

**Notes on strain-gradient plasticity. Finite strain
invariant modelling and global existence in the
infinitesimal rate-independent case.**

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Plan of the talk

- Notation, Motivation
- Second law of thermodynamics with gradients of inner variables
- Higher order boundary conditions - insulation condition
- Invariance, dislocation density measure G
- Finite strain elasto-plastic model, linearized model, Legendre-transformation
- Reformulation as a variational inequality of mixed type
- Bilinear form and new Hilbert space: existence and uniqueness
- Open problems

Notation

- $\varphi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$, deformation, Ω reference configuration
- $F = \nabla\varphi \in \text{GL}^+(3, \mathbb{R})$, deformation gradient, $\mathbb{1}$ identity tensor
- $W(F)$ elastic free energy density, $S_1(F) = D_F W(F)$ first P. K. stresses
- $\|A\|$, $\langle A, B \rangle$, $\text{tr}[A]$ norm, scalar product and trace
- $\text{sym } X = \frac{1}{2}(X + X^T)$, $\text{skew } X = \frac{1}{2}(X - X^T)$, $\text{dev } X = X - \frac{1}{3}\text{tr}[X] \cdot \mathbb{1}$
- $F = F_e F_p$, **multiplicative decomposition**
- $\text{Curl } F_p = \begin{pmatrix} \text{curl}[F_p^1 & - & -] \\ \text{curl}[F_p^2 & - & -] \\ \text{curl}[F_p^3 & - & -] \end{pmatrix} \in \mathbb{M}^{3 \times 3} \quad \text{for } F_p = \begin{pmatrix} F_p^1 & - & - \\ F_p^2 & - & - \\ F_p^3 & - & - \end{pmatrix}$
- Note: $\text{Curl } F_p$ transpose of Gurtin's definition!

Motivation

- Apparent size-dependence of yield stress: Hall-Petch: $\sigma_y = \sigma_y^0 + \frac{k^+}{\sqrt{\text{grainsize}}}$ and Taylor scaling: $\sigma_y = \sigma_y^0 + k^+ \|\text{Curl } F_p\|$
- Model proposals: Acharya, Fleck, Garikipati, Geers, Gurtin, Needleman, Svendsen, Steinmann, Tvergaard, ...
- Regularization of discrete time-step minimization problems: Ortiz, Mielke/Müller, Miehe ..., regularization of softening plasticity: Reddy, ...
- Here: purely phenomenological approach based on **multiplicative decomposition** $F = F_e F_p$, but **no crystallographic model**
- **No plastic metric** C_p , basic variable plastic distortion $F_p \in \text{SL}(3)$. Full spatial, intermediate and referential invariance under superposed rigid rotations - based on F_p **with plastic spin**
- Simplifying assumption: **normality of flow rule** and **rate-independency**: $F_p \frac{d}{dt} F_p^{-1} \in -\partial \chi(\Sigma)$ with **χ indicator function of elastic domain** K

Thermodynamics with internal variables: global form

Free energy $W(\nabla\varphi, P, \nabla P)$ depends on **inner variable** P and **gradient of inner variable** ∇P . Second law of thermodynamics: extended principle of positive dissipation. For all "nice" test volumes $\mathcal{V} \subseteq \Omega$

$$\int_{\mathcal{V}} \underbrace{\langle S_1(F(t), P(t), \nabla P(t)), \nabla\varphi_t \rangle - \frac{d}{dt} W(\nabla\varphi(t), P(t), \nabla P(t))}_{\geq 0 \text{ classical 2. law}} + \int_{\partial\mathcal{V}} \langle q(\nabla\varphi, P, \dot{P}, \nabla P), N \rangle dV \geq 0.$$

Here, $\int_{\partial\mathcal{V}} \langle q(\nabla\varphi, P, \dot{P}, \nabla P), N \rangle dV$ **working of the gradient of inner variable** over the (internal) boundary $\partial\mathcal{V}$ (extra energy flux).

Constitutive assumption for flux q ?

