultant loading functional** Π is given by

$$\Pi(m, \overline{R}_3) = \int_\omega \langle \overline{f}, m\rangle + \langle \overline{M}, \overline{R}_3\rangle\,\mathrm{d}\omega + \int_{\gamma_s} \langle \overline{N}, m\rangle + \langle \overline{M}_c, \overline{R}_3\rangle\,\mathrm{ds}\,. \tag{3.9.102}$$

If we denote the dependence of Π on the loads of the underlying three-dimensional problem as $\Pi(f, N;\, m, \overline{R}_3)$, then it is easily seen that frame-indifference of the external loading functional is satisfied in the sense that $\Pi(Q.f, Q.N;\, Q.m, Q.\overline{R}_3) = \Pi(f, N;\, m, \overline{R}_3)$ for all rigid rotations $Q \in \mathrm{SO}(3,\mathbb{R})$. It is possible to use the **same functional form** of the loading functional **for all finite and linearized models**. We only need to replace $(m, \overline{R}_3)$ by $(m, \vec{d})$, $(m, \vec{n})$, $(v, \overline{A}_3)$ for the different finite and linearized models, respectively.

3.9.2 Transformation of the domain and scaling

Classical finite elasticity

Set $\Omega_L^{\mathrm{rel.thin}} = [0, L[\mathrm{m}]] \times [0, L[\mathrm{m}]] \times [-\frac{h}{2}\cdot L, \frac{h}{2}\cdot L]$ with h a small parameter indicating the **relative thickness** of the domain, e.g. $h \in (0, \frac{1}{20}[\mathrm{m}]]$ with dimension length. The three-dimensional problem with respect to the relatively thin domain $\Omega_L^{\mathrm{rel.thin}}$ reads

$$\int_{\xi\in\Omega_L^{\mathrm{rel.thin}}} W_{3D}(\nabla_\xi \varphi_L(\xi)) - \langle f_L(\xi), \varphi_L(\xi)\rangle\,\mathrm{d}\xi - \int_{\partial\Omega_L^{\mathrm{rel.thin}}} \langle N_L, \varphi_L\rangle\,\mathrm{dS}_L \mapsto \min\,. \quad \text{w.r.t. } \varphi_L\,, \tag{3.9.103}$$

where we are looking for a function $\varphi_L : \Omega_L^{\mathrm{rel.thin}} \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$. Introducing the **scaling transformation**

$$\zeta : \Omega_h = [0,1[\mathrm{m}]] \times [0,1[\mathrm{m}]] \times [-\frac{h}{2}, \frac{h}{2}] \subset \mathbb{R}^3 \mapsto \Omega_L^{\mathrm{rel.thin}} \subset \mathbb{R}^3\,, \qquad \zeta(x) = L\cdot x\,, \tag{3.9.104}$$

(note that L itself is non-dimensional here) this turns into

$$\int_{x\in\Omega_h} \left[W_{3D}(\nabla\zeta(x)\nabla\varphi(x)\nabla\zeta^{-1}(x)) - \langle f_L(\zeta(x)), L\cdot\varphi(x)\rangle\right]\,\det[\nabla\zeta(x)]\,\mathrm{dV} \tag{3.9.105}$$

$$- \int_{\partial\Omega_h} \langle N_L(\zeta(x), L\cdot\varphi(x)\rangle\,\|\operatorname{Cof}\nabla\zeta.e_3\|\,dS_h \mapsto \min\,. \quad \text{w.r.t. } \varphi.\,, \tag{3.9.106}$$

for a function $\varphi : \Omega_h \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined implicitly through $\varphi_L(\xi) = \zeta(\varphi(\zeta^{-1}(\xi)))$. With $f(x) = L\cdot f_L(\zeta(x))$, $N(x) = N_L(\zeta(x))$ we have

$$\int_{x\in\Omega_h} \left[W_{3D}(\nabla\varphi) - \langle f, \varphi\rangle\right]\,L^3\,\mathrm{dV} - \int_{\partial\Omega_h} L\,\langle N, \varphi\rangle\,L^2\,\mathrm{dS} \mapsto \min\,. \quad \text{w.r.t. } \varphi.\,, \tag{3.9.107}$$

or equivalently

$$\int_{x\in\Omega_h} \left[W_{3D}(\nabla\varphi) - \langle f, \varphi\rangle\right]\,\mathrm{dV} - \int_{\partial\Omega_h} \langle N, \varphi\rangle\,\mathrm{dS} \mapsto \min\,. \quad \text{w.r.t. } \varphi.\,, \tag{3.9.108}$$

which shows how the scaling from a domain which is relatively thin to a domain which is absolutely thin is to be performed in order to apply the subsequent dimensional reduction procedure.

Scaling relations for finite Cosserat models with internal length

For completeness let us summarize the scaling relations appearing in a finite elastic Cosserat theory. Our goal is to relate the response of large and small samples of the same material and to asses the influence of the characteristic length L_c.

First, in our definition, the **characteristic length** L_c is a given material parameter, corresponding to the **smallest discernable distance** to be accounted for in the model. A simple consequence is that actual geometrical dimensions L of the bulk material must be larger than L_c, indeed for a continuum theory to apply L should be significantly larger than L_c.

Now let $\Omega_L = [0, L[\mathrm{m}]] \times [0, L[\mathrm{m}] \times [0, L[\mathrm{m}]]$ be the cube with (non-dimensional) edge length L, representing the bulk material. Consider a deformation $\varphi_L : \xi \in \Omega_L \mapsto \mathbb{R}^3$ and microrotation $\overline{R}_L(\xi) : \Omega_L \mapsto \mathrm{SO}(3,\mathbb{R})$ as solution of the simplified minimization problem

$$\int_{\xi\in\Omega_L} \mu\,\|\overline{R}_L^T(\xi)F(\xi) - 1\!\!1\|^2 + \mu\,L_c^q\,\|\mathrm{D}_\xi\overline{R}_L(\xi)\|^q\,\mathrm{d}\xi \mapsto \ \min.\,\text{w.r.t. } (\varphi_L, \overline{R}_L)\,. \tag{3.9.109}$$

The simple scaling transformation $\zeta : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $\zeta(x) = L\cdot x$ maps the unit cube $\Omega_1 = [0,1[\mathrm{m}]] \times [0,1[\mathrm{m}]] \times [0,1[\mathrm{m}]]$ into Ω_L. Defining the related deformation $\varphi : x \in \Omega_1 \mapsto \mathbb{R}^3$ and microrotation $\overline{R}(x) : \Omega_1 \mapsto \mathrm{SO}(3,\mathbb{R})$ as

$$\varphi(x) := \zeta^{-1}\left(\varphi_L(\zeta(x))\right)\,, \quad \overline{R}(x) := \overline{R}_L(\zeta(x))\,, \tag{3.9.110}$$

shows

$$\nabla_x \varphi(x) = \frac{1}{L}\nabla_\xi \varphi_L(\zeta(x))\nabla_x \zeta(x) = \nabla_\xi \varphi_L(\xi)\,, \qquad \mathrm{D_x}\overline{R}(x) = \mathrm{D}_\xi \overline{R}_L(\zeta(x))\cdot \nabla_x \zeta(x) = \mathrm{D}_\xi \overline{R}_L(\xi)\cdot L\,. \tag{3.9.111}$$

Hence, the minimization problem can be transformed

$$\int_{\xi\in\Omega_L} \mu\, \|\overline{R}_L^T(\xi)\nabla_\xi\varphi_L(\xi) - 1\!\!1\|^2 + \mu\, L_c^q\, \|\mathrm{D}_\xi \overline{R}_L(\xi)\|^q \,\mathrm{d}\xi = \int_{x\in\Omega_1} \mu\, \|\overline{R}^T(x)\nabla_x \varphi(x) - 1\!\!1\|^2 \det[\nabla_x\zeta(x)] + \mu\, L_c^q\, \|\frac{1}{L}\mathrm{D_x}\overline{R}(x)\|^q \det[\nabla_x \zeta(x)]\,\mathrm{dx}$$

$$= \int_{x\in\Omega_1} \mu\, \|\overline{R}^T(x)\nabla_x\varphi(x) - 1\!\!1\|^2\, L^3 + \mu\, L_c^q\, L^{3-q}\, \|\mathrm{D_x}\overline{R}(x)\|^q\,\mathrm{dx}\,, \tag{3.9.112}$$

and we may consider at last the problem defined on the unit cube Ω_1:

$$\int_{x\in\Omega_1} \mu\, \|\overline{R}^T(x)\nabla_x\varphi(x) - 1\!\!1\|^2 + \mu\, L_c^q\, L^{3-q-3}\, \|\mathrm{D_x}\overline{R}(x)\|^q\,\mathrm{dx} \mapsto \ \min.\ \text{w.r.t.}\ (\varphi, \overline{R}). \tag{3.9.113}$$

Comparison of different sample sizes is afforded by transformation to the unit cube respectively, e.g. we compare two samples of the same material with bulk sizes $L_1 > L_2$. Transformation to the unit cube shows that the response of sample two is stiffer than the response of sample one.

It is plain to see that for L large compared to L_c, the influence of the rotations will be small and in the limit $\frac{L_c}{L} \to 0$, classical behaviour results. Otherwise, the larger $\frac{L_c}{L} < 1$, the more pronounced the Cosserat effects become and a small sample is relatively stiffer than a large one.

Scaling relations for finite Cosserat plates

As a consequence for relatively thin shells of the former development we consider the finite problem on the relative thin domain $\Omega_L^{\mathrm{rel.thin}}$ in simplified form:

$$\int_{\xi\in\Omega_L^{\mathrm{rel.thin}}} \mu\, \|\overline{R}_L^T(\xi)\nabla_\xi\varphi_L(\xi) - 1\!\!1\|^2 + \mu\, L_c^q\, \|D_\xi \overline{R}_L(\xi)\|^q\,\mathrm{d}\xi \mapsto\ \min.\ \text{w.r.t.}\ (\varphi_L, \overline{R}_L). \tag{3.9.114}$$

This implies on $\Omega_h = \omega\times[-\frac{h}{2}, \frac{h}{2}]$ for the correspondingly transformed variables

$$\int_{x\in\Omega_h} \mu\, \|\overline{R}^T(x)\nabla_x\varphi(x) - 1\!\!1\|^2 + \mu\, \frac{L_c^q}{L^q}\, \|\mathrm{D_x}\overline{R}(x)\|^q\,\mathrm{dx} \mapsto\ \min.\ \text{w.r.t.}\ (\varphi, \overline{R}). \tag{3.9.115}$$

Inserting the reduced kinematics and integrating over the thickness we should consider on ω

$$\int_\omega \mu\, h\, \|\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1\|^2 + \frac{h^3}{12}\,\mu\|\overline{R}^T(\nabla\overline{R}_3|0)\|^2 + \mu\, h\, \frac{L_c^q}{L^q}\, \|\mathrm{D_x}\overline{R}(x)\|^q\,\mathrm{d}\omega \mapsto\ \min.\ \text{w.r.t.}\ (m, \overline{R}). \tag{3.9.116}$$

Comparing domains with the **same relative thickness** $h > 0$, but different in-plane elongation L, we see that the **smaller sample is relatively stiffer** for the same relative thickness.

For very large samples with the same relative thickness, the classical bending terms are retrieved.[36] In this sense, classical plate formulations represent the limit behaviour of ever larger, thin structures with the same relative thickness..

3.9.3 Generalized convexity conditions

For the convenience of the reader we collect some of the most useful convexity conditions. Let an elastic free energy density $W : \mathbb{M}^{n\times m} \mapsto \mathbb{R}$, $n \le m$ be given. We say that W considered in $\nabla\varphi = F \in \mathbb{M}^{n\times m}$ is

1. **uniformly stable**, if $D_F^2 W(F).(H,H) \ge c^+\, \|H\|^2$, $H \in \mathbb{M}^{n\times m}$
2. **strictly Legendre elliptic**, if $D_F^2 W(F).(H,H) > 0$, $\forall\, H \ne 0$
3. **pre-stable**, if $D_F^2 W(x,F).(H,H) \ge c^+\, \|(H|0)^T G(x) + G(x)^T (H|0)\|^2$, $\forall\,(H|0) \in \mathbb{M}^{m\times m}$, $H \ne 0$ with $G \in \mathrm{GL}(m,\mathbb{R})$.
4. **polyconvex**, if there exists a convex function $P : \mathbb{M}^{n\times m} \times \mathbb{M}^{n\times m} \times \mathbb{R} \mapsto \mathbb{R}$ such that $W(F) = P(F, \mathrm{Minors}_{ij})$.
5. **quasiconvex**, if
$$\forall\, \hat{F} \in \mathbb{M}^{n\times m}\ :\ |D|\cdot W(\hat{F}) \le \int_D W(\hat{F} + \nabla\phi(x))\,\mathrm{dx} \quad \forall\,\phi\in C_0^\infty(D,\mathbb{R}^m)\,, \tag{3.9.117}$$
which implies that the homogeneous deformation $\hat{F}$ is absolute minimizer to its own boundary conditions and excludes **internal failure**.
6. **uniformly Legendre-Hadamard elliptic**, if $D_F^2 W(F).(\xi\otimes\eta, \xi\otimes\eta) \ge c^+\, \|\xi\|^2_{\mathbb{R}^m}\cdot\|\eta\|^2_{\mathbb{R}^n}$
7. **Legendre-Hadamard elliptic**, if $D_F^2 W(F).(\xi\otimes\eta, \xi\otimes\eta) \ge 0$
8. **rank-one convex**, if $f(t) := W(F + t(\xi\otimes\eta))$ is convex in t for all $F \in \mathbb{M}^{n\times m}$.

It is known [Dac89] that

$$\textbf{convexity} \Rightarrow \textbf{polyconvexity} \Rightarrow \textbf{quasiconvexity} \Rightarrow \textbf{rank-one convexity} \Leftrightarrow \textbf{Legendre-Hadamard ellipticity}\,, \tag{3.9.118}$$

but the reverse implications are false in general. For the **scalar case** $\varphi(x_1,\ldots,x_n) \in \mathbb{R}$ and the **one dimensional case** $\varphi(x_1) \in \mathbb{R}^m$, all conditions coincide if correctly identified and simplify to the requirement of convexity of W.

