

CISM-Lecture 2007

The Γ -limit of a finite-strain **Cosserat** model for asymptotically thin domains versus a formal dimensional reduction

Patrizio Neff

Technische Universität Darmstadt

Plan of the lecture

- Notation
- The finite-strain Cosserat model
- The rescaled Cosserat problem
- Γ -convergence and the Γ -limit
- Comparison: formal finite-strain Cosserat plate
- Linearization and the linear Reissner-Mindlin model
- Open problems and Conclusions

Notation: Bulk behaviour

- $\varphi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$, the deformation, Ω reference configuration.
- $F = \nabla \varphi \in \text{GL}^+(3, \mathbb{R})$, the deformation gradient, $\mathbb{1}$ the identity.
- $W(F, \overline{R})$ elastic free energy density, $D_F W(F, \overline{R})$ first P. K. stresses.
- $\|A\|$, $\langle A, B \rangle$, $\text{tr}[A]$ norm, scalar product and trace.
- $F = RU = \text{polar}(F) U$, $R \in \text{SO}(3)$, polar decomposition.
- $\overline{R} : \Omega \mapsto \text{SO}(3, \mathbb{R})$ Cosserat-rotation.
- $\overline{U} = \overline{R}^T F$, non-symmetric micropolar-stretch tensor.

The Cosserat approach

- Idea: relax constraint on the rotations $R(\nabla\varphi)$ to coincide with **continuum rotations** $\text{polar}(\nabla\varphi)$. Allow for **independent (micro) rotations** $\bar{R} \in \text{SO}(3, \mathbb{R})$ (Cosserat 1906, motivated by "**triedre caché**" of **surface theory**).
- Attribute energy to spatial variations of rotations (**curvature**): $W_{\text{curv}}(\text{D}_x \bar{R}) = \mu L_c^p \|\text{D}_x \bar{R}\|^p$.
- Look for **independent** fields $\varphi, \bar{R}$ in the minimization problem. $\bar{R} \in \text{SO}(3, \mathbb{R})$ **nonlinear manifold**.
- Suited for material with granular substructure (**individual rotations of particles**), spin, magnetization, homogenisation of heterogeneous materials. Explore lower energy paths, avoid stability problems.

The finite **Cosserat** problem in variational form

Thin domain $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, $\omega \subset \mathbb{R}^2$.

$$I(\varphi, \overline{R}) = \int_{\Omega_h} W_{\text{mp}}(\overline{U}) + \mu L_c^p \|D_x \overline{R}\|^p dV \mapsto \min . \text{ w.r.t. } (\varphi, \overline{R})$$

$$\varphi|_{\Gamma_h} = g_d, \quad \overline{R}|_{\Gamma_h} \text{ free}$$

$$W_{\text{mp}}(\overline{U}) = \mu \|\text{sym}(\overline{U} - \mathbb{1})\|^2 + \frac{\lambda}{2} \text{tr} [\text{sym}(\overline{U} - \mathbb{1})]^2 \\ + \mu_c \|\text{skew}(\overline{U} - \mathbb{1})\|^2$$

$$\overline{U} = \overline{R}^T F, \quad \text{non-symmetric micropolar stretch tensor.}$$

Existence: $\mu_c \geq 0$ Neff (PRSE06), general micromorphic model:

$\overline{R} \in \text{SO}(3, \mathbb{R}) \rightarrow P \in \text{GL}^+(3, \mathbb{R})$, Neff/Forest (JEL07). Infinitesimal elasto-plastic extension, $\mu_c > 0$: Neff/Chełmiński (PRSE05).

The **Cosserat** balance equations

E-L: linear momentum+angular momentum: nonlinear second order PDE, non-constant coefficients

$$0 = \text{Div} \left(S_1(F, \bar{R}) + 2 \mu_c \bar{R} \text{skew}(\bar{R}^T F) \right)$$

$$0 = \text{skew}(D_{\bar{U}} W_{\text{mp}}(\bar{U}) \bar{U}^T) + \text{skew} \left(\bar{R}^T \text{Div} [\bar{R} D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K})] \right) \\ + \text{skew} (D_{\mathfrak{K}} W_{\text{curv}}(\mathfrak{K}) \mathfrak{K}^T)$$