Thermodynamics with internal variables: local form

Free energy $W(\nabla\varphi, P, \nabla P)$ depends on **inner variable** P and **gradient of inner variable** ∇P . Second law of thermodynamics: extended principle of positive dissipation. For all "nice" test volumes $\mathcal{V} \subseteq \Omega$

$$\int_{\mathcal{V}} \langle S_1(F(t), P(t), \nabla P(t)), \nabla\varphi_t \rangle - \frac{d}{dt} W(\nabla\varphi(t), P(t), \nabla P(t)) + \operatorname{Div} q(\nabla\varphi, P, \dot{P}, \nabla P) \, dV \geq 0 \Leftrightarrow$$
$$\int_{\mathcal{V}} \frac{d}{dt} W(\nabla\varphi(t_0), P(t), \nabla P(t)) - \operatorname{Div} q(\nabla\varphi, P, \dot{P}, \nabla P) \, dV \leq 0 \quad \text{at frozen } t_0 \in \mathbb{R}.$$

Introduce **extra energy flux** q suitably! (I. Müller, ARMA67)

Maugin99: "... select (the **extra energy flux** q) in such a form as to eliminate any true divergence term in the dissipation inequality".

Chosen energy W induces q !

Global energy inequality and insulation condition

Assume: Rate of change of the free energy is bounded by classical power of external forces, leads to a **priori-energy inequality** and motivates bc on inner variables:

$$\begin{aligned} \int_{\Omega} W(\nabla\varphi(t), P(t), \nabla P(t)) \, dV &\leq \int_{\Omega} W(\nabla\varphi(0), P(0), \nabla P(0)) \, dV \\ &+ \underbrace{\int_0^t \int_{\Omega} \langle f(t), \varphi_t(t) \rangle \, dV \, dt + \int_0^t \int_{\partial\Omega} \langle \varphi_t(t), S_1(\nabla\varphi(t), P(t), \nabla P(t)) \cdot \vec{n} \rangle \, dS \, dt}_{\text{classical external work of forces}} . \end{aligned}$$

Sufficient (variational derivative)

$$\frac{d}{dt}P = \partial\chi \left(-[D_P W - \text{Div } D_{\nabla P} W] \right), \quad \langle \partial\chi(X), X \rangle \geq 0,$$

$$\int_{\partial\Omega} \left\langle \frac{d}{dt}P, D_{\nabla P} W \cdot \vec{n} \right\rangle \, dS = \int_{\Omega} \text{Div} \left(\underbrace{D_{\nabla P} W^T \cdot \frac{d}{dt}P}_q \right) \, dV = 0, \quad \text{insulation condition}$$

no nonlocal energy exchange across the external boundary is permitted.

Formulation in a variational setting

Multiplicative decomposition $F = F_e F_p$. Define thermodynamic potential $W(F_e, F_p, \text{Curl } F_p)$

$$W(F_e, F_p, \text{Curl } F_p) = W_e(F_e) + W_{\text{sh}}(F_p) + W_{\text{curl}}(F_p, \text{Curl } F_p)$$

How to choose $W_e(F_e)$, $W_{\text{sh}}(F_p)$ **and** $W_{\text{curl}}(F_p, \text{Curl } F_p)$?

Guiding principles:

- do not induce anisotropic behaviour unless explicitly accounted for (e.g. with structural tensors, invariant setting, crystallographic glide-planes).
- be consistent with established finite-strain formulations in the zero-plastic length scale limit $L_c \rightarrow 0$.
- ensure downwards compatibility with classical infinitesimal plasticity ($\varepsilon = \varepsilon_e + \varepsilon_p$). External working only through φ , induce no Dirichlet boundary conditions on ε_p .

Postulate of referential, intermediate and spatial invariance

Multiplicative decomposition $F = F_e F_p$. **Elastic energy** W_e .

Plastic indifference (Mielke, Miehe),
elastic isomorphy (Bertram) $\Rightarrow W_e = W_e(F_e)$

Frame-indifference: $W_e(F_e) = W_e(F_e^T F_e)$.

Elastic isotropy: $\forall Q \in \text{SO}(3) : W_e(F_e Q) = W_e(F_e)$.

Invariance of local hardening: $\forall Q_2, Q_3 \in \text{SO}(3) : W_{\text{sh}}(Q_3^T F_p Q_2) = W_{\text{sh}}(F_p)$.

Invariance of nonlocal hardening: $\forall Q_2, Q_3 \in \text{SO}(3) :$
 $W_{\text{curl}}(Q_3^T X Q_2, Q_3^T Y Q_2) = W_{\text{curl}}(X, Y)$ with $W_{\text{curl}}(F_p, \text{Curl } F_p)$.

