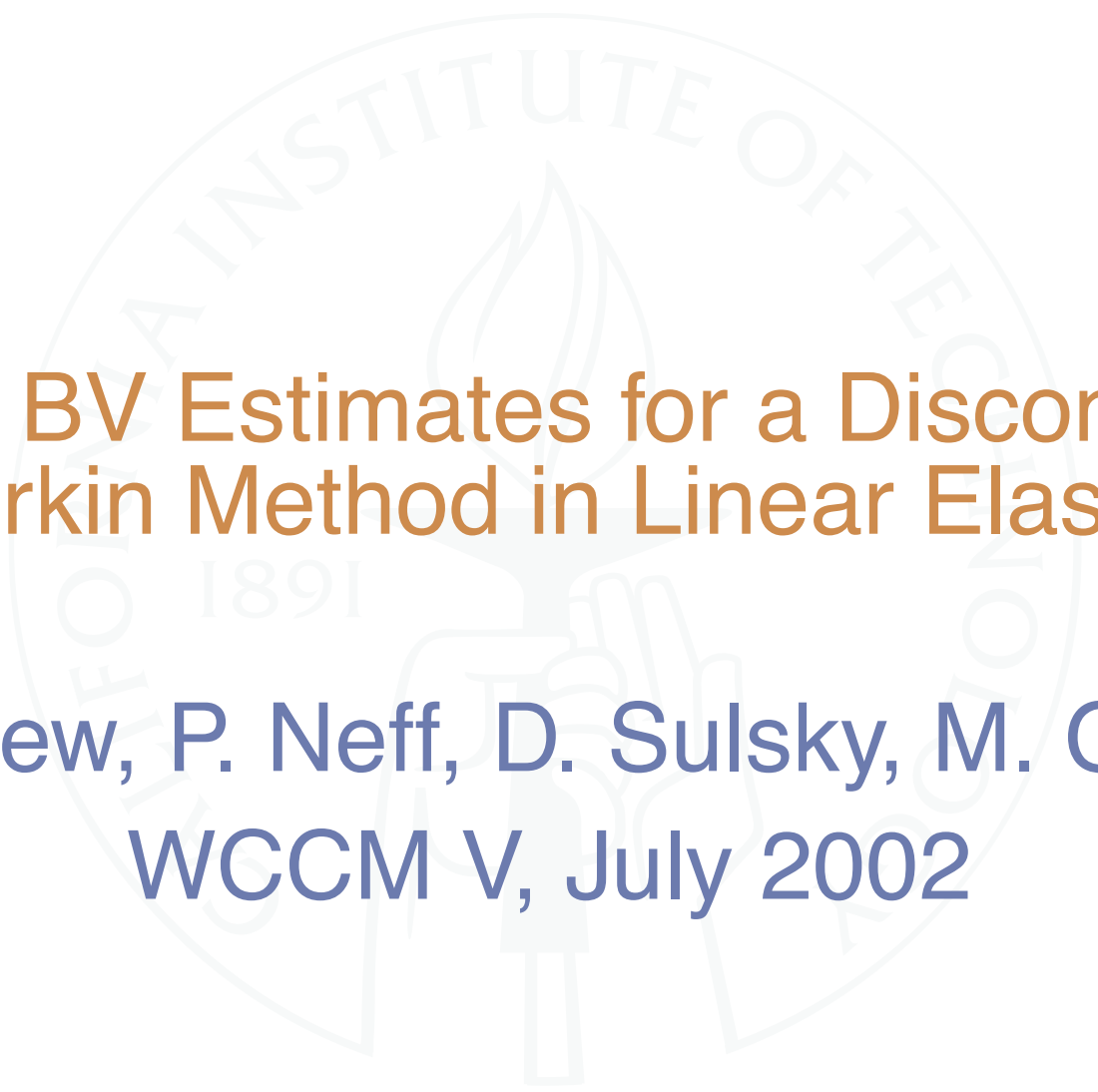




Optimal BV Estimates for a Discontinuous Galerkin Method in Linear Elasticity

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Introductory Remarks

◇ *Discontinuous Galerkin Methods:*

- Uses discontinuous interpolation across elements
- Traditionally used for 1st and 2nd order hyperbolic eqns.
- Little or no advantages over conforming FE for elliptic problems (For “*traditional*” situations)

◇ *Motivations:*

- **Linear Fracture Mechanics**
 - Crack propagation path and convergence of solutions
- **Adaptive Mesh Refinement**
 - Efficient hp-adaptivity
- **Dynamical response of materials**
 - Shock-Capturing schemes