$$S_1(F, \bar{R}) = \bar{R} \left[\mu (F^T \bar{R} + \bar{R}^T F - 2 \mathbb{1}) + \lambda \text{tr} [F^T \bar{R} - \mathbb{1}] \mathbb{1} \right]$$

$$W_{\text{curv}}(D_x \bar{R}) = \mu L_c^p \|D_x \bar{R}\|^p$$

The Cosserat curvature energy

The curvature term has the form

$$W_{\text{curv}}(D_x \bar{R}) = \mu L_c^p \|D_x \bar{R}\|^p ,$$
$$D_x \bar{R} = (\nabla(\bar{R}.e_1), \nabla(\bar{R}.e_2), \nabla(\bar{R}.e_3)) .$$

Coercivity of curvature

$$\exists c^+ > 0 \exists r > 1 : \forall \mathcal{K} \in \mathfrak{T}(3) : \quad W_{\text{curv}}(\mathcal{K}) \geq c^+ \|\mathcal{K}\|^r$$
$$\exists r > 1 : \quad \frac{W_{\text{curv}}(\mathcal{K})}{\|\mathcal{K}\|^r} \rightarrow \infty \quad \text{as } \|\mathcal{K}\| \rightarrow \infty$$
$$\Rightarrow \quad p > 1$$

$L_c > 0$: internal length, characteristic for the material (grain size).

The Cosserat couple modulus μ_c

- $\mu_c > 0$: $D_F^2 W_{\text{mp}}(\bar{R}^T F).(H, H) \geq 2 \mu_c \|H\|^2$
- $\mu_c = 0$: $D_F^2 W_{\text{mp}}(\bar{R}^T F).(H, H) \geq 2 \mu \|\frac{1}{2}(\bar{R}^T H + H^T \bar{R})\|^2$
- $\mu_c = 0$: $D_F^2 W_{\text{mp}}(\bar{R}^T F).(\xi \otimes \eta, \xi \otimes \eta) \geq \mu \|\xi\|^2 \|\eta\|^2$
- $\mu_c = 0$: uniformly Legendre-Hadamard independent of $\bar{R}(x)$.
- $\mu_c = 0$: linearization coincides with classical linear elasticity.
- thinking linear one may believe: $\mu_c > 0$ necessary

The infinitesimal, linear **Cosserat** problem

Expand $F = \mathbb{1} + \nabla u$, $\bar{R} = \mathbb{1} + \bar{A} + \dots$, $\bar{A} \in \mathfrak{so}(3, \mathbb{R})$

$$\int_{\Omega_h} W_{\text{mp}}(\nabla u, \bar{A}) + \mu L_c^2 \|\text{D}_x \bar{A}\|^2 dV \mapsto \min . \text{ w.r.t. } (u, \bar{A})$$

$$W_{\text{mp}}(\nabla u, \bar{A}) = \mu \|\text{sym } \nabla u\|^2 + \frac{\lambda}{2} \text{tr} [\text{sym } \nabla u]^2 + \mu_c \|\text{skew } \nabla u - \bar{A}\|^2$$

Cosserat couple modulus $\mu_c = 0 \Rightarrow$ no coupling, linear elasticity.

$L_c = 0 \Rightarrow$ linear elasticity.

Well posed linear infinitesimal **Cosserat** problem.

The rescaled **Cosserat** problem

Follow Le Dret/Raoult (Membrane-model):

domain with constant thickness $\Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}]$

$$\varphi^\sharp(x, y, z) := \varphi(x, y, h z), \quad \overline{R}^\sharp(x, y, z) := \overline{R}(x, y, h z).$$

$$I^\sharp(\varphi^\sharp, \overline{R}^\sharp) = h \int_{\Omega_1} W_{\text{mp}}(\overline{U}_h^\sharp) + \mu L_c^p \|D_x^h \overline{R}^\sharp\|^p dV \mapsto \min . \text{ w.r.t. } (\varphi^\sharp, \overline{R}^\sharp)$$

$$\overline{U}_h^\sharp := \overline{R}^{\sharp, T} \nabla^h \varphi^\sharp, \quad \varphi^\sharp|_{\Gamma^1} = g_d^\sharp, \quad \overline{R}^\sharp|_{\Gamma^1} \text{ free}$$

$$\nabla^h \varphi^\sharp := (\partial_x \varphi^\sharp | \partial_y \varphi^\sharp | \frac{1}{h} \partial_z \varphi^\sharp)$$

$$D_x^h \overline{R}^\sharp := (\nabla^h(\overline{R}^\sharp . e_1) | \nabla^h(\overline{R}^\sharp . e_2) | \nabla^h(\overline{R}^\sharp . e_3))$$

Consider sequence of variational problems $I_h^\sharp(\varphi^\sharp, \overline{R}^\sharp) := \frac{1}{h} I^\sharp(\varphi^\sharp, \overline{R}^\sharp)$.