Contrary to purely local model, these requirements do not imply that theory can be reduced to $C_p = F_p^T F_p$!

Similar conclusions based on more general invariance requirements:
Cermelli/Gurtin00, Parry/Silhavy99.

The referential dislocation density measure

Gurtin, Parry/Silhavy, Svendsen, Mielke/Müller, Menzel/Steinmann, Miehe:
geometric dislocation density tensor in the intermediate configuration

$$G = \frac{1}{\det[F_p]} (\text{Curl } F_p) F_p^T .$$

Referential measure of dislocation density G_R (Gurtin00) given by

$$G_R = F_p^{-1} \text{Curl } F_p , \quad G = \frac{1}{\det[F_p]} (\text{Curl } F_p) F_p^T = \frac{1}{\det[F_p]} F_p G_R F_p^T .$$

Introduces in a natural way nonlocal kinematical backstress due to dislocation processes.

On the reference configuration, use $W_{\text{curl}}(F_p, \text{Curl } F_p) = W_{\text{curl}}(G_R)$ and for control of plastic spin add: $sp_0 \|R_p^T \text{Curl } R_p\|^2$, where $F_p = R_p U_p$.

The thermodynamic free energy

$$W(F_e, F_p, \text{Curl } F_p) = W_e(F_e) + W_{\text{sh}}(F_p) + W_{\text{curl}}(F_p, \text{Curl } F_p)$$

$$W_e(F_e) = \frac{\mu}{4} \left\| \frac{F_e^T F_e}{\det[F_e]^{2/3}} - \mathbb{1} \right\|^2 + \frac{\lambda}{4} \left((\det[F_e] - 1)^2 + \left(\frac{1}{\det[F_e]} - 1 \right)^2 \right),$$

$$W_{\text{sh}}(F_p) = \frac{\mu H_0}{4} \left\| \frac{F_p^T F_p}{\det[F_p]^{2/3}} - \mathbb{1} \right\|^2,$$

$$W_{\text{curl}}(F_p^{-1} \text{Curl } F_p) = \frac{\mu L_c^2}{2} \left(\|F_p^{-1} \text{Curl } F_p\|^2 + sp_0 \|R_p^T \text{Curl } R_p\|^2 \right).$$

μ, λ Lamé-parameters, H_0 dimensionless hardening modulus, L_c internal plastic length, $sp_0 \in [0, 1]$. Energy is invariant w.r.t. reference, intermediate and spatial configuration and plastically indifferent.

Obtain flow rule by principle of maximal dissipation and thermodynamically conjugate forces: (for zero spin $sp_0 = 0$ only)

The finite strain gradient plasticity model (for $sp_0 = 0$)

$$\text{Div } S_1(F_e) = -f, \quad S_1(F_e) = D_F[W_e(F F_p^{-1})] = DW_e(F_e) \cdot F_p^{-T},$$

$$F_p \frac{d}{dt}[F_p^{-1}] \in -\partial\chi(\Sigma), \quad \Sigma = \Sigma_e + \Sigma_{\text{sh}} + \Sigma_{\text{curl}} \quad (\text{conjugate forces})$$

$$\Sigma_e = F_e^T DW_e(F_e), \quad \Sigma_{\text{sh}} = -DW_{\text{sh}}(F_p) F_p^T,$$

$$\begin{aligned} \Sigma_{\text{curl}} = & F_p^{-T} DW_{\text{curl}}(F_p^{-1} \text{Curl } F_p) (\text{Curl } F_p)^T \\ & - \text{Curl} (F_p^{-T} DW_{\text{curl}}(F_p^{-1} \text{Curl } F_p)) F_p^T, \end{aligned}$$

$$\varphi(x, t) = g_d(x, t), \quad F_p(x, t) \cdot \tau = F_p(x, 0) \cdot \tau, \quad x \in \Gamma_D,$$

$$0 = [\text{Curl } F_p(x, t)] \cdot \tau, \quad x \in \partial\Omega \setminus \Gamma_D, \quad F_p(x, 0) = F_p^0(x).$$

χ indicator of elastic domain K in stress space. τ : tangential directions on Γ_D .