[36] In plane rotations remain unspecified, they cannot be determined from $\overline{R}_3$ alone.

Definition 3.9.1 (Weak lower semicontinuity)
We say that a functional I defined on the Sobolev space $W^{1,p}(\Omega)$ is weakly lower semicontinuous, whenever $\varphi_k \rightharpoonup \varphi \in W^{1,p}(\Omega)$ implies

$$I(\varphi) \le \liminf_k I(\varphi_k)\,. \tag{3.9.119}$$

If $I(\varphi) := \int_\Omega W(\nabla\varphi)\,\mathrm{dx}$, then weak lower semicontinuity is equivalent to quasiconvexity of W. This result is the cornerstone of the classical direct methods of variations.

3.9.4 Macroscopic elastic shear failure for plates

It is convenient to define what we mean by **shear failure** for plates in classical isotropic elasticity. Let $W((\nabla m|\vec{n})) = \hat{W}(\nabla m^T\nabla m)$ be the free elastic energy density of the membrane (intrinsic) part of the plate defined on the **first fundamental form** of the surface $\nabla m^T\nabla m = I_m \in \mathrm{Sym}$. If for some regular $m:\omega\subset\mathbb{R}^2\mapsto\mathbb{R}^3$ with $(\nabla m|\vec{n}_m)\in \mathrm{GL}^+(3,\mathbb{R})$

$$\exists\,\xi,\eta\in\mathbb{R}^3:\quad D_F^2W((\nabla m|\vec{n})).(\xi\otimes\eta,\xi\otimes\eta)<0\,, \tag{3.9.120}$$

we say that the material fails or looses **Legendre-Hadamard ellipticity (LH)**, also called a **material instability**.[37] This failure can give rise to highly localized deformation patterns, subsumed under the notion of **microstructure**. Related is the possible emergence of discontinuous deformations since **Hadamard´s jump relations** are violated. However, loss of ellipticity may already occur for midsurface deformations which are not related to in-plane shear, e.g. uniaxial situations and pure in plane dilations. Thus we say that W suffers from **genuine elastic shear failure** whenever

$$\begin{aligned}
&\exists\,F\in\mathrm{GL}^+(3,\mathbb{R})\quad \exists\,\xi,\eta\in\mathbb{R}^3:\quad D^2W(F).(\xi\otimes\eta,\xi\otimes\eta)<0\,,\quad\text{but}\\
&\forall\,F\in\mathrm{diag}(\lambda_1^+,\lambda_2^+,1)\quad \forall\,\xi,\eta\in\mathbb{R}^3:\quad D^2W(F).(\xi\otimes\eta,\xi\otimes\eta)\ge 0\,.
\end{aligned}\tag{3.9.121}$$

It seems that failure of a material on a macroscale other than shear failure is unphysical and rather due to the idiosyncrasy of the constitutive equations, as long as the bulk is modelled as elastic. In fact, Legendre-Hadamard ellipticity for $F=\mathrm{diag}(\lambda_1^+,\lambda_2^+,1)$ of the membrane energy implies immediately the **Baker-Ericksen (BE)** inequalities [MH83, p.19] for the membrane and genuine elastic shear failure happens, if BE is satisfied but LH is violated.[38]

In this sense the following non exhaustive list of free energy terms should be avoided for the membrane since they are not only failing under shear (already BE is not satisfied): with $\overline{C}=(\nabla m|\vec{n})^T(\nabla m|\vec{n})$, $\overline{U}=\sqrt{\overline{C}}$, $F=(\nabla m|\vec{n})$ the list reads

$$\|\overline{C}-1\!\!1\|^2,\ \langle\overline{C}-1\!\!1,1\!\!1\rangle^2,\ \langle\ln\overline{C},1\!\!1\rangle^2,\ \langle\ln\overline{C},1\!\!1\rangle^2+\|\operatorname{dev}\ln\overline{C}\|^2,\ \langle\ln\overline{U},1\!\!1\rangle^2,\ -\ln\det[F]+(\ln\det[F])^2,\ \|\frac{\overline{C}}{\det[\overline{C}]^{1/3}}-1\!\!1\|^2\,, \tag{3.9.122}$$

and it is obvious that the membrane part of the finite Kirchhoff-Love plate model (3.7.83) and the finite Reissner-Mindlin model (3.6.82) is failing, not only in shear! Of course, combination with other terms could remove the problem. Terms which genuinely fail only in shear are e.g.

$$\|\overline{U}-1\!\!1\|^2,\ \langle\overline{U}-1\!\!1,1\!\!1\rangle^2,\ \|\frac{\overline{U}}{\det[\overline{U}]^{1/3}}-1\!\!1\|^2,\ \mathrm{tr}\left[\frac{\overline{U}}{\det[\overline{U}]^{1/3}}-1\!\!1_2\right]^2. \tag{3.9.123}$$

3.9.5 Linearized plate models

The classical infinitesimal Reissner-Mindlin model

Let us linearize a modification of case II ($\mu_c=0$, $\alpha_4=0$, $q=0$, $p>1$) for situations of small midsurface deformations and small curvature. We write $m(x,y)=(x,y,0)^T+v(x,y)$, with the displacement of the midsurface of the plate $v:\omega\mapsto\mathbb{R}^3$ and $\overline{R}=1\!\!1+\overline{A}+\ldots$ with $\overline{A}\in\mathfrak{so}(3,\mathbb{R})$ the infinitesimal microrotation. For the boundary deformation we write $g_\mathrm{d}(x,y,z)=(x,y,z)^T+u^\mathrm{d}(x,y,z)$, with the consequence, that $\nabla g_\mathrm{d}.e_3=(u^\mathrm{d}_{1,z},u^\mathrm{d}_{2,z},1+u^\mathrm{d}_{3,z})$. The curvature tensors are expanded as

$$\begin{aligned}
\mathfrak{K}_b&=\overline{R}^T(\nabla\overline{R}_3|0)=(1\!\!1+\overline{A}+\ldots)^T\,(\nabla[\overline{A}_3+\overline{A}^2.e_3+\ldots]|0)\approx(\nabla\overline{A}_3|0)+\ldots\\
\mathfrak{K}_s&\approx\left((\nabla(\overline{A}.e_1)|0),(\nabla(\overline{A}.e_2)|0),(\nabla(\overline{A}.e_3)|0)\right)\,,
\end{aligned}\tag{3.9.124}$$

and the Cosserat micropolar plate stretch tensor expands like

$$\overline{U}=\overline{R}^TF_s=\overline{R}^T(\nabla m|\overline{R}_3)=(1\!\!1+\overline{A}+\ldots)^T\left(\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}+\nabla v|(1\!\!1+\overline{A}+\ldots).e_3)\right)\approx 1\!\!1+(\nabla v|\overline{A}_3)+\ .\tag{3.9.125}$$

Since $p>1$, the Cosserat curvature contribution has an exponent strictly bigger than two such that a linearization w.r.t zero curvature does not yield any contribution of this term. Moreover, for $\mu_c=0$, in-plane rotations (drilling degrees of freedom) do

[37] Material instability should be carefully distinguished from **geometrical instabilities** occurring in buckling or necking and which are fully consistent with Legendre-Hadamard ellipticity. In this sense, **polyconvex** materials are **unconditionally** materially stable and certainly appropriate for rubber and soft-tissues [SN02, HN03].

[38] One version of the **BE**-inequalities for membranes can be stated as follows: for $\lambda_i^2\ge 0$, $i=1,2$, $\lambda_3^2=1$ the (generalized) principal stretches (here λ_i^2 are the eigenvalues of $(\nabla m|\vec{n})^T(\nabla m|\vec{n})$), the free energy $\Phi(\lambda_1,\lambda_2,1):=\hat{W}(\nabla m^T\nabla m)$ is separately convex in λ_i. No mathematical existence results based only on BE are known. Note also that BE is enough to effectively exclude phase-transformations, modelled with multi-well potentials.

not survive the linearization process! We are indeed left with the minimization problem for $v \in \mathbb{R}^3$ and $A_3 \in \mathbb{R}^3$

$$\begin{aligned}
&\int_\omega h\,\left(\mu\,\|\operatorname{sym}((\nabla v|\overline{A}_3))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}((\nabla v|\overline{A}_3))\right]^2\right) \\
&\qquad + \frac{h^3}{12}\left(\mu\,\|\operatorname{sym}((\nabla \overline{A}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}((\nabla \overline{A}_3|0))\right]^2\right)\,\mathrm{d}\omega - \Pi(v,\overline{A}_3) \mapsto \min.\ \text{w.r.t. } (v,\overline{A}_3), \\
&v_{|\gamma_0} = u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad\qquad (3.9.126) \\
&\overline{A}_{|\gamma_0} = \operatorname{skew}((\nabla v|\partial_z u^{\mathrm{d}}))_{|\gamma_0}\,, \quad \text{linearized consistent coupling} \Rightarrow \overline{A}_{3|\gamma_0} = (\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0)^T \\
&\overline{A}_{3|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{rigid prescription}\,.
\end{aligned}$$

Abbreviating now $\theta = (\theta_1, \theta_2, 0)^T = -\overline{A}_3$, we are left with the following set of equations for the displacement of the midsurface of the plate $v : [0,T] \times \overline{\omega} \mapsto \mathbb{R}^3$ and the infinitesimal increment of the 'normal', $\theta : \omega \mapsto \mathbb{R}^3$

$$\begin{aligned}
&\int_\omega h\left(\mu\,\|\operatorname{sym}\nabla(v_1,v_2)\|^2 + \underbrace{\frac{\mu}{2}\,\|\nabla v_3 - \theta\|^2}_{\text{transverse shear energy}} + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\nabla(v_1,v_2)\right]^2\right) \\
&\qquad + \frac{h^3}{12}\left(\mu\,\|\operatorname{sym}\nabla\theta\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\nabla\theta\right]^2\right)\,\mathrm{d}\omega - \Pi(v,-\theta) \mapsto \min.\ \text{w.r.t. } (v,\theta), \\
&v_{|\gamma_0} = u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad\qquad (3.9.127) \\
&-\theta_{|\gamma_0} = (\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0)^T\,, \quad \text{linearized consistent coupling} \\
&-\theta_{|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{rigid prescription}\,.
\end{aligned}$$

A further reduction arises if we assume only normal displacements: $v_1 = v_2 = 0$. The resulting minimization problem is

$$\int_\omega h\,\frac{\mu}{2}\,\|\nabla v_3 - \theta\|^2 + \frac{h^3}{12}\left(\mu\,\|\operatorname{sym}\nabla\theta\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\nabla\theta\right]^2\right)\,\mathrm{d}\omega - \Pi(v_3\cdot e_3, -\theta) \mapsto \min.\ \text{w.r.t. } (v_3,\theta), \qquad (3.9.128)$$

$$\begin{aligned}
&v_{3|\gamma_0} = u^{\mathrm{d}}_3\,, \qquad \text{simply supported} \\
&-\theta_{|\gamma_0} = (\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0)^T \qquad \text{linearized coupling}\,, \qquad -\theta_{|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T \quad \text{rigid}\,.
\end{aligned}$$

The elastic free energy should be compared with

$$W_{\mathrm{RM,class}}(\nabla v_3, \theta) = h\,\frac{\kappa\,\mu}{2}\,\|\nabla v_3 - \theta\|^2 + \frac{h^3}{12}\left(\frac{\mu}{4}\,\|\nabla\theta^T + \nabla\theta\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\operatorname{tr}\left[\nabla\theta^T + \nabla\theta\right]^2\right)\,, \qquad (3.9.129)$$

where $\kappa = \frac{5}{6}$ is the so called **shear correction factor**. In this last form, the Reissner-Mindlin problem can be found in many textbooks, e.g. [Bra92, p.281] or [Ste95]. It should be noted, however, that in our variationally based finite derivation with subsequent linearization there is no imminent reason to introduce $\kappa \neq 1$. In fact, the shear correction factor κ can be seen as a tuning parameter of the infinitesimal model which, for certain types of loading,[39] allows to **improve** the **order of convergence** of the infinitesimal Reissner-Mindlin solution to the three-dimensional linear elasticity solution [Rös99].[40]

The classical infinitesimal Kirchhoff-Love plate (Koiter model)

For the convenience of the reader we also supply the similar system of equations for the classical infinitesimal Kirchhoff-Love plate (also the Koiter model) which we derive as linearization of the finite Kirchhoff-Love plate. In terms of the midsurface displacement v we have to find a solution of the minimization problem for $v \in \mathbb{R}^3$

$$\begin{aligned}
&\int_\omega h\,\left(\mu\,\|\operatorname{sym}\nabla(v_1,v_2)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}\nabla(v_1,v_2)\right]^2\right) \\
&\qquad + \frac{h^3}{12}\left(\mu\,\|D^2 v_3\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[D^2 v_3\right]^2\right)\,\mathrm{d}\omega - \Pi(v, -\nabla v_3) \mapsto \min.\ \text{w.r.t. } v, \\
&v_{|\gamma_0} = u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad\qquad (3.9.130) \\
&-\nabla v_{3|\gamma_0} = (\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0)^T\,, \quad \text{linearized consistent coupling} \Rightarrow \nabla v_3 = -1/2\,(u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T \\
&-\nabla v_{3|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{rigid prescription, linearized Kirchhoff}\,.
\end{aligned}$$

This energy can be obtained formally from (3.9.129) by setting $\theta = \nabla v_3$.

[39]Hence the shear correction factor κ shows some similarity to the Cosserat couple modulus μ_c, whose influence on the solution of the three-dimensional problem is also strongly dependent on boundary conditions. For rather thick plates, it is known that the shear energy in RM_{lin} is overestimated, therefore, one is led to reduce the shear energy contribution a posteriori by taking $\kappa < 1$.

[40]It would be interesting to know the optimal shear correction factor $0 < \kappa \leq 1$ of the infinitesimal Reissner-Mindlin model with our reduced consistent coupling boundary condition. Such an optimized parameter should also be beneficial for the finite Cosserat plate!