Γ -convergence- variational convergence

X metric space. $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ -converges in X to $I_0 : X \mapsto \overline{\mathbb{R}}$, if for all $x \in X$ we have

$$\forall x_{h_j} \rightarrow x : \quad I_0(x) \leq \liminf_{h_j \rightarrow 0} I_{h_j}(x_{h_j}), \quad (\text{lim inf-inequality})$$

$$\exists x_{h_i} \rightarrow x : \quad I_0(x) \geq \limsup_{h_i \rightarrow 0} I_{h_i}(x_{h_i}), \quad (\text{recovery sequence}) .$$

Γ -convergence ensures that limit of minimizers to I_{h_j} is a minimizer of the limit functional I_0 .

Here: $X = L^{\textcolor{red}{r}}(\Omega_1, \mathbb{R}^3) \times W^{1,\textcolor{red}{p}}(\Omega_1, \text{SO}(3, \mathbb{R}))$, $\textcolor{red}{r} = \textcolor{red}{p}'$, $\textcolor{red}{p} > 3$.
 $H^{1,2}(\Omega_1, \mathbb{R}^3) \subset L^{\textcolor{red}{r}}(\Omega_1, \mathbb{R}^3)$ compact.

Notation: Thin structures

- $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, midsurface-deformation.
- ω planar reference configuration, γ_0 Dirichlet boundary.
- $\nabla m = (\partial_x m | \partial_y m) \in \mathbb{M}^{3 \times 2}$ the midsurface-deformation gradient.
- $(\xi | \eta | \theta)$ matrix composed of columns $\xi, \eta, \theta \in \mathbb{R}^3$.
- $\mathbb{1}_2$ the identity on $\mathbb{M}^{2 \times 2}$.
- $\overline{R}_i = \overline{R} \cdot e_i$, i.th column of matrix $\overline{R}$.

The Γ -limit "membrane" plate

Find midsurface deformation $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and microrotation of the plate $\bar{R} : \omega \subset \mathbb{R}^2 \mapsto \text{SO}(3, \mathbb{R})$:

$$I_0^\sharp(m, \bar{R}) = \int_{\omega} W_{\text{mp}}^{\text{hom}}(\nabla m, \bar{R}) + \mu L_c^{\textcolor{blue}{p}} \|\mathfrak{K}_s\|^{\textcolor{red}{p}} d\omega \mapsto \min . \text{ w.r.t. } (m, \bar{R})$$

$$m|_{\gamma_0} = g_d(x, y, 0), \quad \text{simply supported}$$

$$\bar{R}|_{\gamma_0} \quad \text{free: Neumann condition}$$

$$\mathfrak{K}_s = \left(\bar{R}^T(\nabla(\bar{R}.e_1)|0), \bar{R}^T(\nabla(\bar{R}.e_2)|0), \bar{R}^T(\nabla(\bar{R}.e_3)|0) \right) \in \mathfrak{T}(3)$$

third order curvature tensor

The homogenized reduced energy density

$$\begin{aligned}
 W_{\text{mp}}^{\text{hom}}(\nabla m, \bar{R}) &= \mu \underbrace{\|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)\|^2}_{\text{"intrinsic" shear-stretch energy}} + \mu_c \underbrace{\|\text{skew}((\bar{R}_1 | \bar{R}_2)^T \nabla m)\|^2}_{\text{"intrinsic" first order drill energy}} \\
 &+ \underbrace{2\mu \frac{\mu_c}{\mu + \mu_c} \left(\langle \bar{R}_3, m_x \rangle^2 + \langle \bar{R}_3, m_y \rangle^2 \right)}_{\text{homogenized transverse shear energy}} \\
 &+ \frac{\mu\lambda}{2\mu + \lambda} \underbrace{\text{tr} \left[\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2) \right]^2}_{\text{homogenized elongational stretch energy}}.
 \end{aligned}$$

Not coercive if $\mu_c = 0$, missing transverse shear energy. Loss of coercivity not related to missing drill energy.