Plastic indifference of dissipation (Mielke) $\Rightarrow$ rate of plastic deformation given by $F_p \frac{d}{dt}[F_p^{-1}]$.

The quadratic energy for the geometrically linear model

Expand $F = \mathbb{1} + \nabla u$, $F_p = \mathbb{1} + p + \dots$, $R_p = \mathbb{1} + \text{skew } p + \dots$, $F_e = \mathbb{1} + e + \dots$

Additive decomposition: $\nabla u = e + p$.

Infinitesimal plastic distortion p need not be symmetric! Quadratic energy

$$W(\nabla u, p, \text{Curl } p) = W_e^{\text{lin}}(\nabla u - p) + W_{\text{sh}}(p) + W_{\text{curl}}^{\text{lin}}(\text{Curl } p),$$

$$W_e^{\text{lin}}(\nabla u - p) = \mu \|\text{sym}(\nabla u - p)\|^2 + \frac{\lambda}{2} \text{tr} [\nabla u - p]^2,$$

$$W_{\text{sh}}^{\text{lin}}(p) = \mu H_0 \|\text{dev sym } p\|^2,$$

$$W_{\text{curl}}^{\text{lin}}(\text{Curl } p) = \frac{\mu L_c^2}{2} (\|\text{Curl } p\|^2 + sp_0 \|\text{Curl skew } p\|^2).$$

extra energy flux: $q = \mu L_c^2 \left(\frac{d}{dt} p \times \text{Curl } p + sp_0 \frac{d}{dt} [\text{skew } p] \times \text{Curl skew } p \right)$

insulation condition: no extra energy flux across external boundary: $\langle q, \vec{n} \rangle_{\partial\Omega} = 0$
(locally)

Mielke: No net energy flux across the external boundary globally!

The geometrically linear model: strong form

$$\operatorname{Div} \sigma = -f, \quad \sigma = 2\mu \operatorname{sym}(\nabla u - p) + \lambda \operatorname{tr} [\nabla u - p] \mathbb{1},$$

$$\dot{p} \in \partial \mathcal{X}(\Sigma^{\operatorname{lin}}), \quad \Sigma^{\operatorname{lin}} = \Sigma_{\mathrm{e}}^{\operatorname{lin}} + \Sigma_{\mathrm{sh}}^{\operatorname{lin}} + \Sigma_{\mathrm{curl}}^{\operatorname{lin}},$$

$$\Sigma_{\mathrm{e}}^{\operatorname{lin}} = 2\mu \operatorname{sym}(\nabla u - p) + \lambda \operatorname{tr} [\nabla u - p] \mathbb{1} = \sigma,$$

$$\Sigma_{\mathrm{sh}}^{\operatorname{lin}} = -2\mu H_0 \operatorname{dev} \operatorname{sym} p \quad (\text{local Prager backstress})$$

$$\Sigma_{\mathrm{curl}}^{\operatorname{lin}} = -\mu L_c^2 (\operatorname{Curl} \operatorname{Curl} p + sp_0 \operatorname{skew}(\operatorname{Curl} \operatorname{Curl} \operatorname{skew} p)),$$

$$u(x, t) = u_{\mathrm{d}}(x), \quad p(x, 0) = p^0(x),$$

.... and nonlocal boundary conditions .

If elastic domain $K := \{\|\operatorname{dev} \operatorname{sym} \Sigma\| \leq \sigma_{\mathrm{y}}\}$ then $\partial \mathcal{X}_K(\Sigma) = \mathbb{R}^+ \operatorname{dev} \operatorname{sym} \Sigma$ and $p \in \operatorname{Sym}(3)$ (**irrotational**). Suitable space for **symmetric** p is well-known $H_{\mathrm{curl}}(\Omega) := \{v \in L^2(\Omega), \operatorname{Curl} v \in L^2(\Omega)\}$.

Higher order boundary conditions

extra energy flux: $q = \mu L_c^2 \left(\frac{d}{dt} p \times \text{Curl } p + sp_0 \frac{d}{dt} [\text{skew } p] \times \text{Curl skew } p \right)$

insulation condition: no extra energy flux across external boundary: $\langle q, \vec{n} \rangle_{\partial\Omega} = 0$

Satisfy insulation condition (irrotational=no plastic spin= $p \in \text{Sym}$). Sufficient:

$$0 = [\text{Curl } p(x, t)] \cdot \tau, \quad x \in \partial\Omega$$

Consistent with $L_c \rightarrow 0$: no boundary condition on **plastic strain** $\varepsilon_p := \text{sym } p$!