Scalar Case

◇ *DG for scalar diffusion equation:*

- **Developed for convection dominated diffusion problems**
 - Ej: Bassi and Rebay(1997), Arnold, Brezzi, Cockburn and Marini (2002), etc.
- **Optimal energy, *mesh-dependent*, error estimates for a stabilized formulation. (Brezzi, Manzini, et. al (2000))**
 - Completely discontinuous approximation

$$\begin{aligned} \|v - v_h\|^2 &\leq C h^{m-1} |v|_m \\ &= \sum_{E \in \mathcal{T}_h} \|\nabla v - \nabla v_h\|_{0,E}^2 + \underbrace{\sum_{e \in \mathcal{E}_h} \|r_e(\llbracket v - v_h \rrbracket)\|_{0,B}^2}_{\text{mesh-dependent}} \end{aligned}$$

- $\|r_e(\llbracket v \rrbracket)\|_{0,B}^2$ measures the jump of v across face e
- Optimal L^2 -estimates: $\|v - v_h\| \leq C h^m |v|_m$

Vectorial Case

◇ *DG for Linear Elasticity:*

- An analog stabilized method to the scalar case can be formulated and analyzed for the *vectorial* case
- Error estimates obtained in $(L^2)^d$ and $\|\cdot\|_s$, given by

$$\|v\|_s^2 = \sum_{E \in \mathcal{T}_h} \|\nabla_s v\|_{0,E}^2 + \sum_{e \in \mathcal{E}_h} \|r_e(\llbracket v \rrbracket)\|_{0,B}^2$$

- $\|\cdot\|_s$ is not a *norm*.
- Need a discrete Korn's-like inequality, i. e.,

$$\|v\| \leq C \|v\|_s \quad \forall v \in V_h + (H_0^1(B))^d$$

with C independent of h (it does not necessarily hold).

- Want a *mesh-independent* error estimate $\rightarrow$ BV norm.

Problem Formulation

◇ $\sigma - u$ formulation of Linear Elasticity:

● **Hellinger-Reissner Energy:**

$$I[u, \sigma] = \int_B \left(\frac{1}{2} \sigma : \mathbb{C}^{-1} : \sigma - \sigma : \nabla_s u + f \cdot u \right) + \int_{\partial_d B} n \cdot \sigma \cdot (u - \bar{u}) + \int_{\partial_\tau B} \bar{T} \cdot u$$

◇ *Discrete formulation allowing jumps:*

● **Let E be an element, consider the energy:**

$$I_E = \int_E \left(\frac{1}{2} \sigma : \mathbb{C}^{-1} : \sigma - \sigma : \nabla_s u + f \cdot u \right) + \int_{\partial E \cap \partial_\tau B} \bar{T} \cdot u + \int_{\partial E \setminus \partial B} \frac{1}{2} n^- \cdot \sigma^- \cdot (u^- - u^+) + \int_{\partial E \cap \partial_d B} n \cdot \sigma \cdot (u - \bar{u})$$

● **The total energy is given by:** $I_h = \sum_E I_E$

Euler-Lagrange equations

◇ *Discrete Spaces:*

- $u_h|_E \in V_h^E$, **where** $(\mathcal{P}_k^E)^d \subseteq V_h^E \subset (H^1(E))^d$
- $\sigma_h|_E \in W_h^{E,s}$, **where** $(\mathcal{P}_{k-1}^E)^{d \times d, s} \subseteq W_h^E \subset (H^{1,div}(E))^d$

◇ *Euler-Lagrange equations:*

- **For all** $\gamma \in W_h^s$ **and** $v \in V_h$

$$0 = \sum_{E \in \mathcal{T}_h} \int_E \left(\gamma : \mathbb{C}^{-1} : \sigma - \gamma : \nabla_s u \right) + \int_{\Gamma} \llbracket u \rrbracket : \{\gamma\} - \int_{\partial_d B} n \cdot \gamma \cdot \bar{u}$$

$$0 = \sum_{E \in \mathcal{T}_h} \int_E \left(-\sigma : \nabla_s v + f \cdot v \right) + \int_{\Gamma} \llbracket v \rrbracket : \{\sigma\} + \int_{\partial_{\tau} B} \bar{T} \cdot v$$

where

$$\llbracket v \rrbracket = v^+ \otimes n^+ + v^- \otimes n^- \text{ **and** } \{\sigma\} = \frac{1}{2}(\sigma^+ + \sigma^-)$$

across any face e .