3.10 Appendix B

3.10.1 Prerequisites from differential geometry

A given mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, describing a surface imbedded in the three-dimensional space is called **regular** whenever $\mathrm{rank}(\nabla m) = 2$. The vector

$$\vec{n} := \frac{m_x \times m_y}{\|m_x \times m_y\|}, \tag{3.10.131}$$

is the **Gauss unit normal field** on the surface. The map $n : \omega \subset \mathbb{R}^2 \mapsto \mathbb{S}^2$ is called the **Gauss map** and the moving 3-frame $(m_x|m_y|n)$ is called the **Gauss frame** of the surface m which in general is not orthonormal. The matrix representation of the **first fundamental form (metric)** is given through

$$I_m := \nabla m^T \nabla m = \begin{pmatrix} \|m_x\|^2 & \langle m_x, m_y\rangle \\ \langle m_x, m_y\rangle & \|m_y\|^2 \end{pmatrix} \in \mathbb{M}^{2\times 2}, \quad I_m + e_3 \otimes e_3 = (\nabla m|n)^T(\nabla m|n) =: \widehat{I}_m \tag{3.10.132}$$

$$\det[I_m] = \det[\widehat{I}_m] = \det[(\nabla m|\vec{n})]^2 .$$

The metric alone is not sufficient to describe the shape of a surface in the ambient three-dimensional Euclidean space, the curvature is also needed, although in the **rigid** case $(\nabla m|\vec{n}) \in \mathrm{SO}(3,\mathbb{R})$, the metric is indeed enough.

The matrix representation of the **second fundamental form** providing a measure for curvature of the surface is given by

$$II_m := -\nabla m^T Dn = -(m_x|m_y)^T \cdot (n_x|n_y) = -\begin{pmatrix} \langle m_x, D_x n\rangle & \langle m_x, D_y n\rangle \\ \langle m_y, D_x n\rangle & \langle m_y, D_y n\rangle \end{pmatrix} \in \mathbb{M}^{2\times 2} \tag{3.10.133}$$

$$(\nabla m|n)^T(D_x n|D_y n|0) = \begin{pmatrix} \langle m_x, D_x n\rangle & \langle m_x, D_y n\rangle & 0 \\ \langle m_y, D_x n\rangle & \langle m_y, D_y n\rangle & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widehat{II}_m := -(\nabla m|\vec{n})^T(\nabla \vec{n}|\vec{n}), \quad \det[II_m] = \det[\widehat{II}_m].$$

Since n is orthogonal to the tangent space $T_x m$, the relation $0 = \partial_x \langle m_y, \vec{n}\rangle = \partial_y \langle m_x, \vec{n}\rangle$ shows easily that II_m is **symmetric**. The **third fundamental form** of the surface in matrix representation is defined as

$$III_m := Dn^T Dn = \begin{pmatrix} \|D_x n\|^2 & \langle D_x n, D_y n\rangle \\ \langle D_y n, D_x n\rangle & \|D_y n\|^2 \end{pmatrix} \in \mathbb{M}^{2\times 2}, \quad \widehat{III}_m := (\nabla \vec{n}|\vec{n})^T(\nabla \vec{n}|\vec{n}). \tag{3.10.134}$$

The matrix representation of the **Weingarten map** (shape operator) L is given by

$$L(x,y) := -Dn(x,y)\, \nabla_\xi m^{-1}(m(x,y) \in \mathbb{M}^{3\times 3}, \quad L = -(\nabla \vec{n}|0)(\nabla m|n)^{-1}, \tag{3.10.135}$$

representing the variation of the normal in the metric of the surface. In order to see that $L = -(Dn|0)(\nabla m|n)^{-1}$ we extent m to $\mathbb{R}^3$ by setting $\Theta(x,y,z) = m(x,y) + z\, n(x,y)$. This yields $\Theta(x,y,0) = m(x,y)$ and $\nabla\Theta(x,y,0) = (\nabla m|n)$ while $\Theta^{-1}(\Theta(x,y,z)) = (x,y,z)^T$ and the chain rule shows $\nabla_\xi \Theta^{-1}(\Theta(x,y,z)]\nabla\Theta(x,y,z) = \mathbb{1}$. Hence

$$\nabla_\xi \Theta^{-1}(\Theta(x,y,0))\nabla\Theta(x,y,0) = \mathbb{1}$$

but $\nabla_\xi \Theta^{-1}(\Theta(x,y,0)) = \nabla_\xi m^{-1}(m(x,y))$ which finishes the argument. The **Gauss curvature** K of the surface is determined by

$$K(x,y) := \frac{\det[II_m]}{\det[I_m]} = \det[L] = \det[Dn\, \nabla_\xi m^{-1}(\xi)], \tag{3.10.136}$$

and the **mean curvature** H through

$$2\, H(x,y) := \mathrm{tr}\,[L] = \mathrm{tr}\left[Dn\, \nabla_\xi m^{-1}(\xi)\right]. \tag{3.10.137}$$

The relation $III_m - 2H\, II_m + K\, I_m = 0$ ([Kli78, Prop. 3.5.6]) is a consequence of the Caley-Hamilton theorem and shows that III_m is not independent of I_m, II_m. The principal curvatures κ_1, κ_2 are the solutions of the characteristic equation of $-L$, i.e. $\kappa^2 - \mathrm{tr}\,[L]\kappa + \det[L] = \kappa^2 - 2H\kappa + K = 0$. The Caley-Hamilton theorem on $\mathbb{M}^{2\times 2}$ implies for the second fundamental form on account of its symmetry

$$II_m^2 - \mathrm{tr}\,[II_m]\, II_m + \det[II_m]\, \mathbb{1}_2 = 0 \Rightarrow \|II_m\|^2 - \mathrm{tr}\,[II_m]^2 + 2\det[II_m] = 0 \quad \Rightarrow \mathrm{tr}\,[II_m]^2 - \|II_m\|^2 = 2\det[II_m]. \tag{3.10.138}$$

Thus the Gauss curvature K can be expressed equivalently as

$$K = \frac{\mathrm{tr}\,[II_m]^2 - \|II_m\|^2}{2\det[I_m]}. \tag{3.10.139}$$

Of major importance is the following classification

Definition 3.10.1 (Intrinsicality)
A property or a set of equations is **intrinsic** *whenever it can be reduced to the first fundamental form, i.e. depends only on local length and local angles on the surface. (Or the change of local length and local angles.) Intrinsic properties remain invariant under isometries.*

For example, the mean curvature H is not intrinsic, since bending a surface changes H but leaves length and angles invariant (bending belongs to the outer geometry of the surface); or take the normal of the surface n: this is not an intrinsic quantity, since bending changes the normals but leaves length and angles invariant.

Gauss' celebrated **Theorema Egregium** states that contrary to appearance (it involves the normals!), the Gauss curvature K is an intrinsic quantity: it can be computed through the first and second derivatives of the first fundamental form. The same is trivially true for $\|(\nabla m|\vec{n})^T(\nabla m|\vec{n}) - \mathbb{1}\|^2$- it is a purely intrinsic strain measure (the dependence on $\vec{n}$ cancels out algebraically).

In the thin shell limit of $h \to 0$ it is expedient to get a model which is purely two-dimensional, i.e. intrinsic.

Theorem 3.10.2 (Fundamental theorem of surface theory)
Any two surfaces $m, \tilde{m} : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, which have the same first and second fundamental form, differ only by an **isometry**, *i.e. $\tilde{m}(x,y) = Q.m(x,y)$, $Q \in \mathrm{SO}(3)$.*

Proof. Well known in differential geometry, e.g.[Kli78, p.64]. ■

Lemma 3.10.3 (Developable surfaces)
A surface m with no planar points ($II_m \neq 0$) is developable (on the plane, without stretch) if and only if the Gauss curvature K vanishes.

Proof. Theorem 3.7.9 in [Kli78]. ■

Lemma 3.10.4 (Isometric surfaces)
Two surfaces with different Gauss curvature K cannot be mapped isometrically into each other.

Proof. Well known. ■
The following classification is standard. The surface m is locally

$$\begin{cases}\textbf{elliptic} \\ \textbf{parabolic} \\ \textbf{hyperbolic}\end{cases} \quad \text{at } (x,y) \in \omega \text{ if } \det[II_m(x,y)] \text{ is } \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} . \tag{3.10.140}$$

The surface m is locally **uniformly elliptic** if

$$\exists\, c^+ > 0 : \forall\, \xi \in \mathbb{R}^2 : \quad \langle II_m.\xi, \xi\rangle_{\mathbb{R}^2} = \langle \nabla m^T Dn.\xi, \xi\rangle_{\mathbb{R}^2} = \langle Dn.\xi, \nabla m.\xi\rangle_{\mathbb{R}^3} \geq c^+ \, \|\xi\|^2 . \tag{3.10.141}$$

Definition 3.10.5 (Christoffel symbols)
Let the regular surface m be given. The Christoffel symbols of the first kind of the surface are defined by

$$\Gamma^i_{jk} := \langle \partial_j a_k, a_i\rangle, \quad a_1 = m_x, a_2 = m_y, a_3 = n,\ j = 1,2,\ i,k = 1,2,3 . \tag{3.10.142}$$

They are **not independent of the choice of coordinates (not covariant), but intrinsic quantities**, *belonging to the inner geometry of the surface, see [Lau60, p.36].*

Let us look at $\|\nabla_x[\varrho^2]\|^2$. It is clear that this defines an intrinsic quantity, since it can be expressed as partial derivatives of the metric. We have

$$\nabla_x\|\nabla m\|^2 = \nabla_x\langle \nabla m^T \nabla m, 1\!\!1_2\rangle = \nabla_x(\|m_x\|^2 + \|m_y\|^2) = 2\begin{pmatrix}\langle m_x, m_{xx}\rangle + \langle m_y, m_{yx}\rangle \\ \langle m_x, m_{xy}\rangle + \langle m_y, m_{yy}\rangle\end{pmatrix} = 2\begin{pmatrix}\Gamma^1_{11} + \Gamma^2_{12} \\ \Gamma^1_{21} + \Gamma^2_{22}\end{pmatrix} . \tag{3.10.143}$$

Hence, $\|\nabla_x[\varrho^2]\|^2 = 4\,(\Gamma^1_{11} + \Gamma^2_{12})^2 + 4\,(\Gamma^1_{21} + \Gamma^2_{22})^2$.

3.10.2 Additional material

Lemma 3.10.6 (Normality and polar decomposition)
Let $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ be regular and assume for some $R \in \mathrm{SO}(3)$ that $R = \mathrm{polar}(\nabla m|\,\varrho\, R.e_3)$, where $\varrho > 0$ is given. Then $R_3 = \vec{n}_m$ and $R = \mathrm{polar}(\nabla m|\varrho\,\vec{n})$.

Proof. Since $F = R\,U$ we must have $F^T R \in \mathrm{Sym}$. But

$$U = U^T = F^T R = (\nabla m|\varrho\, R_3)^T \cdot \mathrm{polar}(\nabla m|\varrho\, R_3) = (\nabla m|\varrho\, R_3)^T \cdot R = \begin{pmatrix}\langle m_x, R_1\rangle & \langle m_x, R_2\rangle & \langle m_x, \varrho\, R_3\rangle \\ \langle m_y, R_1\rangle & \langle m_y, R_2\rangle & \langle m_y, \varrho\, R_3\rangle \\ 0 & 0 & \varrho\end{pmatrix} , \tag{3.10.144}$$

which implies $\varrho\,\langle m_y, R_3\rangle = \varrho\,\langle m_x, R_3\rangle = 0$ by symmetry of U. Thus R_3 coincides with the unit normal $\vec{n}_m$ on m. ■

For $m = (x,y,0)^T + v(x,y)$ we have

$$(\nabla v|\overline{A}_3) = \begin{pmatrix} v_{1,x} & v_{1,y} & \beta \\ v_{2,x} & v_{2,y} & \gamma \\ v_{3,x} & v_{3,y} & 0\end{pmatrix} , \quad \mathrm{sym}((\nabla v|\overline{A}_3)) = \begin{pmatrix} v_{1,x} & \frac{v_{1,y}+v_{2,x}}{2} & \frac{\beta + v_{3,x}}{2} \\ \frac{v_{1,y}+v_{2,x}}{2} & v_{2,y} & \frac{\gamma + v_{3,y}}{2} \\ \frac{\beta + v_{3,x}}{2} & \frac{\gamma + v_{3,y}}{2} & 0\end{pmatrix} \tag{3.10.145}$$

$$\|\,\mathrm{sym}((\nabla v|\overline{A}_3))\|^2 = \|\,\mathrm{sym}_2(\nabla(v_1,v_2))\|^2 + \|\nabla v_3 - (-\overline{A}_3)\|^2 , \quad \mathrm{tr}\left[\mathrm{sym}((\nabla v|\overline{A}_3))\right]^2 = \mathrm{tr}\,[\mathrm{sym}(\nabla v|0)]^2 .$$

Lemma 3.10.7 (Rigidity coincidence II)
If $(\nabla m|n) \in \mathrm{SO}(3)$ then

$$\begin{aligned} \|II_m\|^2 &= \mathrm{tr}\,[III_m] = \|D_x n\|^2 + \|D_y n\|^2 \\ \mathrm{tr}\,[II_m]^2 &= (\|m_{xx}\| + \|m_{yy}\|)^2 \\ \|II_m\|^2 &= \|m_{xx}\|^2 + 2\|m_{xy}\|^2 + \|m_{yy}\|^2 . \end{aligned} \tag{3.10.146}$$

Proof.

$$(\nabla m|n)^T \cdot (D_x n|D_y n|0) = \begin{pmatrix}\langle m_x, D_x n\rangle & \langle m_x, D_y n\rangle & 0 \\ \langle m_y, D_x n\rangle & \langle m_y, D_y n\rangle & 0 \\ \underbrace{\langle n, D_x n\rangle}_{=0} & \underbrace{\langle n, D_y n\rangle}_{=0} & 0\end{pmatrix} . \tag{3.10.147}$$