Satisfy insulation condition ($p \notin \text{Sym}$, with plastic spin). Sufficient:

$$0 = [\text{Curl } p(x, t)] \cdot \tau, \quad x \in \partial\Omega,$$

$$0 = [\text{skew } p(x, t)] \cdot \tau, \quad x \in \Gamma_D,$$

$$0 = [\text{Curl skew } p(x, t)] \cdot \tau, \quad x \in \partial\Omega \setminus \Gamma_D$$

τ : tangential directions at boundary.

Uniqueness of strong solutions with these boundary conditions but also with other choices.

Legendre Transformation of the flow rule

Legendre transformation of indicator function χ of elastic domain K

$$R(\xi) = \chi^*(\xi) = \begin{cases} \sigma_y \|\xi\| & \text{tr} [\xi] = 0 \\ \infty & \text{else} \end{cases},$$

with R the **dissipation potential** (Carstensen, Hackl, Mielke, Reddy, ...). **R is one-homogeneous due to rate-independency** of χ . Rate-independency is appropriate for homogenized samples.

Convex analysis: **flow rule can be formulated equivalently**

$$\dot{p} \in \partial\chi(\Sigma^{\text{lin}}) \quad \Leftrightarrow \quad \Sigma^{\text{lin}} \in [\partial\chi]^{-1}(\dot{p}) \quad \Leftrightarrow \quad \Sigma^{\text{lin}} \in \partial R(\dot{p}) \quad \Leftrightarrow$$

$$R(q) \geq R(\dot{p}) + \langle \Sigma^{\text{lin}}, q - \dot{p} \rangle \quad \forall q \in L^2([0, T], \mathfrak{sl}(3)),$$

$$\Sigma^{\text{lin}} = \dots -\mu L_c^2 (\text{Curl Curl } p + sp_0 \text{skew}(\text{Curl Curl skew } p))$$

Weak reformulation: Variational inequality

Use insulation boundary conditions, partial integration and add weak form of elastic equilibrium. This yields mixed variational inequality: Strong solution (u, p) having "enough" regularity satisfies **weak primal problem of gradient-plasticity formulation**:

find $w = (u, p) \in Z = H^1((0, T); H_0^1(\Omega, \Gamma_D, \mathbb{R}^3)) \times \mathcal{H}_\#$ such that for almost all $t \in (0, T)$ it holds

$$\forall (v, q) \in H(\Omega) := H_0^1(\Omega, \Gamma_D, \mathbb{R}^3) \times \mathcal{H}_\# :$$
$$a \left(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v - \dot{u} \\ q - \dot{p} \end{pmatrix} \right) + \int_{\Omega} R(q) - R(\dot{p}) \, dV \geq \int_{\Omega} \langle f(x, t), v - \dot{u} \rangle \, dV .$$

Remaining local boundary condition for plastic distortion: skew $p \cdot \tau = 0$ on Γ_D .
 $\mathcal{H}_\#$ linear space for plastic distortion p .

Abbreviate $j(\dot{p}) := \int_{\Omega} R(\dot{p}) \, dV = \int_{\Omega} \mathcal{X}^*(\dot{p}) \, dV$ (integrated dissipation) and
 $\ell(w) := \int_{\Omega} \langle f(x, t), w \rangle \, dV$ (external body loads)

The associated bilinear form

$a : H(\Omega) \times H(\Omega) \mapsto \mathbb{R}^+$ for $w = (u_1, p_1)$, $z = (u_2, p_2)$

$$\begin{aligned} a(w, z) = & \int_{\Omega} 2\mu \langle \text{sym}(\nabla u_1 - p_1), \text{sym}(\nabla u_2 - p_2) \rangle + \lambda \text{tr} [\nabla u_1 - p_1] \text{tr} [\nabla u_2 - p_2] \\ & + 2\mu H_0 \langle \text{dev sym } p_1, \text{dev sym } p_2 \rangle \\ & + \mu L_c^2 (\langle \text{Curl } p_1, \text{Curl } p_2 \rangle + sp_0 \langle \text{Curl skew } p_1, \text{Curl skew } p_2 \rangle) \text{dV} . \end{aligned}$$