Sketch of proof for $\mu_c > 0$: **equi-coercivity**

- $\mu_c > 0$: elastic energy estimates strain part: $\|\overline{R}^{k,T} \nabla^h \varphi_k - \mathbb{1}\|^2 = \|\overline{R}^{k,T} \nabla^h \varphi_k\|^2 - 2\langle \overline{R}^k, \nabla^h \varphi_k \rangle + 3 \geq \|\nabla^h \varphi_k\|^2 - 2\sqrt{3}\|\nabla^h \varphi_k\| + 3.$
- Control of φ_k in $H^{1,2}(\Omega_1, \mathbb{R}^3)$ is **independent** of $\overline{R}^k$.
- Since $\|D_x^h \overline{R}\|^p$ in the energy, control $\overline{R}^k$ in $W^{1,p}(\Omega_1)$.
- Restrict attention to strongly convergent subsequence of rotations $\overline{R}^k \rightarrow \overline{R}$ in $L^p(\omega, \text{SO}(3, \mathbb{R}))$.
- Use compact embedding $H^{1,2}(\Omega_1, \mathbb{R}^3) \subset L^r(\Omega_1, \mathbb{R}^3)$.
- $I_h^\sharp$ is **equi-coercive** over $X = L^r(\Omega_1, \mathbb{R}^3) \times W^{1,p}(\Omega_1, \text{SO}(3, \mathbb{R}))$.

Sketch of proof for $\mu_c > 0$: **lower-bound**

- Local minimization w.r.t. "transverse" derivatives $\partial_z \varphi$.
- $W_{\text{mp}}^{\text{hom}}(\nabla m, \bar{R}) := W_{\text{mp}}(\bar{R}^T (\nabla m | b^*)) := \inf_{b \in \mathbb{R}^3} W_{\text{mp}}(\bar{R}^T (\nabla m | b)).$
- Determine "director" b^* explicitly: $D_F W_{\text{mp}}((\nabla m | b^*), \bar{R}) \cdot e_3 = 0.$
"Plane stress", membrane state.
- Use convexity of energy w.r.t. $\nabla^h \varphi$ and weak convergence of $\nabla^h \varphi_k$.
- Use convexity of energy w.r.t. $D_x^h \bar{R}$ and strong convergence of $\bar{R}^k$.
- Obtain lim inf-inequality.

Sketch of proof $\mu_c > 0$: recovery sequence

- Given $m \in L^r(\omega, \mathbb{R}^3), \bar{R} \in W^{1,p}(\omega, \text{SO}(3, \mathbb{R}))$.
- Reconstruction: $\varphi_{h_j}(x, y, z) = m(x, y) + h_j z b^*(\nabla m(x, y), \bar{R}(x, y))$.
- Reconstruction: $\bar{R}_{h_j}(x, y, z) = \bar{R}(x, y)$, constant in thickness.
- Approximation and continuity yield upper bound: lim sup-inequality.

Ideas of proof for $\mu_c = 0$:

- **Unusual difficulty**: sequence I_h is **not equi-coercive**. (equi-coercivity: uniform coercivity independent of $h > 0$).
- Construct suitable comparison functionals. Establish $\Gamma - \limsup$ estimate.
- Circumvent lack of compactness in φ_k by looking at in-plane components $(\overline{R}_1 | \overline{R}_2)^{k,T} \nabla_{(x,y)} \varphi_k$.
- Use Korn's second inequality for a suitable quantity.
- Establish weak convergence of $(\overline{R}_1 | \overline{R}_2)^{k,T} \nabla_{(x,y)} \varphi_k$, which does not imply weak convergence of $\nabla_{(x,y)} \varphi_k$! Establish $\Gamma - \liminf$ estimate. Show $\Gamma - \liminf \geq \Gamma - \limsup$.