Therefore

$$\|II_m\|^2 = \|(\nabla m|n)^T (D_x n|D_y n|0)\|^2 = \|(D_x n|D_y n|0)\|^2 = \|D_x n\|^2 + \|D_y n\|^2 = \mathrm{tr}\,[III_m] , \tag{3.10.148}$$

which finishes the first part. Now $\mathrm{tr}\,[II_m]^2 = (\langle m_x, n_x\rangle + \langle m_y, n_y\rangle)^2$. Using $0 = \frac{\mathrm{d}}{\mathrm{dx}}\langle m_x, n\rangle = \frac{\mathrm{d}}{\mathrm{dx}}\langle m_y, n\rangle$ we have $\mathrm{tr}\,[II_m]^2 = (\langle m_{xx}, n\rangle + \langle m_{yy}, n\rangle)^2$. But if $(\nabla m|n) \in \mathrm{SO}(3,\mathbb{R})$ we get in addition, on account of

$$0 = \frac{\mathrm{d}}{\mathrm{dx}}\langle m_x, m_x\rangle, \quad 0 = \frac{\mathrm{d}}{\mathrm{dy}}\langle m_y, m_y\rangle, \quad 0 = \frac{\mathrm{d}}{\mathrm{dx}}\langle m_x, m_y\rangle\,, \tag{3.10.149}$$

that $\langle m_{xx}, m_x\rangle = \langle m_{xx}, m_y\rangle = 0$ and $\langle m_{yy}, m_x\rangle = \langle m_{yy}, m_y\rangle = 0$ which implies that $m_{xx} = \|m_{xx}\|\, n$ and $m_{yy} = \|m_{yy}\|\, n$. The same reasoning implies altogether

$$(\nabla m|n)^T \cdot (D_x n|D_y n|0) = \begin{pmatrix} -\|m_{xx}\| & -\|m_{xy}\| & 0 \\ -\|m_{yx}\| & -\|m_{yy}\| & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.10.150}$$

■

Corollary 3.10.8 (Rigidity coincidence III)
If $\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1 = 0$ then $R = (\nabla m|\vec{n}_m) \in \mathrm{SO}(3)$ and $R_{1,y} = R_{2,x}$ together with

$$\overline{R}^T(\nabla \overline{R}_3|0) = II_m = \begin{pmatrix} -\|\overline{R}_{1,x}\| & -\|\overline{R}_{1,y}\| & 0 \\ -\|\overline{R}_{2,x}\| & -\|\overline{R}_{2,y}\| & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{3.10.151}$$

Moreover,

$$\|\,\mathrm{sym}\begin{pmatrix} -\|\overline{R}_{1,x}\| & -\|\overline{R}_{1,y}\| & 0 \\ -\|\overline{R}_{2,x}\| & -\|\overline{R}_{2,y}\| & 0 \\ 0 & 0 & 0 \end{pmatrix}\|^2 \geq \|\nabla \overline{R}_1\|^2 + \|\nabla \overline{R}_2\|^2\,. \tag{3.10.152}$$

Proof. The first part is a consequence of Lemma 3.10.7. The second part is an algebraic computation. ■

3.10.3 Linearized quantities

At various places we are interested in the linearization of the proposed systems with respect to the reference plane. Let therefore $m(x,y) = (x,y,0)^T + (v_1,v_2,v_3)^T$ and $g_{\mathrm{d}} = (x,y,z)^T + u^{\mathrm{d}}(x,y,z)^T$. Then upon expanding to first order

$$\begin{aligned}
\nabla m &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} v_{1,x} & v_{1,y} \\ v_{2,x} & v_{2,y} \\ v_{3,x} & v_{3,y} \end{pmatrix} \\
m_x \times m_y &= e_3 + e_1 \times v_y + v_x \times e_2 + v_x \times v_y \\
n &\approx e_3 + (-v_{3,x}, -v_{3,y}, 0)^T + o(\|\nabla v\|) \\
\nabla g_{\mathrm{d}} &= 1\!\!1 + \nabla u^{\mathrm{d}}, \quad \nabla g_{\mathrm{d}}.e_3 = e_3 + \nabla u^{\mathrm{d}}.e_3 = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 1 + u^{\mathrm{d}}_{3,z})^T \\
\frac{\nabla g_{\mathrm{d}}.e_3}{\|\nabla g_{\mathrm{d}}.e_3\|} &= \nabla g_{\mathrm{d}}.e_3 \cdot \|\nabla g_{\mathrm{d}}.e_3\|^{-1} = (e_3 + \nabla u^{\mathrm{d}}.e_3) \cdot \|e_3 + \nabla u^{\mathrm{d}}.e_3\|^{-1} \\
&\approx (e_3 + \nabla u^{\mathrm{d}}.e_3)\left[\|e_3\|^{-1} + (-1)\|e_3\|^{-2}\langle \frac{e_3}{\|e_3\|}, \nabla u^{\mathrm{d}}.e_3\rangle + \ldots\right] \\
&\approx (e_3 + \nabla u^{\mathrm{d}}.e_3)\left[1 - \langle e_3, \nabla u^{\mathrm{d}}.e_3\rangle\right] \approx e_3 + \nabla u^{\mathrm{d}}.e_3 - u^{\mathrm{d}}_{3,z} = e_3 + (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T \\
(\nabla m|n) &\approx 1\!\!1 + \begin{pmatrix} v_{1,x} & v_{1,y} & -v_{3,x} \\ v_{2,x} & v_{2,y} & -v_{3,y} \\ v_{3,x} & v_{3,y} & 0 \end{pmatrix} + o(\|\nabla v\|) \\
(\nabla m|n)^T(\nabla m|n) - 1\!\!1 &\approx \begin{pmatrix} 2v_{1,x} & v_{1,y} + v_{2,x} & 0 \\ v_{2,x} + v_{1,y} & 2v_{2,y} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(\|\nabla v\|) \\
\|\nabla m\|^2 &\approx 2 + 2(v_{1,x} + v_{2,y}) + \|\nabla v\|^2 = 2 + 2\,\mathrm{Div}\, v + \|\nabla v\|^2 \\
\mathrm{polar}((\nabla m|n)) &\approx 1\!\!1 + \mathrm{skew}\begin{pmatrix} v_{1,x} & v_{1,y} & -v_{3,x} \\ v_{2,x} & v_{2,y} & -v_{3,y} \\ v_{3,x} & v_{3,y} & 0 \end{pmatrix} + o(\|\nabla v\|) \\
U((\nabla m|n)) &\approx 1\!\!1 + \frac{1}{2}\begin{pmatrix} 2v_{1,x} & v_{1,y} + v_{2,x} & 0 \\ v_{2,x} + v_{1,y} & 2v_{2,y} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(\|\nabla v\|) \\
(Dn|0) &\approx \begin{pmatrix} -v_{3,xx} & -v_{3,xy} & 0 \\ -v_{3,yx} & -v_{3,yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(\|\nabla v\|) = -D^2 v_3 + o(\|\nabla v\|) \\
(\nabla m|n)^T(Dn|0) &\approx -D^2 v_3 \\
\mathrm{tr}\left[(\nabla m|n)^T(Dn|0)\right]^2 &\approx \mathrm{tr}\left[D^2 v_3\right]^2 \\
\|(\nabla m|n)^T(Dn|0)\|^2 &\approx \|D^2 v_3\|^2 \\
\varrho_m^2(\nabla m) &\approx 1 - \frac{2\lambda}{2\mu + \lambda}\,\mathrm{Div}\, v - \frac{\lambda}{2\mu + \lambda}\|\nabla v\|^2
\end{aligned} \tag{3.10.153}$$

$$
\begin{aligned}
\nabla_x[\varrho_m^2(\nabla m)] &\approx -\frac{2\lambda}{2\mu+\lambda}\begin{pmatrix} v_{1,xx}+v_{2,yx} \\ v_{1,xy}+v_{2,yy}\end{pmatrix} \\
\|\nabla_x[\varrho_m^2(\nabla m)]\|^2 &\approx \frac{4\lambda^2}{(2\mu+\lambda)^2}\left((v_{1,xx}+v_{2,yx})^2+(v_{1,xy}+v_{2,yy})^2\right) \\
\operatorname{Curl}(\nabla m|n) &\approx \begin{pmatrix} -v_{3,xy} & v_{3,xx} & 0 \\ -v_{3,yy} & v_{3,yx} & 0 \\ 0 & 0 & 0\end{pmatrix}, \quad \|\operatorname{Curl}(\nabla m|n)\|^2 \approx v_{3,xx}^2+2v_{3,xy}^2+v_{3,yy}^2 \\
\overline{R} &= 1\!\!1+\overline{A}+\frac{1}{2}\overline{A}^2+\ldots, \quad \overline{R}.e_3 \approx e_3+\overline{A}.e_3
\end{aligned}
$$

3.10.4 Detailed derivations

Detailed computations for the new Cosserat model

The equilibrium equations of the three-dimensional Cosserat problem given in [Nef03c] show that on the transverse boundary (exact)

$$
\begin{aligned}
S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3 &= N^{\text{trans}}(x,y,+h/2) \\
S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).(-e_3) &= N^{\text{trans}}(x,y,-h/2)\,, \qquad (3.10.154)
\end{aligned}
$$

where N^{trans} are the prescribed tractions N on the transverse boundary given globally in the basis (e_1,e_2,e_3). This implies (exact)

$$
\begin{aligned}
\overline{R}^{3d}(x,y,+h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3 &= \overline{R}^{3d}(x,y,+h/2))^T N^{\text{trans}}(x,y,+h/2) \qquad (3.10.155) \\
\overline{R}^{3d}(x,y,-h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).(-e_3) &= \overline{R}^{3d}(x,y,-h/2))^T N^{\text{trans}}(x,y,-h/2)\,.
\end{aligned}
$$

Abbreviate

$$
N^+ := N^{\text{trans}}(x,y,+h/2), \quad N^- := N^{\text{trans}}(x,y,-h/2)\,, \qquad (3.10.156)
$$

and define

$$
N_{\text{res}} := N^{\text{trans}}(x,y,+h/2)+N^{\text{trans}}(x,y,-h/2)\,,\ N_{\text{diff}} := \frac{1}{2}\left[N^{\text{trans}}(x,y,+h/2)-N^{\text{trans}}(x,y,-h/2)\right]\,. \qquad (3.10.157)
$$

Then also (exact)

$$
\begin{aligned}
\langle\overline{R}^{3d}(x,y,+h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,+h/2),\overline{R}^{3d}(x,y,+h/2)).e_3,e_3\rangle &= \langle N^+,\overline{R}^{3d}(x,y,+h/2)).e_3\rangle \qquad (3.10.158) \\
\langle\overline{R}^{3d}(x,y,-h/2))^T S_1^{3d}(\nabla\varphi^{3d}(x,y,-h/2),\overline{R}^{3d}(x,y,-h/2)).e_3,e_3\rangle &= -\langle N^-,\overline{R}^{3d}(x,y,-h/2)).e_3\rangle\,.
\end{aligned}
$$

We determine ϱ_m, ϱ_b from the corresponding requirement in terms of the assumed kinematics $(\varphi_s,\overline{R}_s)$, yielding

$$
\begin{aligned}
\langle\overline{R}_s^T(x,y,\pm h/2)S_1(\nabla\varphi_s(x,y,\pm h/2),\overline{R}_s).e_3,e_3\rangle &= \pm\langle N^{\text{trans}}(x,y,\pm h/2),\overline{R}_s(x,y,\pm h/2).e_3\rangle \quad \Rightarrow \\
\langle\overline{R}^T S_1(\nabla\varphi_s(x,y,\pm h/2),\overline{R}).e_3,e_3\rangle &= \pm\langle N^{\text{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle\,, \qquad (3.10.159)
\end{aligned}
$$

which condition reduces to **zero normal tractions on the transverse free boundary** (in the absence of tractions N^{trans}) in the classical continuum limit of $\overline{R}=\operatorname{polar}(\nabla\varphi)$. We compute

$$
\begin{aligned}
\overline{R}^T F_s &= \overline{R}^T\left[(\nabla m|\varrho_m\overline{R}_3)+z(\nabla(\varrho_m\,\overline{R}_3)|\varrho_b\,\overline{R}_3)+\frac{z^2}{2}(\nabla(\varrho_b\,\overline{R}_3)|0)\right] \\
&= \overline{R}^T\left[(\nabla m|0)+(0|0|\varrho_m\overline{R}_3)+z(\nabla(\varrho_m\,\overline{R}_3)|0)+(0|0|\varrho_b\,\overline{R}_3)+\frac{z^2}{2}(\nabla(\varrho_b\,\overline{R}_3)|0)\right] \\
&= \overline{R}^T(\nabla m|0)+\varrho_m(0|0|e_3)+z\overline{R}^T(\nabla(\varrho_m\,\overline{R}_3)|0)+z\varrho_b(0|0|e_3)+\frac{z^2}{2}\overline{R}^T(\nabla(\varrho_b\,\overline{R}_3)|0) \\
F_s^T\overline{R}+\overline{R}^T F_s-2 1\!\!1 &= \overline{R}^T(\nabla m|0)+(\nabla m|0)^T\overline{R}+2\varrho_m(0|0|e_3)-2 1\!\!1 \\
&\quad+z\left(\overline{R}^T(\nabla(\varrho_m\,\overline{R}_3)|0)+(\nabla(\varrho_m\,\overline{R}_3)|0)^T\overline{R}+2\varrho_b(0|0|e_3)\right) \\
&\quad+\frac{z^2}{2}\left(\overline{R}^T(\nabla(\varrho_b\,\overline{R}_3)|0)+(\nabla(\varrho_b\,\overline{R}_3)|0)^T\overline{R}\right) \qquad (3.10.160) \\
\langle\left[F_s^T\overline{R}+\overline{R}^T F_s-2 1\!\!1\right].e_3,e_3\rangle &= 2(\varrho_m-1)+2z\,\varrho_b \\
\operatorname{tr}\left[F_s^T\overline{R}+\overline{R}^T F_s-2 1\!\!1\right] &= 2\left(\langle\overline{R}^T(\nabla m|0),1\!\!1\rangle+\varrho_m+z\langle(\nabla(\varrho_m\,\overline{R}_3)|0)^T\overline{R},1\!\!1\rangle+z\,\varrho_b-3\right. \\
&\quad\left.+\frac{z^2}{2}\langle\overline{R}^T(\nabla(\varrho_b\,\overline{R}_3)|0),1\!\!1\rangle\right) \\
\langle(\nabla(\varrho_b\,\overline{R}_3)|0),\overline{R}\rangle &= \langle(\varrho_{b,x}\overline{R}_3|(\varrho_{b,y}\overline{R}_3|0),(\overline{R}_1|\overline{R}_2|\overline{R}_3)\rangle+\varrho_b\,\langle(\nabla\overline{R}_3)|0),\overline{R}\rangle=\varrho_b\,\langle(\nabla\overline{R}_3)|0),\overline{R}\rangle \\
\langle(\nabla(\varrho_m\,\overline{R}_3)|0),\overline{R}\rangle &= \varrho_m\,\langle(\nabla\overline{R}_3)|0),\overline{R}\rangle\,.
\end{aligned}
$$