Simple local observation: $p \in \mathfrak{sl}(3)$ and local kinematical backstress yield

$$a(w, w) \geq c^+ \int_{\Omega} \underbrace{\|\text{sym } \nabla u\|^2}_{\text{Korn}} + \|\text{sym } p\|^2 + \|\text{Curl } p\|^2 + sp_0 \|\text{Curl skew } p\|^2 \text{dV} .$$

No local control of skew p ! Is a positive definite/coercive on a suitable Hilbert-space $\mathcal{H}_{\sharp}$ with Dirichlet-boundary conditions skew $p \cdot \tau = 0$ on Γ_D ?

Global Existence and Uniqueness

Theorem [Han/Reddy99]

W closed convex subset of Hilbert space $H(\Omega)$. Find $w \in W$ s.t. t a.e.

$$\forall z \in W : \quad a(w, z - w) + j(z) - j(w) \geq \ell(z - w). \quad (1)$$

Assume $a(\cdot, \cdot)$ is symmetric, continuous on $H(\Omega)$, and coercive on W .

Functional j non-negative, convex, positively homogeneous of grade one and Lipschitz-continuous, i.e.,

$$\forall s \in \mathbb{R} \quad j(sw) = |s| j(w), \quad |j(z) - j(w)| \leq L \|z - w\|_W.$$

Then (1) has unique global solution $w(t) \in W$. Method: time-discretization and passage to the limit. **Rate-independency of j essential.**

Solution of V-inequality satisfies global a priori energy inequality automatically!

Important finite strain generalization: Time-incremental energetic formulation based on dissipation distances by Mielke.

Discussion of boundary conditions only essential for linearized case!

$\mathcal{H}_\#$ - A new Hilbertspace for plastic spin

Set $\mathcal{X}_\# := \{p \in C^\infty(\overline{\Omega}, \mathbb{M}^{3 \times 3}) \mid \text{skew } p(x) \cdot \tau = 0, x \in \Gamma_D\}$. Define quadratic form for non-symmetric p

$$\|p\|_\#^2 := \|\text{sym } p\|_{L^2(\Omega)}^2 + \|\text{Curl } p\|_{L^2(\Omega)}^2 + \|\text{Curl skew } p\|_{L^2(\Omega)}^2$$

Norm on $\mathcal{X}_\#$ (control of skew p with Curl p only after integration!).

Completion $\mathcal{H}_\# := \overline{\mathcal{X}_\#}^{\|\cdot\|_\#}$ is a Hilbert-space with scalar product

$$\begin{aligned} \langle p_1, p_2 \rangle_\# &= \int_\Omega \langle \text{sym } p_1, \text{sym } p_2 \rangle + \langle \text{Curl } p_1, \text{Curl } p_2 \rangle \\ &\quad + \langle \text{Curl skew } p_1, \text{Curl skew } p_2 \rangle \, dV. \end{aligned}$$

$\mathcal{H}_\#$ continuously embedded in $L^2(\Omega, \mathbb{M}^{3 \times 3})$.

This shows that a is positive (strain part is controlled locally with symmetric kinematic hardening) on $H(\Omega) := H_0^1(\Omega, \Gamma_D, \mathbb{R}^3)) \times \mathcal{H}_\#$.

Open problems

- Repeat analysis for multiplicative decomposition $F = F_e F_p$ with quadratic energy $W(F_e) = \|\text{sym } F_e - \mathbb{1}\|^2 = \frac{1}{4} \|\nabla \varphi F_p^{-1} + F_p^{-T} \nabla \varphi^T - 2 \mathbb{1}\|^2$ and dislocation density $\|\text{Curl } F_p\|^2 \Rightarrow$ extended Korn's first inequality
- Local isotropic hardening (SSD - statistically stored dislocations) instead of local kinematical hardening? $\Rightarrow$ (Neff/Ebobisse/Reddy)
- Physically motivated choice of boundary conditions for infinitesimal plastic distortion p ? Status of 2.nd law of thermodynamics (insulation condition, strong formulation) versus a priori energy inequality satisfied by weak solution?
- Is the unique weak solution a strong solution if smooth? Relation to $\|\text{Curl } F_p\|$ -augmented time-incremental formulation (Ortiz, Mielke/Müller)?
- FEM for irrotational small strain model $\Rightarrow$ (Neff/Sydow/Wiener, submitted to Int. J. Num. Meth. Engrg.)