Lifting Operators

- ◇ *Face Lifting Operator* : For a face e , let $r_e : (L^2(e))^{d \times d} \rightarrow W_h^s$ be such that

$$\int_B r_e(\tau) : \gamma = - \int_e \tau : \{\gamma\}, \quad \forall \gamma \in W_h^s$$

- ◇ *Lifting Operator* : Let $R : (L^2(\cup_{e \in \mathcal{T}_h} e))^{d \times d} \rightarrow W_h^s$ such that for any element $E \in \mathcal{T}_h$,

$$R(\tau) = \sum_{e \subset \partial E} r_e(\tau) \quad \text{on } E$$

- ◇ *It holds* : There exist $C_1 > 0, C_2 > 0$ independent of the face e and h such that for all $v \in V_h + (H_0^1(B))^d$

$$C_2 h^{1/2} \|r_e(\llbracket v \rrbracket)\|_{0,B} \leq \|\llbracket v \rrbracket\|_{0,e} \leq C_1 h^{1/2} \|r_e(\llbracket v \rrbracket)\|_{0,B},$$

- ◇ For simplicity, we assume $u|_{\partial B} = 0$

One Field Equation

◇ *Equivalent one-field problem :*

- Find $u \in V_h$ such that

$$a(u, v) = \int_B f \cdot v \quad \forall v \in V_h$$

where the symmetric bilinear form $a : V_h \times V_h \rightarrow \mathbb{R}$ is given by

$$\begin{aligned} a(u, v) = & \sum_{E \in \mathcal{T}_h} \int_E (\nabla_s v + R([v]_s)) : \mathbb{C} : (\nabla_s u + R([u]_s)) \\ & + \underbrace{\beta \sum_{e \in \mathcal{E}_h} \int_B r_e([u]_s) : \mathbb{C} : r_e([v]_s)}_{\text{Stabilization term}} \end{aligned}$$

- The stress σ is given by

$$\sigma = \sigma(u) = \mathbb{C} : \nabla_s u + \mathbb{C} : R([u]_s)$$

Summary of Results

◇ *DG for Linear Elasticity. We proved:*

- Optimal error estimates in $(L^2(B))^d$ and in $\|\cdot\|_s$.
- An elemental version of Korn's inequality for the discretization.
- Therefore, optimal error estimates in $\|\cdot\|$.
- Then we also proved that

$$\|v\|_{BV} \leq C \|v\| \quad \forall v \in V_h + (H_0^1(B))^d$$

with C independent of h .

- Altogether we have the chain of inequalities

$$\|v\|_{BV} \leq C \|v\| \leq \bar{C} \left(\|v\|_s + \frac{1}{h} \|v\|_{0,B} \right) \quad \forall v \in V_h + (H_0^1(B))^d$$

from where the optimal and *mesh-independent* BV estimate follows.

Mesh-dependent estimate

◇ The proof follows the steps in Brezzi, Manzini et. al. (2001).

◇ *From here it follows that:*

- $a(\cdot, \cdot)$ is coercive and continuous in $\|\cdot\|_s$ for $\beta > N_e$. (respectively $\beta > 0$).
- If $u \in (H^m(B))^d$ is the exact solution, $1 \leq m \leq k + 1$, then

$$\|u - u_h\|_s < C h^{m-1} |u|_m$$

and

$$\|u - u_h\|_{0,B} < C h^m |u|_m$$

and

$$\|\sigma - \sigma_h\|_{0,B} < C h^{m-1} |u|_m$$

Discrete Korn's Inequality

◇ *Discrete Elemental Korn's inequality:*

- For a regular family of triangulations $\{\mathcal{T}_h\}$ composed of affine equivalent elements

$$\exists \quad C > 0 : \quad \forall E \in \mathcal{T}_h, \quad \forall u \in H^1(E),$$
$$\|\nabla u^T + \nabla u\|_{0,E}^2 + \frac{1}{h_E^2} \|u\|_{0,E}^2 \geq C \left(\|\nabla u\|_{0,E}^2 + \frac{1}{h_E^2} \|u\|_{0,E}^2 \right)$$

- Obtained by considering the effect of affine transformations on the reference elements.
 - Consider the distortion and volumetric deformations separately

Convergence in $\|\cdot\|$

◇ *It holds that:*

$$\|u - u_h\| < Ch^{m-1} |u|_m$$

● **Proof:**

$$\begin{aligned} \sum_{E \in \mathcal{T}_h} \|\nabla u^T + \nabla u\|_{0,E}^2 &+ \frac{1}{h^2} \|u\|_{0,E}^2 + \sum_{e \in \mathcal{E}_h} \|r_e([u]_s)\|_{0,B}^2 \\ &\geq C \sum_{E \in \mathcal{T}_h} \left(\|\nabla u\|_{0,E}^2 + \frac{1}{h^2} \|u\|_{0,E}^2 + \sum_{e \in \mathcal{E}_h} \|r_e([u]_s)\|_{0,B}^2 \right) \\ \|u\|_s^2 + \frac{1}{h^2} \|u\|_{0,B}^2 &\geq C \left(\|u\|^2 + \frac{1}{h^2} \|u\|_{0,B}^2 \right) \\ Ch^{2m-2} = C \left(h^{2m-2} + \frac{1}{h^2} h^{2m} \right) &\geq \|u\|^2 \end{aligned}$$

BV Estimate

◇ *Natural BV estimate:*

$$\exists C > 0 : \quad \|u\|_{BV} \leq C \|u\|,$$

$\forall u \in V_h + (H_0^1(B))^d$ with C independent of h .