Since

$$
S_1(F,\overline{R})=\overline{R}\left[\mu\left(F^T\overline{R}+\overline{R}^T F-2 1\!\!1\right)+2\mu_c\,\operatorname{skew}(\overline{R}^T F)+\frac{\lambda}{2}\operatorname{tr}\left[F^T\overline{R}+\overline{R}^T F-2 1\!\!1\right]1\!\!1\right]\,, \qquad (3.10.161)
$$

the requirement $\langle \overline{R}^T S_1(\nabla\varphi_s(x,y,z),\overline{R}).e_3,e_3\rangle = \pm\langle N^{\text{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle$ turns into

$$\begin{aligned}
&\pm\langle N^{\text{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle = \mu\,(2(\varrho_m-1)+2z\,\varrho_b)\\
&\quad+\lambda\left(\langle\overline{R}^T(\nabla m|0),1\!\!1\rangle+\varrho_m+z\langle(\nabla(\varrho_m\,\overline{R}_3)|0)^T\overline{R},1\!\!1\rangle+z\,\varrho_b-3+\frac{z^2}{2}\langle\overline{R}^T(\nabla(\varrho_b\,\overline{R}_3)|0),1\!\!1\rangle\right)\Rightarrow\\
&\pm\langle N^{\text{trans}}(x,y,\pm h/2),\overline{R}.e_3\rangle = \mu\,(2(\varrho_m-1)+2z\,\varrho_b)\\
&\quad+\lambda\left(\langle\overline{R}^T(\nabla m|0),1\!\!1\rangle+\varrho_m+z\,\varrho_m\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle+z\,\varrho_b-3+\frac{z^2}{2}\,\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),1\!\!1\rangle\right),
\end{aligned}\tag{3.10.162}$$

independent of the Cosserat couple modulus μ_c. Let us evaluate the last equation for $z=\pm h/2$. This yields two **linear** equations in ϱ_m,ϱ_b

$$\begin{aligned}
\langle N^+,\overline{R}.e_3\rangle &= \mu\,(2(\varrho_m-1)+h\,\varrho_b)\\
&\quad+\lambda\left(\langle\overline{R}^T(\nabla m|0),1\!\!1\rangle+\varrho_m+h/2\varrho_m\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle+h/2\,\varrho_b-3+\frac{h^2}{8}\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),1\!\!1\rangle\right)\\
-\langle N^-,\overline{R}.e_3\rangle &= \mu\,(2(\varrho_m-1)-h\,\varrho_b)\\
&\quad+\lambda\left(\langle\overline{R}^T(\nabla m|0),1\!\!1\rangle+\varrho_m-h/2\varrho_m\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle-h/2\,\varrho_b-3+\frac{h^2}{8}\varrho_b\langle\overline{R}^T(\nabla\overline{R}_3|0),1\!\!1\rangle\right).
\end{aligned}\tag{3.10.163}$$

Adding and subtracting shows that

$$\begin{aligned}
&\varrho_m\,[2\mu+\lambda]+\varrho_b\left[\frac{\lambda h^2}{8}\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle\right]=\frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{2}+(2\mu+\lambda)-\lambda\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]\\
&\varrho_m\left[\lambda\,h\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle\right]+\varrho_b\,[(2\mu+\lambda)h]=\langle N_{\text{res}},\overline{R}_3\rangle\\
&\begin{pmatrix}2\mu+\lambda & \frac{\lambda h^2}{8}\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle\\ \lambda\,h\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle & (2\mu+\lambda)h\end{pmatrix}\begin{pmatrix}\varrho_m\\ \varrho_b\end{pmatrix}=\begin{pmatrix}\langle N_{\text{diff}},\overline{R}_3\rangle+(2\mu+\lambda)-\lambda\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]\\ \langle N_{\text{res}},\overline{R}_3\rangle\end{pmatrix}.
\end{aligned}\tag{3.10.164}$$

The exact solution is given by

$$\begin{aligned}
\begin{pmatrix}\varrho_m\\ \varrho_b\end{pmatrix}&=\frac{1}{(2\mu+\lambda)^2\,h-\frac{\lambda^2h^3}{8}\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle^2}\begin{pmatrix}(2\mu+\lambda)\,h & -\frac{\lambda h^2}{8}\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle\\ -\lambda\,h\langle(\nabla\overline{R}_3|0)^T\overline{R},1\!\!1\rangle & (2\mu+\lambda)\end{pmatrix}\\
&\qquad\begin{pmatrix}\langle N_{\text{diff}},\overline{R}_3\rangle+(2\mu+\lambda)-\lambda\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]\\ \langle N_{\text{res}},\overline{R}_3\rangle\end{pmatrix},
\end{aligned}\tag{3.10.165}$$

which will be approximated through

$$\begin{aligned}
\begin{pmatrix}\varrho_m\\ \varrho_b\end{pmatrix}&\approx\frac{1}{(2\mu+\lambda)^2\,h}\begin{pmatrix}(2\mu+\lambda)\,h & -\frac{\lambda h^2}{8}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\\ -\lambda\,h\langle(\nabla\overline{R}_3|0),\overline{R}\rangle & (2\mu+\lambda)\end{pmatrix}\begin{pmatrix}\langle N_{\text{diff}},\overline{R}_3\rangle+(2\mu+\lambda)-\lambda\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]\\ \langle N_{\text{res}},\overline{R}_3\rangle\end{pmatrix}\\
&=\begin{pmatrix}\frac{1}{(2\mu+\lambda)} & -\frac{\lambda\,h}{8(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\\ -\frac{\lambda}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle & \frac{1}{(2\mu+\lambda)\,h}\end{pmatrix}\begin{pmatrix}\langle N_{\text{diff}},\overline{R}_3\rangle+(2\mu+\lambda)-\lambda\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]\\ \langle N_{\text{res}},\overline{R}_3\rangle\end{pmatrix}\\
&=\begin{pmatrix}1-\frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]+\frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)}-\frac{\lambda\,h}{8(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\,\langle N_{\text{res}},\overline{R}_3\rangle\\ -\frac{\lambda}{2(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\langle N_{\text{diff}},\overline{R}_3\rangle-\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\\ +\frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]+\frac{\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}\end{pmatrix}.
\end{aligned}\tag{3.10.166}$$

Hence the leading terms[41] are:

$$\begin{aligned}
\varrho_m&=1-\frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]+\frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)}-\frac{\lambda\,h}{8(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\,\langle N_{\text{res}},\overline{R}_3\rangle\\
\varrho_b&=-\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle+\frac{\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}-\frac{\lambda}{2(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\langle N_{\text{diff}},\overline{R}_3\rangle\\
&\quad\frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\left[\langle(\nabla m|0),\overline{R}\rangle-2\right].
\end{aligned}\tag{3.10.167}$$

The term $\frac{\lambda^2}{(2\mu+\lambda)^2}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]$ represents a nonlinear coupling between midsurface deformationgradient and curvature, an artefact of the derivation not present in the underlying three-dimensional theory where only products of deformation-gradient and rotations occur, we therefore neglect this term.[42] Moreover, for a **rigid** material with $\lambda\gg1$ we have $\frac{\lambda}{(2\mu+\lambda)^2}\ll1$, leading finally to the reduced expressions:

$$\begin{aligned}
\varrho_m&=1-\frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle-2\right]+\frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)},\\
\varrho_b&=-\frac{\lambda}{2\mu+\lambda}\langle(\nabla\overline{R}_3|0),\overline{R}\rangle+\frac{\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}.
\end{aligned}\tag{3.10.168}$$

[41] Note that ϱ_m,ϱ_b have different units. ϱ_m is dimensionless, whereas $[\varrho_b]=\mathrm{m}^{-1}$.

[42] It would be possible to base all further considerations indeed on the exact solution of ϱ_m,ϱ_b and it seems that the resulting two-dimensional model would allow an existence proof. However, the much more involved expressions are not easily interpreted and do not reduce to the classical expressions upon linearization.

The formula (3.10.168) shows the physically reasonable behaviour that to first order, **fibers will be elongated by opposite transverse tractions**.

Having obtained the general form of the relevant coefficients ϱ_m, ϱ_b it is expedient to base the expansion of the three-dimensional elastic Cosserat energy, as far as its bending contribution is concerned, on a further simplified expression, namely

$$F_s = \nabla\varphi_s(x,y,z) \approx (\nabla m|\, \varrho_m\, \overline{R}_3) + z\cdot(\nabla \overline{R}_3|\varrho_b\, \overline{R}_3) = A_m + z\, A_r = \overline{F}_s\,. \tag{3.10.169}$$

This is motivated by our already anounced principle of reduction. The use of (3.3.25) excludes (up to order h^3) exactly those terms which would violate our principle had we used (3.3.9) instead. We compute further

$$\begin{aligned}
\overline{R}^T A_r &= \overline{R}^T(\nabla\overline{R}_3|\tilde{\varrho}_b\,\overline{R}_3) = \begin{pmatrix} \langle\overline{R}_1,\overline{R}_{3,x}\rangle & \langle\overline{R}_1,\overline{R}_{3,y}\rangle & 0\\ \langle\overline{R}_2,\overline{R}_{3,x}\rangle & \langle\overline{R}_2,\overline{R}_{3,y}\rangle & 0\\ \underbrace{\langle\overline{R}_3,\overline{R}_{3,x}\rangle}_{=0} & \underbrace{\langle\overline{R}_3,\overline{R}_{3,y}\rangle}_{=0} & \varrho_b\end{pmatrix},\\
\|\operatorname{skew}(\overline{R}^T A_r)\|^2 &= \frac{1}{2}\left(\langle\overline{R}_1,\overline{R}_{3,y}\rangle - \langle\overline{R}_2,\overline{R}_{3,x}\rangle\right)^2 = \|\operatorname{skew}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2\\
\overline{R}^T A_r + A_r^T\overline{R} &= \begin{pmatrix} 2\langle\overline{R}_1,\overline{R}_{3,x}\rangle & \langle\overline{R}_1,\overline{R}_{3,y}\rangle+\langle\overline{R}_2,\overline{R}_{3,x}\rangle & 0\\ \langle\overline{R}_1,\overline{R}_{3,y}\rangle+\langle\overline{R}_2,\overline{R}_{3,x}\rangle & 2\langle\overline{R}_2,\overline{R}_{3,y}\rangle & 0\\ 0 & 0 & 2\varrho_b\end{pmatrix},\\
\varrho_b^2 &= \frac{\lambda^2\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2}{(2\mu+\lambda)^2} - \frac{2\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] + \frac{\langle N_{\text{res}},\overline{R}_3\rangle^2}{(2\mu+\lambda)^2\,h^2}\\
\|\overline{R}^T A_r + A_r^T\overline{R}\|^2 &= 4\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + 4\,\varrho_b^2\\
&= 4\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{4\lambda^2}{(2\mu+\lambda)^2}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\\
&\quad - \frac{8\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] + \frac{4\langle N_{\text{res}},\overline{R}_3\rangle^2}{(2\mu+\lambda)^2\,h^2}
\end{aligned} \tag{3.10.170}$$

$$\begin{aligned}
\operatorname{tr}\left[\overline{R}^T A_r + A_r^T\overline{R}\right]^2 &= \left(2\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] + 2\,\varrho_b\right)^2\\
&= 4\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 + 8\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\varrho_b + 4\,\varrho_b^2\\
&= 4\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 - \frac{8\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 + \frac{8\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\\
&\quad + \frac{4\lambda^2}{(2\mu+\lambda)^2}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 - \frac{8\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] + \frac{4\langle N_{\text{res}},\overline{R}_3\rangle^2}{(2\mu+\lambda)^2\,h^2}\\
&= \operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\cdot\left(4 - \frac{8\lambda}{2\mu+\lambda} + \frac{4\lambda^2}{(2\mu+\lambda)^2}\right) - \frac{8\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\\
&\quad + \frac{8\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] + \frac{4\langle N_{\text{res}},\overline{R}_3\rangle^2}{(2\mu+\lambda)^2\,h^2}\,.
\end{aligned} \tag{3.10.171}$$

Since

$$\begin{aligned}
&\frac{\mu}{4}\left(-\frac{8\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\right) + \frac{\lambda}{8}\left(-\frac{8\lambda\,\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)^2\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\right.\\
&\quad\left.+\frac{8\langle N_{\text{res}},\overline{R}_3\rangle}{(2\mu+\lambda)\,h}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\right) = \left(\frac{-2\mu\lambda}{(2\mu+\lambda)^2\,h} - \frac{\lambda^2}{(2\mu+\lambda)^2\,h} + \frac{\lambda}{(2\mu+\lambda)\,h}\right)\langle N_{\text{res}},\overline{R}_3\rangle\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]\\
&= \left(\frac{-(2\mu+\lambda)\lambda}{(2\mu+\lambda)^2\,h} + \frac{\lambda}{(2\mu+\lambda)\,h}\right)\langle N_{\text{res}},\overline{R}_3\rangle\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right] = 0\,,
\end{aligned} \tag{3.10.172}$$

therefore (the mixed term just cancels!)