Extended Korn's first inequality

Theorem [N. PRSE02]

Let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL^+(3, \mathbb{R}))$ be given with $\det[F_p(x)] \geq c > 0$. Suppose **dislocation density** $\text{Curl } F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3 \times 3})$. Then

$$\exists c^+ > 0 \quad \forall \phi \in H^1(\Omega), \varphi|_{\Gamma} = 0 :$$

$$\|\nabla \phi F_p^{-1}(x) + F_p^{-T}(x) \nabla \phi^T\|_{L^2(\Omega)}^2 \geq c^+ \|\phi\|_{H^{1,2}(\Omega)}^2 .$$

Recent improvement: $F_p \in C^0(\overline{\Omega})$. Work in progress suggests $F_p \in L^\infty(\Omega)$, $\text{Curl } F_p \in L^2(\Omega)$ suffices. Counterexample for $F_p \in L^\infty(\Omega)$.

Classical Korn's first inequality:

$$\exists c^+ > 0 \quad \forall \phi \in H^1(\Omega), \varphi|_{\Gamma} = 0 : \quad \|\nabla \phi + \nabla \phi^T\|_{L^2(\Omega)}^2 \geq c^+ \|\phi\|_{H^{1,2}(\Omega)}^2 .$$

Definition of the Norm

Define $\mathcal{X} := \{p \in C^\infty(\bar{\Omega}, \mathbb{M}^{3 \times 3}) \mid p(x) \cdot \tau = 0, x \in \Gamma_D\}$. Why defines

$$\|p\|_{\sharp}^2 := \int_{\Omega} \|\operatorname{sym} p\|^2 + \|\operatorname{Curl} p\|^2 \, dV$$

a norm on $\mathcal{X}$? Need to show $\|p\|_{\sharp} = 0 \Rightarrow p = 0$. First: $\|p\|_{\sharp} = 0 \Rightarrow p \in \mathfrak{so}(3)$.
Two linear independent tangential vectors τ : $p(x) \cdot \tau = 0, x \in \Gamma_D$ and
 $p \in \mathfrak{so}(3) \Rightarrow p(x) = 0, x \in \Gamma_D$.

$$\begin{aligned} 0 &= \int_{\Omega} \|\operatorname{Curl} p\|^2 \, dV = \int_{\Omega} \|\operatorname{Curl} \operatorname{skew} p\|^2 \, dV \geq \frac{1}{2} \int_{\Omega} \|\nabla \operatorname{skew} p\|^2 \, dV, \\ \int_{\Omega} \|\nabla [\operatorname{skew} p]\|^2 \, dV &\geq c^+ \int_{\Omega} \|\operatorname{skew} p\|^2 \, dV \end{aligned}$$

Poincaré-inequality $\Rightarrow \operatorname{skew} p = 0 \Rightarrow p = 0$.

Problem: with this norm no continuous embedding in L^2 ? (skew-symmetric part is not properly controlled). No possibility to define weak traces $p \cdot \tau$? Remedy: add **plastic spin control**: $sp_0 \|\operatorname{Curl} \operatorname{skew} p\|^2$.

Rigidity estimates

Qualitative Version with relation to well known rigidity statements (John, Sverak, Müller, Friesecke)

Theorem [N./Münch ESAIM:COCV07]

$$\forall \mathbf{R} \in C^1(\mathbb{R}^3, \mathrm{SO}(3)) : \quad \|\mathrm{Curl} \mathbf{R}\|^2 \geq \frac{1}{2} \|\nabla \mathbf{R}\|^2.$$

No integration! Special case: $\mathbf{R} = \nabla \varphi$ and $\mathrm{Curl} \nabla \varphi = 0$ yields $\varphi(x) = \mathbf{R} \cdot x + b$.