- **BV norm**

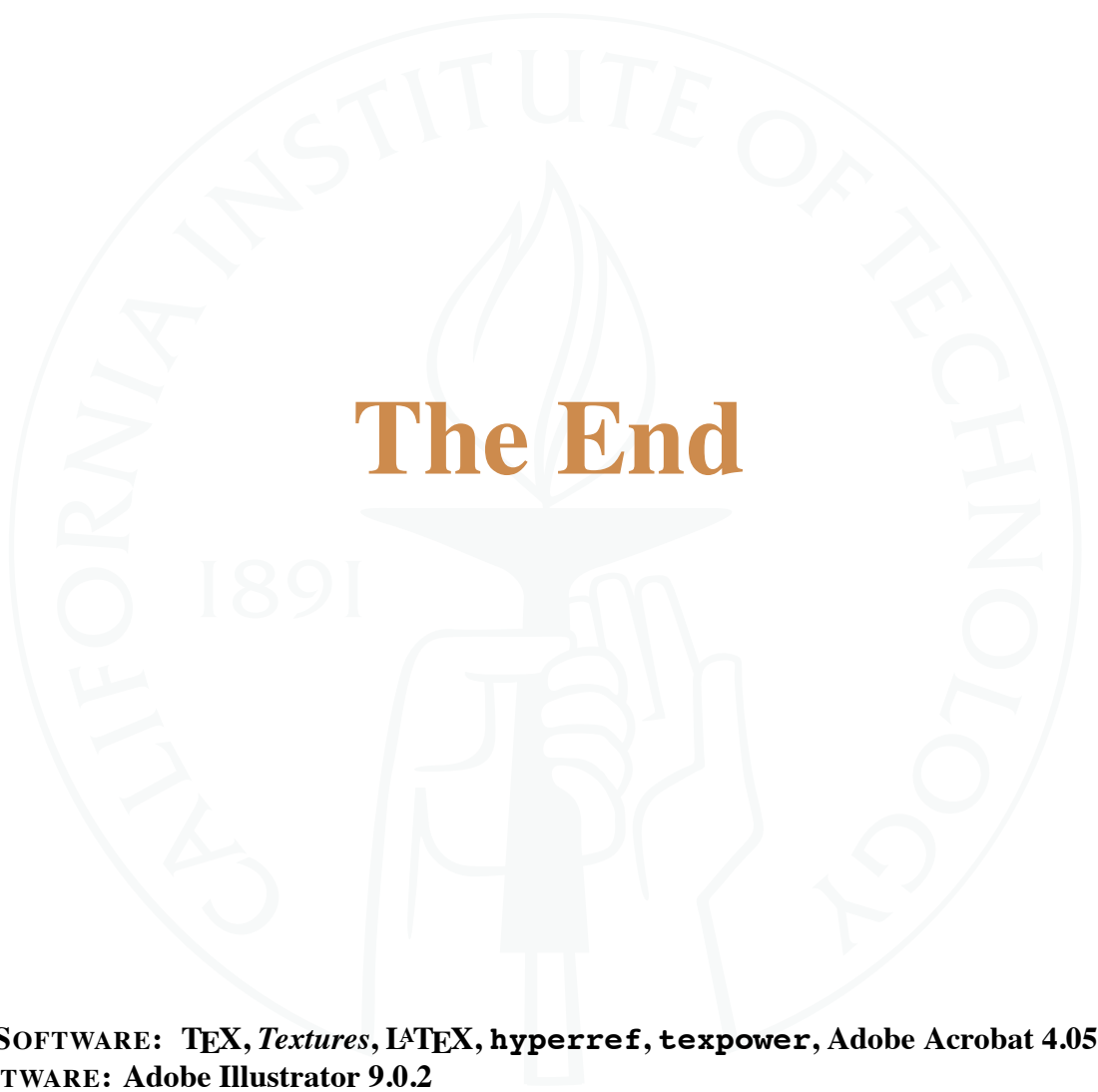
$$\|u\|_{BV(B)} = \|u\|_{L^1(B)} + \|Du\|(B)$$

- **Poincaré for BV:**

$$\exists C > 0 : \quad \forall u \in BV(\mathbb{R}^d), \quad \|u\|_{L^{d/(d-1)}(\mathbb{R}^d)} \leq C \|Du\|(\mathbb{R}^d).$$

- **From the definition of $\|\cdot\|_{BV}$ and the fact that $u \in V_h + (H_0^1(B))^d$ it is possible to obtain**

$$\|Du\|(B) \leq C \|u\|$$



The End

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