$$\begin{aligned}
&\frac{\mu}{4}\|\overline{R}^T A_r + A_r^T\overline{R}\|^2 + \frac{\lambda}{8}\operatorname{tr}\left[\overline{R}^T A_r + A_r^T\overline{R}\right]^2\\
&= \mu\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{\mu\lambda^2}{(2\mu+\lambda)^2}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\\
&\quad + \frac{\lambda}{8}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\cdot\left(4 - \frac{8\lambda}{2\mu+\lambda} + \frac{4\lambda^2}{(2\mu+\lambda)^2}\right) + \frac{\langle N_{\text{res}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)\,h^2}\\
&= \mu\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \left[\frac{\mu\lambda^2}{(2\mu+\lambda)^2} + \frac{\lambda}{8}\cdot\left(4 - \frac{8\lambda}{2\mu+\lambda} + \frac{4\lambda^2}{(2\mu+\lambda)^2}\right)\right]\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2\\
&\quad + \frac{\langle N_{\text{res}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)\,h^2}\\
&= \mu\|\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}\left[\operatorname{sym}(\overline{R}^T(\nabla\overline{R}_3|0))\right]^2 + \frac{\langle N_{\text{res}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)\,h^2}\,.
\end{aligned} \tag{3.10.173}$$

A similar computation is now performed for the membrane contribution. Set

$$\begin{aligned}
\varrho_m &= 1 - \frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)} = 1 + \hat{\varrho}_m \\
\hat{\varrho}_m &= -\frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|0),\overline{R}\rangle - 2\right] + \frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)} = -\frac{\lambda}{2\mu+\lambda}\left[\langle(\nabla m|\overline{R}_3),\overline{R}\rangle - 3\right] + \frac{\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)}\,. \qquad (3.10.174)
\end{aligned}$$

Then

$$\begin{aligned}
\hat{\varrho}_m^2 &= \frac{\lambda^2\,\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\right]^2}{(2\mu+\lambda)^2} - \frac{2\lambda\,\langle N_{\text{diff}},\overline{R}_3\rangle}{(2\mu+\lambda)^2}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\right] + \frac{\langle N_{\text{diff}},\overline{R}_3\rangle^2}{(2\mu+\lambda)^2} \\
&\|\overline{R}^T A_m + A_m^T\overline{R} - 2 1\!\!1\|^2 = 4\|\,\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\|^2 + 4\,\hat{\varrho}_m^2 \qquad (3.10.175) \\
&\mathrm{tr}\left[\overline{R}^T A_m + A_m^T\overline{R} - 2 1\!\!1\right]^2 = (2\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\right] + 2\,\hat{\varrho}_m)^2\,.
\end{aligned}$$

Exactly the same computations as for the bending term allows us to conclude that

$$\begin{aligned}
&\frac{\mu}{4}\|\overline{R}^T A_r + A_r^T\overline{R} - 2 1\!\!1\|^2 + \frac{\lambda}{8}\mathrm{tr}\left[\overline{R}^T A_r + A_r^T\overline{R}\right]^2 \qquad (3.10.176) \\
&\quad = \mu\|\,\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\mathrm{tr}\left[\mathrm{sym}(\overline{R}^T(\nabla m|\overline{R}_3)) - 1\!\!1\right]^2 + \frac{\langle N_{\text{diff}},\overline{R}_3\rangle^2}{2(2\mu+\lambda)}\,.
\end{aligned}$$

Detailed computations for the new finite Reissner-Mindlin model

We compute

$$\begin{aligned}
F_s^T F_s &= \left(\tilde{A}_m + z\,\tilde{A}_r + \frac{z^2}{2}\tilde{B}_r\right)^T\left(\tilde{A}_m + z\,\tilde{A}_r + \frac{z^2}{2}\tilde{B}_r\right) \\
&= \tilde{A}_m^T\tilde{A}_m + z\,\tilde{A}_m^T\tilde{A}_r + \frac{z^2}{2}\tilde{A}_m^T\tilde{B}_r + z\,\tilde{A}_r^T\tilde{A}_m + z^2\,\tilde{A}_r^T\tilde{A}_r + \frac{z^3}{2}\tilde{A}_r^T\tilde{B}_r + \frac{z^2}{2}\tilde{B}_r^T\tilde{A}_m + \frac{z^3}{2}\tilde{B}_r^T\tilde{A}_r + \frac{z^4}{4}\tilde{B}_r^T\tilde{B}_r \\
&= \underbrace{\tilde{A}_m^T\tilde{A}_m}_{=:A} + z\underbrace{\left(\tilde{A}_m^T\tilde{A}_r + \tilde{A}_r^T\tilde{A}_m\right)}_{=:B} + \frac{z^2}{2}\underbrace{\left(\tilde{A}_m^T\tilde{B}_r + 2\,\tilde{A}_r^T\tilde{A}_r + \tilde{B}_r^T\tilde{A}_m\right)}_{=:C} + O(h^3)\,, \qquad (3.10.177)
\end{aligned}$$

and further

$$\begin{aligned}
\tilde{A}_m^T\tilde{A}_m &= (\nabla m|\,\varrho_m\,\vec{d})^T(\nabla m|\,\varrho_m\,\vec{d}) = \begin{pmatrix} \|m_x\|^2 & \langle m_x,m_y\rangle & \varrho_m\langle m_x,\vec{d}\rangle \\ \langle m_x,m_y\rangle & \|m_y\|^2 & \varrho_m\langle m_y,\vec{d}\rangle \\ \varrho_m\langle m_x,\vec{d}\rangle & \varrho_m\langle m_y,\vec{d}\rangle & \varrho_m^2 \end{pmatrix} \\
\mathrm{tr}\,[B] &= \mathrm{tr}\left[\tilde{A}_m^T\tilde{A}_r + \tilde{A}_r^T\tilde{A}_m\right] = 2\mathrm{tr}\left[\tilde{A}_m^T\tilde{A}_r\right] = 2\langle\tilde{A}_m,\tilde{A}_r\rangle = 2\left(\langle(\nabla m|\varrho_m\,\vec{d}),(\nabla(\varrho_m\vec{d})|0)\rangle + \varrho_m\,\varrho_b\right) \\
\langle B.e_3,e_3\rangle &= 2\langle\tilde{A}_m^T\tilde{A}_r.e_3,e_3\rangle = 2\langle\tilde{A}_m.e_3,\tilde{A}_r.e_3\rangle = 2\langle\varrho_m\,\vec{d},\varrho_b\,\vec{d}\rangle = 2\,\varrho_m\,\varrho_b\,. \qquad (3.10.178)
\end{aligned}$$

We neglect the $O(h^3)$ contribution and insert $z = \pm h/2$. This yields two equations

$$\begin{aligned}
&\langle\left[\mu(A - 1\!\!1 + (h/2)B + h^2/4C) + \frac{\lambda}{2}\mathrm{tr}\left[A - 1\!\!1 + (h/2)B + h^2/4C\right]1\!\!1\right].e_3,e_3\rangle = \langle N^+,(\varrho_m + (h/2)\,\varrho_b)\,\vec{d}\rangle \qquad (3.10.179) \\
&\langle\left[\mu(A - 1\!\!1 - (h/2)B + h^2/4C) + \frac{\lambda}{2}\mathrm{tr}\left[A - 1\!\!1 - (h/2)B + h^2/4C\right]1\!\!1\right].e_3,e_3\rangle = -\langle N^-,(\varrho_m - (h/2)\,\varrho_b)\,\vec{d}\rangle\,.
\end{aligned}$$

Adding and subtracting shows that

$$\begin{aligned}
2\langle\left[\mu(A - 1\!\!1 + h^2/4C) + \frac{\lambda}{2}\mathrm{tr}\left[A - 1\!\!1 + h^2/4C\right]1\!\!1\right].e_3,e_3\rangle &= \langle N^+ - N^-,\varrho_m\vec{d}\rangle + h/2\,\varrho_b\langle N^+ + N^-,\vec{d}\rangle \qquad (3.10.180) \\
\langle\left[\mu(hB) + \frac{\lambda}{2}\mathrm{tr}\,[hB]1\!\!1\right].e_3,e_3\rangle &= \langle N^+ - (-N^-),\varrho_m\,\vec{d}\rangle + h/2\,\langle N^+ - N^-,\varrho_b\,\vec{d}\rangle\,,
\end{aligned}$$

and to leading order in both equations

$$\begin{aligned}
2\langle\left[\mu(A - 1\!\!1) + \frac{\lambda}{2}\mathrm{tr}\,[A - 1\!\!1]1\!\!1\right].e_3,e_3\rangle &= \langle N^+ - N^-,\varrho_m\,\vec{d}\rangle \\
\langle\left[\mu B + \frac{\lambda}{2}\mathrm{tr}\,[B]1\!\!1\right].e_3,e_3\rangle &= \frac{1}{h}\langle N^+ + N^-),\varrho_m\,\vec{d}\rangle \qquad (3.10.181)
\end{aligned}$$

or

$$\langle\left[\mu(A - 1\!\!1) + \frac{\lambda}{2}\mathrm{tr}\,[A - 1\!\!1]1\!\!1\right].e_3,e_3\rangle = \langle N_{\text{diff}},\varrho_m\,\vec{d}\rangle, \quad \langle\left[\mu B + \frac{\lambda}{2}\mathrm{tr}\,[B]1\!\!1\right].e_3,e_3\rangle = \langle\frac{N_{\text{res}}}{h},\varrho_m\,\vec{d}\rangle\,. \qquad (3.10.182)$$

This yields for ϱ_m

$$
\begin{aligned}
&\varrho_m \,\langle N_{\rm diff}, \vec{d}\rangle = \left[\mu(\varrho_m^2-1) + \frac{\lambda}{2}\left(\|\nabla m\|^2 + \varrho_m^2 - 3\right)\right] \Rightarrow \\
&\left[2\mu(\varrho_m^2-1) + \lambda\left(\|\nabla m\|^2 + (\varrho_m^2-1) - 2\right)\right] = 2\varrho_m\,\langle N_{\rm diff}, \vec{d}\rangle \\
&(\varrho_m^2-1)(2\mu+\lambda) + \lambda[\|\nabla m\|^2 - 2] = 2\varrho_m\,\langle N_{\rm diff}, \vec{d}\rangle \\
&(\varrho_m^2-1) - 2\varrho_m\,\frac{\langle N_{\rm diff}, \vec{d}\rangle}{(2\mu+\lambda)} + \frac{\lambda}{(2\mu+\lambda)}[\|\nabla m\|^2 - 2] = 0 \\
&\varrho_m^2 - 2\varrho_m\,\frac{\langle N_{\rm diff}, \vec{d}\rangle}{(2\mu+\lambda)} + \frac{\lambda}{(2\mu+\lambda)}[\|\nabla m\|^2 - 2] - 1 = 0 \\
&\varrho_m = +\frac{\langle N_{\rm diff}, \vec{d}\rangle}{(2\mu+\lambda)} \pm \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}[\|\nabla m\|^2 - 2] + \frac{\langle N_{\rm diff}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}}\,, \qquad (3.10.183)
\end{aligned}
$$

and for ϱ_b

$$
\begin{aligned}
&\langle\left[\mu B + \frac{\lambda}{2}\operatorname{tr}[B]1\!\!1\right].e_3, e_3\rangle = \frac{1}{h}\langle N_{\rm res}, \varrho_m\,\vec{d}\rangle \\
&\frac{1}{h}\langle N_{\rm res}, \varrho_m\,\vec{d}\rangle = 2\mu\varrho_m\,\varrho_b + \lambda\,\varrho_m\,\varrho_b + \lambda\,\langle(\nabla m|\varrho_m\,\vec{d}), (\nabla(\varrho_m\vec{d})|0)\rangle \Rightarrow \qquad (3.10.184) \\
&(2\mu+\lambda)\,\varrho_m\,\varrho_b + \lambda\,\langle(\nabla m|\varrho_m\,\vec{d}), (\varrho_m\nabla\vec{d}|0) + (\varrho_{m,x}\vec{d}|\varrho_{m,y}\vec{d}|0)\rangle = \varrho_m\frac{1}{h}\langle N_{\rm res}, \vec{d}\rangle \qquad (3.10.185) \\
&(2\mu+\lambda)\,\varrho_b + \lambda\,\langle(\nabla m|\vec{d}), (\nabla\vec{d}|0) + \frac{1}{\varrho_m}(\varrho_{m,x}\vec{d}|\varrho_{m,y}\vec{d}|0)\rangle = \frac{1}{h}\langle N_{\rm res}, \vec{d}\rangle \\
&\varrho_b = -\frac{\lambda}{2\mu+\lambda}\langle(\nabla m|\vec{d}), (\nabla\vec{d}|0)\rangle + \frac{1}{(2\mu+\lambda)\,h}\langle N_{\rm res}, \vec{d}\rangle + \frac{1}{\varrho_m\,(2\mu+\lambda)}\langle(\nabla m|\varrho_m\vec{d}), (\varrho_{m,x}\vec{d}|\varrho_{m,y}\vec{d}|0)\rangle\,.
\end{aligned}
$$

Since we do not want to consider space variations in the thickness-stretch ϱ_m we take finally

$$
\begin{aligned}
\varrho_m &= \frac{\langle N_{\rm diff}, \vec{d}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\,[\|\nabla m\|^2 - 2] + \frac{\langle N_{\rm diff}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}} \\
\varrho_b &= -\frac{\lambda}{2\mu+\lambda}\langle(\nabla m|\vec{d}), (\nabla\vec{d}|0)\rangle + \frac{\langle N_{\rm res}, \vec{d}\rangle}{(2\mu+\lambda)\,h}\,. \qquad (3.10.186)
\end{aligned}
$$

3.10.5 Units and elastic constants

The body force f has units [N/m^3], the surface traction N has units [N/m^2], of course. Note that a typical value of the elastic moduli for steel is $\mu = 80.000[\mathrm{N/mm}^2] = 8\cdot 10^{10}[\mathrm{N/m}^2]$ and $\lambda = 100.000[\mathrm{N/mm}^2] = 10\cdot 10^{10}[\mathrm{N/m}^2] = 80.000MPa = 80GPa$.