Known "linear" result: $\mathbf{R} = \mathbb{1} + \mathbf{A} + \dots$, $\mathbf{A}^T + \mathbf{A} = 0$

$$\forall \mathbf{A} \in C^1(\mathbb{R}^3, \mathfrak{so}(3)) : \quad \|\mathrm{Curl} \mathbf{A}\|^2 \geq \frac{1}{2} \|\nabla \mathbf{A}\|^2.$$

Is implicitly used in the proof of Korn's first inequality:

$$\|\nabla u^T + \nabla u\|_{\Omega}^2 = 0 \Rightarrow \nabla u = \mathbf{A} \in \mathfrak{so}(3) \Rightarrow 0 = \mathrm{Curl} \nabla u = \mathrm{Curl} \mathbf{A} \Rightarrow \nabla \mathbf{A} = 0.$$

Kill constant skew-symmetric matrices $\mathbf{A}$ with boundary conditions!

Some useful relations

The convex, one-homogeneous dissipation functional R (Mielke/Hackl)

No-spin:
$$R(\dot{\xi}) = \begin{cases} \sigma_y \|\dot{\xi}\| & \dot{\xi} \in \text{Sym}(3) \cap \mathfrak{sl}(3) \\ \infty & \text{skew}(\dot{\xi}) \neq 0 \text{ or } \text{tr} [\dot{\xi}] \neq 0 \end{cases}$$

Determines the plastic distortion rate $\dot{p}$ to remain **symmetric** and trace-free, leads to **irrotational** model.

Free-spin :
$$R(\dot{\xi}) = \begin{cases} \sigma_y \|\text{sym } \dot{\xi}\| & \dot{\xi} \in \mathfrak{sl}(3) \\ \infty & \text{tr} [\dot{\xi}] \neq 0 \end{cases}$$

Determines the plastic distortion rate to remain trace-free, but local plastic rotational spin is not controlled.

For smooth fields $A, B \in \mathbb{M}^{3 \times 3}$ Stokes Theorem reads:

$$\int_{\Omega} \langle \text{Curl } A, \text{Curl } B \rangle \, dV = \int_{\Omega} \langle \text{Curl } \text{Curl } A, B \rangle \, dV + \int_{\partial\Omega} \langle \text{Curl } A \times B, N \rangle \, dS.$$

Selected References

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The irrotational geometrically linear model: no spin

Elastic domain $K := \{\|\operatorname{dev} \operatorname{sym} \Sigma\| \leq \sigma_y\}$ then $\partial\mathcal{X}_K(\Sigma) = \mathbb{R}^+ \operatorname{dev} \operatorname{sym} \Sigma$ and $p \in \operatorname{Sym}(3)$ (**irrotational**). Define **plastic strain** $\varepsilon_p := \operatorname{sym} p$.

$$\operatorname{Div} \sigma = -f, \quad \sigma = 2\mu \operatorname{sym}(\nabla u - \varepsilon_p) + \lambda \operatorname{tr}[\nabla u - \varepsilon_p] \mathbb{1},$$

$$\dot{\varepsilon}_p \in \partial\mathcal{X}(\Sigma^{\operatorname{lin}}), \quad \Sigma^{\operatorname{lin}} = \Sigma_e^{\operatorname{lin}} + \Sigma_{\operatorname{sh}}^{\operatorname{lin}} + \Sigma_{\operatorname{curl}}^{\operatorname{lin}},$$

$$\Sigma_e^{\operatorname{lin}} = 2\mu \operatorname{sym}(\nabla u - \varepsilon_p) + \lambda \operatorname{tr}[\nabla u - \varepsilon_p] \mathbb{1} = \sigma,$$

$$\Sigma_{\operatorname{sh}}^{\operatorname{lin}} = -2\mu H_0 \operatorname{dev} \varepsilon_p \quad (\text{linear Prager hardening})$$

$$\Sigma_{\operatorname{curl}}^{\operatorname{lin}} = -\mu L_c^2 \operatorname{sym}(\operatorname{Curl} \operatorname{Curl} \varepsilon_p),$$

$$u(x, t) = u_d(x), \quad \varepsilon_p(x, 0) = \varepsilon_p^0(x), \quad [\operatorname{Curl} \varepsilon_p] \cdot \tau = 0.$$

Suitable space for **symmetric** ε_p is well-known

$H_{\operatorname{curl}}(\Omega) := \{\varepsilon_p \in L^2(\Omega), \operatorname{Curl} \varepsilon_p \in L^2(\Omega)\}$ instead of $\mathcal{H}_{\sharp}$.