The Youngs-modulus E and the Poisson number ν are defined in terms of the Lamé constants as follows:

$$
\begin{aligned}
&E := \mu\,\frac{2\mu+3\lambda}{\mu+\lambda}, \quad \nu := \frac{\lambda}{2(\mu+\lambda)}, \quad \lambda\to\infty \Leftrightarrow \nu\to\frac{1}{2} \\
&\lambda = \frac{E\,\nu}{(1+\nu)\,(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}\,. \qquad (3.10.187)
\end{aligned}
$$

This implies the well known relations

$$
\frac{1}{2}\frac{E}{1-\nu^2} = 2\mu\,\frac{(2\mu+3\lambda)(\mu+\lambda)}{(2\mu+\lambda)(2\mu+3\lambda)} = 2\mu\,\frac{(\mu+\lambda)}{(2\mu+\lambda)}\,. \qquad (3.10.188)
$$

It is also useful to have the physical properties of various very thin samples at hand. For A4-paper (80g/m^2), the thickness of a 20cm quadrangle is roughly 0.08mm which gives a characteristic value $h \approx \frac{1}{1000}$. Representative values for elastic moduli for isotropic standardized paper are $E = 5840[\mathrm{N/mm}^2]$, $\nu = 0.24$ or $\mu = 2.6GPa$, $\lambda = 2.34GPa$.

For kitchen plastic wrap one has the thickness 0.03mm which implies $h \approx \frac{3}{10000}$ and standard Aluminum foil has a thickness of 0.01mm implying $h \approx \frac{1}{10000}$. A typical thin film, for which we consider a 20mm rectangle with thickness as small as 5 micrometers ($5\cdot 10^{-6}$m) yields a characteristic thickness of $h \approx \frac{5}{10000}$. In the special case of e.g. a steel rod of length 1m and radius 2mm we obtain the characteristic variable $h \approx \frac{4}{1000}$. For such small values of h it seems to be clear that classical bending cannot play a prominent role.

3.10.6 The penalized finite Cosserat plate

While the treatment of the microrotations $\overline{R}$ is conceptually clear, any numerical implementation has the burden that the rotations live on a nonlinear manifold. In order to circumvent this difficulty, we propose a simplified variant of our new Cosserat plate model, where we relax the constraint of exact rotations and add a penalizing term. The new minimization problem reads: find the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and the relaxed 'microrotation' of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathbb{M}^{3\times 3}$ solving on ω:

$$\begin{aligned}
I &= \int_\omega h\, W_{\mathrm{mp}}(\overline{U}) + h\, W_{\mathrm{curv}}(\mathfrak{K}_s) + \frac{h^3}{12}\, W_{\mathrm{bend}}(\mathfrak{K}_b) + \frac{\aleph}{4}\|\overline{R}^T\overline{R} - 1\!\!1\|^2 \,\mathrm{d}\omega - \Pi(m, \overline{R}_3) \mapsto \min .\ \text{w.r.t. } (m, \overline{R}),\\
&\overline{U} = \overline{R}^T \hat{F}, \quad \hat{F} = (\nabla m|\overline{R}_3), \quad \textbf{penalty:} \quad \aleph \to \infty\\
m_{|\gamma_0} &= g_{\mathrm{d}}(x,y,0)\,, \quad \overline{R}_{|\gamma_0} = \mathrm{polar}((\nabla m|\nabla g_{\mathrm{d}}(x,y,0).e_3))_{|\gamma_0}\,, \qquad (3.10.189)\\
W_{\mathrm{mp}}(\overline{U}) &= \mu\,\|\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\overline{U} - 1\!\!1)\right]^2, \quad W_{\mathrm{curv}}(\mathfrak{K}_s) = \mu\,\frac{L_c^2}{12}\,\|\mathfrak{K}_s\|^2\,,\\
\mathfrak{K}_s &= \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right)\\
W_{\mathrm{bend}}(\mathfrak{K}_b) &= \mu\,\|\mathrm{sym}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}(\mathfrak{K}_b)\right]^2, \quad \mathfrak{K}_b = \overline{R}^T(\nabla\overline{R}_3|0) = \mathfrak{K}_s^3\,.
\end{aligned}$$

It should be observed that the penalized model is still frame-indifferent, a welcome feature.

3.10.7 The partially linearized Cosserat plate

Another method of reducing the complexity of the ensuing model consists in **partially linearizing** the equations. Let us reduce a modification of case II ($\mu_c = 0$, $\alpha_4 = 0$, $q = 0$, $p = 1$, $\alpha_5 = \alpha_6 = 1$, $\alpha_7 = 0$) for situations in which we expect the curvature and microrotations to remain small but the midsurface deformations are unrestricted. We write $m(x,y) = (x,y,0)^T + v(x,y)$, with the (finite) displacement of the midsurface of the plate $v : \omega \mapsto \mathbb{R}^3$ and $\overline{R} = 1\!\!1 + \overline{A} + \ldots$ with $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$ the infinitesimal microrotation. For the boundary deformation we write $g_{\mathrm{d}}(x,y,z) = (x,y,z)^T + u^{\mathrm{d}}(x,y,z)$, with the consequence, that $\nabla g_{\mathrm{d}}.e_3 = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 1 + u^{\mathrm{d}}_{3,z})$. The curvature tensors are expanded as

$$\begin{aligned}
\mathfrak{K}_b &= \overline{R}^T(\nabla\overline{R}_3|0) = (1\!\!1 + \overline{A} + \ldots)^T\,(\nabla[\overline{A}_3 + \overline{A}^2.e_3 + \ldots]|0) \approx (\nabla\overline{A}_3|0) + \ldots\\
\mathfrak{K}_s &\approx \left((\nabla(\overline{A}.e_1)|0), (\nabla(\overline{A}.e_2)|0), (\nabla(\overline{A}.e_3)|0)\right)\,, \qquad (3.10.190)
\end{aligned}$$

and the Cosserat micropolar plate stretch tensor expands like

$$\begin{aligned}
\overline{U} &= \overline{R}^T F_s = \overline{R}^T(\nabla m|\overline{R}_3) = (1\!\!1 + \overline{A} + \ldots)^T \left(\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix} + \nabla v|(1\!\!1 + \overline{A} + \ldots).e_3)\right) \approx 1\!\!1 + (\nabla v|\overline{A}_3) + \overline{A}^T + \overline{A}^T(\nabla v|\overline{A}_3)\\
&\approx 1\!\!1 + (\nabla v|\overline{A}_3) + \overline{A}^T + \underbrace{\overline{A}^T(\nabla v|0)}_{\text{drill rotations}} + \overline{A}^T(0|0|\overline{A}_3)\,. \qquad (3.10.191)
\end{aligned}$$

Neglecting the quadratic term $\overline{A}^T(0|0|\overline{A}_3)$ in view of the expected smallness of rotations, we are indeed left with the minimization problem for $v \in \mathbb{R}^3$ and $A \in \mathfrak{so}(3,\mathbb{R})$

$$\begin{aligned}
&\int_\omega h\,\left(\mu\,\|\mathrm{sym}((\nabla v|\overline{A}_3) + \overline{A}^T(\nabla v|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}((\nabla v|\overline{A}_3) + \overline{A}^T(\nabla v|0))\right]^2 + \mu\,\frac{L_c^2}{12}\|\mathrm{D_x}\overline{A}\|^2\right)\\
&\quad + \frac{h^3}{12}\,\left(\mu\,\|\mathrm{sym}((\nabla\overline{A}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\left[\mathrm{sym}((\nabla\overline{A}_3|0))\right]^2\right)\,\mathrm{d}\omega - \Pi(v, \overline{A}_3) \mapsto \min .\ \text{w.r.t. } (v, \overline{A}),\\
v_{|\gamma_0} &= u^{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported} \qquad (3.10.192)\\
\overline{A}_{|\gamma_0} &= \mathrm{skew}((\nabla v|\partial_z u^{\mathrm{d}}))_{|\gamma_0}\,, \quad \text{linearized consistent coupling} \Rightarrow \overline{A}_{3_{|\gamma_0}} = \left(\frac{u^{\mathrm{d}}_{1,z} - v_{3,x}}{2}, \frac{u^{\mathrm{d}}_{2,z} - v_{3,y}}{2}, 0\right)^T\\
\overline{A}_{3_{|\gamma_0}} &= (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z}, 0)^T\,, \quad \text{rigid prescription}\,.
\end{aligned}$$

The internal length $L_c > 0$ is seen to be necessary to control the **in-plane drill rotations** which appear only as a **second order effect** in the model. The membrane part can be shown to be coercive w.r.t. v in $H^1(\omega, \mathbb{R}^3)$ if $\overline{A} \in C^0(\overline{\omega}, \mathfrak{so}(3,\mathbb{R}))$, since the second derivative w.r.t. v can be estimated through $\|\mathrm{sym}((1\!\!1 + \overline{A})^T(\nabla\phi|0))\|^2$ and $1\!\!1 + \overline{A} \in \mathrm{GL}(3,\mathbb{R})$ for $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$. The corresponding field equations are **semilinear**, more precisely, balance of linear momentum is a uniformly Legendre-Hadamard elliptic linear system w.r.t. v at given $\overline{A}$ and balance of angular momentum is a uniformly Legendre-Hadamard elliptic linear system w.r.t. $\overline{A}$ at given v with constant coefficients. Nevertheless, the resulting model is nonlinear but not frame-indifferent. While it is not entirely clear how to show existence, the simplified model with drill rotations should prove to be easily implemented along the lines of traditional infinitesimal Reissner-Mindlin models taking into account all the available knowledge on non-locking approximations.

Finally, we now re-derive the classical Reissner-Mindlin model in the infinitesimal context, pointing out certain 'inconsistencies' usually encountered and give a short existence proof. Let us sketch briefly the 'direct' derivation of (3.9.129) in the infinitesimal context in order to understand some of the peculiarities of plate modelling.

3.10.8 Derivation of the classical infinitesimal Reissner-Mindlin bending plate

If ε is the symmetrized displacement gradient of the three-dimensional theory, the elastic free energy of an isotropic medium takes the form

$$W_{\mathrm{infin}}(\varepsilon) = \mu\,\|\varepsilon\|^2 + \frac{\lambda}{2}\mathrm{tr}\,[\varepsilon]^2, \quad \sigma = 2\mu\,\varepsilon + \lambda\,\mathrm{tr}\,[\varepsilon]\,1\!\!1\,, \qquad (3.10.193)$$

where σ is the symmetric Cauchy stress tensor. If we assume zero normal traction **across** the thickness on $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, i.e. $\sigma_{33}(x,y,z) = 0$, $z \in [-\frac{h}{2}, \frac{h}{2}]$, then this implies immediately

$$0 = \sigma_{33} = 2\mu\,\varepsilon_{33} + \lambda\,\mathrm{tr}\,[\varepsilon]\,1 \Rightarrow \varepsilon_{33} = -\frac{\lambda}{2\mu+\lambda}\,(\varepsilon_{11} + \varepsilon_{22}) = -\frac{\lambda}{2\mu+\lambda}\,\langle\varepsilon, 1\!\!1_2\rangle\,, \qquad (3.10.194)$$

still on the three-dimensional level. Applying (3.10.194) and eliminating ε_{33} in the elastic energy (so called **condensation of the material law**) yields,

$$\begin{aligned} W_{\mathrm{infin}}^{\mathrm{stress}}(\varepsilon) &= \mu\,\|\varepsilon\|^2 + \frac{\lambda}{2}\mathrm{tr}\,[\varepsilon]^2 \\ &= \mu\,\|\varepsilon\,1\!\!1_2\|^2 + \mu\,(\varepsilon_{31}^2+\varepsilon_{32}^2) + \mu\,\frac{\lambda^2}{(2\mu+\lambda)^2}\langle\varepsilon, 1\!\!1_2\rangle^2 + \frac{\lambda}{2}\left(\langle\varepsilon, 1\!\!1_2\rangle^2 - 2\langle\varepsilon, 1\!\!1_2\rangle\frac{\lambda}{2\mu+\lambda}\langle\varepsilon, 1\!\!1_2\rangle + \frac{\lambda^2}{(2\mu+\lambda)^2}\langle\varepsilon, 1\!\!1_2\rangle^2\right) \\ &= \mu\,\|\varepsilon\,1\!\!1_2\|^2 + \mu\,(\varepsilon_{31}^2+\varepsilon_{32}^2) + \frac{\lambda}{2}\frac{2\mu}{2\mu+\lambda}\langle\varepsilon, 1\!\!1_2\rangle^2 . \end{aligned} \tag{3.10.195}$$

Now the **linear** kinematical ansatz $\varphi_s(x,y,z) = m + z\,\overline{R}.e_3$ together with $\overline{R} = 1\!\!1 + \overline{A} + \ldots$, $\theta = -\overline{A}_3$ implies to leading order

$$\varphi_s(x,y,z) = \begin{pmatrix} x+v_1(x,y) \\ y+v_2(x,y) \\ 0+v_3(x,y)\end{pmatrix} + z\begin{pmatrix}-\theta_1\\ -\theta_2 \\ z\end{pmatrix} + \ldots = \begin{pmatrix}x\\y\\z\end{pmatrix} + u(x,y,z) + \ldots, \quad u(x,y,z) = \begin{pmatrix} v_1(x,y) - z\theta_1(x,y) \\ v_2(x,y) - z\theta_2(x,y) \\ v_3(x,y)\end{pmatrix} . \tag{3.10.196}$$

Hence for $v_1, v_2 = 0$ we get

$$\varepsilon = \frac{1}{2}(\nabla u^T + \nabla u) = \begin{pmatrix} -z\,\theta_{1,x} & -\frac{z}{2}\,(\theta_{1,y}+\theta_{2,x}) & \frac{v_{3,x}-\theta_1}{2} \\ -\frac{z}{2}\,(\theta_{1,y}+\theta_{2,x}) & -z\,\theta_{2,y} & \frac{v_{3,y}-\theta_2}{2} \\ \frac{v_{3,x}-\theta_1}{2} & \frac{v_{3,y}-\theta_2}{2} & 0 \end{pmatrix} . \tag{3.10.197}$$

Explicitly integrating over the thickness with respect to z results in (3.9.129) with $\kappa = 1$. We note that this derivation seems to be not fully consistent: the linear kinematical ansatz yields $\varepsilon_{33} = 0$, while we use $\varepsilon_{33} = -\frac{\lambda}{2\mu+\lambda}\langle\varepsilon, 1\!\!1_2\rangle \neq 0$ in evaluating the elastic free energy. The zero normal traction condition σ_{33} is true for the chosen kinematical ansatz only on the midsurface while in the derivation we have tacitly assumed it to hold uniformly over the thickness. However, the final result is correct.

3.10.9 The classical infinitesimal Kirchhoff bending plate

For the convenience of the reader we also supply the similar system of equations for the classical Kirchhoff bending plate. If only transverse deflections $v_3(x,y)$ are considered, the energy to be minimized is

$$W_{\mathrm{K}}^{\mathrm{class}} = \frac{h^3}{12}\left(\frac{\mu}{4}\,\|\nabla(\nabla v_3)^T + \nabla(\nabla v_3)\|^2 + \frac{2\mu\lambda}{8(2\mu+\lambda)}\mathrm{tr}\left[\nabla(\nabla v_3)^T + \nabla(\nabla v_3)^T\right]^2\right) = \frac{h^3}{12}\left(\mu\,\|D^2 v_3\|^2 + \frac{\lambda}{2}\frac{2\mu}{2\mu+\lambda}\mathrm{tr}\left[D^2 v_3\right]^2\right) , \tag{3.10.198}$$

since $\nabla(\nabla v_3) = D^2 v_3$. This energy can be obtained formally from (3.9.129) by setting $\theta = \nabla v_3$, see [Bra92, p.266]. It should be clear, however, that these bending equations are only appropriate for deflections $v_3 \ll h$. For $v_3 \approx h$ combined membrane/bending needs to be used and for $v_3 \gg h$ the membrane effect dominates.

Let us turn quickly to the existence theory [Dav75] involved in the infinitesimal case:

Theorem 3.10.9 (Existence for infinitesimal Reissner-Mindlin)
Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^2(\omega, \mathbb{R}^3)$. Moreover, let $\overline{f} \in L^2(\omega, \mathbb{R}^3)$ and suppose $\overline{N} \in L^2(\gamma_s, \mathbb{R}^3)$ together with $\overline{M} \in L^1(\omega, \mathbb{R}^3)$ and $\overline{M}_c \in L^1(\gamma_s, \mathbb{R}^3)$, see (3.9.102). Then problem (3.9.127) admits a **unique** *minimizing solution pair $(v, \theta) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \mathbb{R}^3)$.*

Proof. By the direct methods of variations it is a simple matter to establish the existence of a solution: Since the functional is bounded above, we may take infimizing sequences $(v_k, \theta_k) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \mathbb{R}^3)$ and establish weak convergence of $\theta_k \rightharpoonup \theta \in H^1(\omega, \mathbb{R}^3)$, strong in $L^2(\omega, \mathbb{R}^3)$. This implies the boundedness of $v_k \in H^1(\omega, \mathbb{R}^3)$ by Korn's first inequality and establishes as well that the functional is bounded below. We may extract a subsequence v_k not relabelled, converging weakly to $v \in H^1(\omega, \mathbb{R}^3)$. Overall convexity of the functional allows us to pass to the limit. The pair (v, θ) is a minimizer.

The general infinitesimal problem is easily seen to have a **unique** solution (v, θ) on account of the strict positivity of the second derivative of the energy $W_{\mathrm{RM,infin}}$:

$$D^2 W_{\mathrm{RM,infin}}(\nabla v, \theta).((\nabla\phi, \delta\theta), (\nabla\phi, \delta\theta)) \geq h\,\mu\,\|\mathrm{sym}((\nabla\phi|\delta\theta)\|^2 + \frac{h^3}{12}\mu\,\|\nabla\delta\theta\|^2 , \tag{3.10.199}$$

on the linear space $H_0^1(\omega, \mathbb{R}^3) \times H_0^1(\omega, \mathbb{R}^3)$. Strict positivity is a consequence of the classical Korn's inequality for the membrane part and full control of the skew-symmetric increment in the bending part. In this case, the drill rotations, which are associated to α, remain unspecified. Since only two independent simple rotations are required to orient a unit director field, a distinctive feature of classical plate and shell theories is a rotation field defined in terms of only two independent degrees of freedom: rotations about the director itself-the so called drill rotations, are irrelevant and for that matter undefined in classical shell theory. ■

The analysis based on (3.9.128) is even simpler and can be done with Poincaré's inequality replacing Korn's inequality. Note, however, that a numerical implementation of the linearized models based on the presented setting (displacement approach) shows to perform badly on coarse meshes [Bra92] for small $h > 0$ due to **shear locking**.

3.10.10 Comparison of formulas for the thickness stretch

The different formulas for the thickness stretch ϱ_m in the plate models will be elucidated. We have

$$\varrho_m := \begin{cases} 1 - \frac{\lambda}{2\mu+\lambda}\mathrm{tr}\left[\overline{R}^T(\nabla m|\overline{R}_3) - 1\!\!1\right] + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle}{(2\mu+\lambda)} & \text{new Cosserat plate (3.4.42)} \\ \frac{1}{1+\frac{\lambda}{2\mu+\lambda}(\det[(\nabla m|\overline{R}_3)]-1)} + \frac{\langle N_{\mathrm{diff}}, \overline{R}_3\rangle}{(2\mu+\lambda)} & \text{modified new Cosserat plate (3.5.70)} \\ \frac{\langle N_{\mathrm{diff}}, \vec{d}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\left[\|\nabla m\|^2 - 2\right] + \frac{\langle N_{\mathrm{diff}}, \vec{d}\rangle^2}{(2\mu+\lambda)^2}} & \text{new finite Reissner-Mindlin plate (3.6.82)} \\ \frac{\langle N_{\mathrm{diff}}, \vec{n}\rangle}{(2\mu+\lambda)} + \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\left[\|\nabla m\|^2 - 2\right] + \frac{\langle N_{\mathrm{diff}}, \vec{n}\rangle^2}{(2\mu+\lambda)^2}} & \text{finite Kirchhoff-Love plate (3.7.83) .} \end{cases} \tag{3.10.200}$$

Now assume that $N_{\text{diff}} = 0$ and $\overline{R} = \text{polar}(\nabla m|\vec{n})$. Let $\lambda_1, \lambda_2 \geq 0$ be the eigenvalues of $\sqrt{\nabla m^T \nabla m}$. In terms of λ_1, λ_2 we distinguish the cases

$$\varrho_m^{\text{COSS}} = 1 - \frac{\lambda}{2\mu+\lambda}\left[\lambda_1 + \lambda_2 - 2\right], \quad \varrho_m^{\text{LARGE}} = \frac{1}{1 + \frac{\lambda}{2\mu+\lambda}\left[\lambda_1\,\lambda_2 - 1\right]}, \quad \varrho_m^{\text{KL}} = \sqrt{1 - \frac{\lambda}{(2\mu+\lambda)}\left[\lambda_1^2 + \lambda_2^2 - 2\right]}\,. \tag{3.10.201}$$

To further simplify the exposition, take $\lambda = 2\mu$ and assume that $\lambda_1 = \lambda_2 = |\zeta|$. Then

$$\varrho_m^{\text{COSS}} = 2 - |\zeta|, \qquad \varrho_m^{\text{LARGE}} = \frac{2}{1+\zeta^2}, \qquad \varrho_m^{\text{KL}} = \sqrt{2-\zeta^2}\,. \tag{3.10.202}$$

All three formulas produce the same tangent at the identity $\zeta = 1$ (no in-plane stretch). In the Kirchhoff-Love model, evaluation of $\varrho_m^{\text{KL}} = \sqrt{2-\zeta^2}$ is only possible for $\zeta \leq \sqrt{2}$, a severe shortcoming of the model. In the new Cosserat plate model, evaluation of ϱ_m^{Coss} is possible for all $\zeta \in \mathbb{R}$ but does make sense only for $\zeta \leq 2$. Finally, the modified Cosserat model allows useful evaluation for all $\zeta \in \mathbb{R}$.

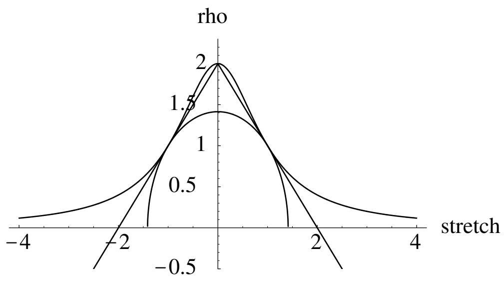

Figure 3.2: The different formulas for the thickness stretch ϱ_m: unphysical response of the Kirchhhoff-Love model and reasonable response $\varrho_m > 0$, $(\nabla m|\varrho_m \overline{R}_3) \in \text{GL}^+(3,\mathbb{R})$ of the modified Cosserat model.

3.10.11 Open questions

Show that

$$\int_\Omega \|\overline{R}^T \nabla\varphi + \nabla\varphi^T \overline{R} - 2\mathbb{1}\|^2 \,\text{dV} \mapsto \min\,.\ \text{w.r.t.}\ (\varphi, \overline{R}).\,, \tag{3.10.203}$$

has the unique family of solutions $\overline{R} = Q = \text{const.}$, $\varphi(x) = Q.x + b$, $Q \in \text{SO}(3,\mathbb{R})$. Without gradient constraint on φ the solution is not unique. The same problem for plates and shells: show that

$$\int_\omega \|\overline{R}^T(\nabla m|\overline{R}_3) + (\nabla m|\overline{R}_3)^T \overline{R} - 2\mathbb{1}\|^2 \,\text{d}\omega \mapsto \min\,.\ \text{w.r.t.}\ (m, \overline{R}).\,, \tag{3.10.204}$$

has the unique family of solutions $\overline{R} = Q = \text{const.}$, $m(x,y) = Q_1 \cdot x + Q_2 \cdot y + b$, $Q \in \text{SO}(3,\mathbb{R})$. Without gradient constraint on m the solution is not unique, but it can be seen that $\overline{R}_3 = \vec{n}$ must hold anyway. The same question turned around: assume that $\varphi \in H^1(\Omega,\mathbb{R}^3)$, $\overline{R} \in \text{SO}(3,\mathbb{R})$ and $\overline{R}^T \nabla\varphi + \nabla\varphi^T \overline{R} - 2\mathbb{1} = 0$. Show that this implies **rigidity**: $\overline{R} = Q = \text{const.}$, $\varphi(x) = Q.x + b$ and that we are dealing with a true strain measure.

Chapter 4

Notation

4.1 Notation for bulk material

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times 3} \;|\det[X] \neq 0\}$ the general linear group, $\mathrm{SL}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|\det[X] = 1\}$, $\mathrm{O}(3) := \{X \in \mathrm{GL}(3,\mathbb{R}) \mid X^T X = 1\!\!1\}$, $\mathrm{SO}(3,\mathbb{R}) := \{X \in \mathrm{GL}(3,\mathbb{R}) \;|X^T X = 1\!\!1,\ \det[X] = 1\}$ with corresponding Lie-algebras $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3} \;|X^T = -X\}$ of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times 3} \;|\operatorname{tr}[X] = 0\}$ of traceless tensors. With $\operatorname{Adj} X$ we denote the tensor of transposed cofactors $\mathrm{Cof(X)}$ such that $\operatorname{Adj} X = \det[X]\,X^{-1} = \mathrm{Cof(X)^T}$ if $X \in \mathrm{GL}(3,\mathbb{R})$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$ and for vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i\,\eta_j$.

We write the polar decomposition in the form $F = RU = \operatorname{polar}(F)\,U$ with $R = \operatorname{polar}(F)$ the orthogonal part of F. In general we work in the context of nonlinear, finite elasticity. For the total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Furthermore, $S_1(F)$ and $S_2(F)$ denote the first and second Piola Kirchhoff stress tensors, respectively. Total time derivatives are written $\frac{\mathrm{d}}{\mathrm{dt}}X(t) = \dot{X}$. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H, H)$, respectively. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H_\circ^{1,2}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. Moreover, we set $\|X\|_\infty = \sup_{x\in\Omega}\|X(x)\|$. For $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ we define $\operatorname{Curl} A(x)$ as the operation curl applied row wise. We define $H_\circ^{1,2}(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$, where $\phi_{|_\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω. We use capital letters to denote possibly large positive constants, e.g. C^+, K and lower case letters to denote possibly small positive constants, e.g. c^+, d^+. The smallest eigenvalue of a positive definite symmetric tensor P is abbreviated by $\lambda_{\min}(P)$.

4.2 Notation for thin structures

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\omega$ and let γ_0 be a smooth subset of $\partial\omega$ with non-vanishing 1-dimensional Hausdorff measure. The thickness of the plate is taken to be $h > 0$ with dimension length (contrary to Ciarlet's definition of the thickness to be 2ε, which difference leads only to various different constants in the resulting formulas). We denote by $\mathbb{M}^{n\times m}$ the set of matrices mapping $\mathbb{R}^n \mapsto \mathbb{R}^m$. For $H \in \mathbb{M}^{2\times 3}$ and $\xi \in \mathbb{R}^3$ we employ also the notation $(H|\xi) \in \mathbb{M}^{3\times 3}$ to denote the matrix composed of H and the column ξ. Likewise $(v|\xi|\eta)$ is the matrix composed of the columns v, ξ, η. The identity tensor on $\mathbb{M}^{2\times 2}$ will be denoted by $1\!\!1_2$. The mapping $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ is the deformation of the midsurface, ∇m is the corresponding deformation gradient and $m_x = (m_{1,x}, m_{2,x}, m_{3,x})^T$, $m_y = (m_{1,y}, m_{2,y}, m_{3,y})^T$. Sometimes, this is also written as ∇m. We write $v : \mathbb{R}^2 \mapsto \mathbb{R}^3$ for the displacement of the midsurface, such that $m(x, y) = (x, y, 0)^T + v(x, y)$. The standard volume element is written $\mathrm{dx\,dy\,dz} = \mathrm{dV} = \mathrm{d}\omega\,\mathrm{dz}$.

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Erklärung

Ich habe diese Habilitationsschrift, abgesehen von den ausdrücklich genannten Hilfsmitteln, selbständig verfaßt.

Darmstadt, den 20.Oktober 2003

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