Comparison: formal **Cosserat** plate

Find $m : \omega \mapsto \mathbb{R}^3$, $\bar{R} \in \text{SO}(3, \mathbb{R})$:

$$I(m, \bar{R}) = \int_{\omega} h W_{\text{mp}}(\nabla m, \bar{R}) + h \mu L_c^p \|\mathfrak{K}_s\|^p + \frac{h^3}{12} W_{\text{bend}}(\mathfrak{K}_b) \, d\omega \mapsto \min .$$

$$\mathfrak{K}_s = \left(\bar{R}^T (\nabla(\bar{R}.e_1)|_0), \bar{R}^T (\nabla(\bar{R}.e_2)|_0), \bar{R}^T (\nabla(\bar{R}.e_3)|_0) \right) \in \mathfrak{T}(3) ,$$

$$m|_{\gamma_0} = g_d(x, y, 0) , \quad \text{simply supported .}$$

$$\bar{R}|_{\gamma_0} \quad \text{free: Neumann condition}$$

$$\varphi(x, y, z) = m(x, y) + \left(\varrho_m(x, y) z + \varrho_b(x, y) \frac{z^2}{2} \right) \bar{R}(x, y).e_3$$

$$\bar{R}(x, y, z) = \bar{R}(x, y).$$

Quadratic ansatz through thickness. ϱ_m thickness stretch, ϱ_b asymmetric shift of the midsurface due to bending.

The formal reduced energy densities: stretch

$$\begin{aligned}
 & W_{\text{mp}}(\nabla m, \bar{R}) \\
 &= \underbrace{\mu \|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)\|^2}_{\text{"intrinsic" shear-stretch energy}} + \underbrace{\mu_c \|\text{skew}((\bar{R}_1 | \bar{R}_2)^T \nabla m)\|^2}_{\text{"intrinsic" first order drill energy}} \\
 &\quad + \underbrace{\frac{\kappa(\mu + \mu_c)}{2} \left(\langle \bar{R}_3, m_x \rangle^2 + \langle \bar{R}_3, m_y \rangle^2 \right)}_{\text{classical transverse shear energy}} \\
 &\quad + \frac{\mu\lambda}{2\mu + \lambda} \underbrace{\text{tr} [\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)]^2}_{\text{"intrinsic" elongational stretch energy}},
 \end{aligned}$$

$0 < \kappa \leq 1$, transverse shear correction coefficient

The formal reduced energy densities: bending

$$W_{\text{bend}}(\mathfrak{K}_b) = \mu \|\text{sym}(\mathfrak{K}_b)\|^2 + \mu_c \|\text{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \text{tr} [\text{sym}(\mathfrak{K}_b)]^2,$$

$$\mathfrak{K}_b = \overline{R}^T (\nabla \overline{R}_3|_0) = \mathfrak{K}_s^3$$

Neff (CMT04), Existence $\mu_c = 0$: Neff (M3AS07). Main idea: new Korn's inequality for shells/plates.

Linearization

$m(x, y) = (x, y, 0)^T + v(x, y)$, $\bar{R} = \mathbb{1} + \bar{A} + \dots$, $\bar{A} \in \mathfrak{so}(3, \mathbb{R})$. First order reduction of formal **Cosserat** plate model ($\kappa = 1$, $p > 2$)

$$\begin{aligned} & \int_{\omega} h \left(\mu \|\operatorname{sym}((\nabla v | \bar{A}_3))\|^2 + \mu_c \|\operatorname{skew}((\nabla v | \bar{A}_3) - \bar{A})\|^2 \right. \\ & \quad \left. + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr} [(\nabla v | \bar{A}_3)]^2 \right) + \frac{h^3}{12} (\mu \|\operatorname{sym}((\nabla \bar{A}_3 | 0))\|^2 \\ & \quad + \mu_c \|\operatorname{skew}((\nabla \bar{A}_3 | 0))\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr} [(\nabla \bar{A}_3 | 0)]^2) \, d\omega \\ & \mapsto \min . \text{ w.r.t. } (v, \bar{A}) \end{aligned}$$

$\mu_c = 0$: infinitesimal in-plane rotations do not "survive" linearization: classical Reissner-Mindlin model, $-\bar{A}_3 = (\theta_1, \theta_2, 0)^T$. Infinitesimal "**director**" $\theta \in \mathbb{R}^2$.

The infinitesimal, linear Reissner-Mindlin plate

Displacement $v : \omega \mapsto \mathbb{R}^3$, infinitesimal "director" $\theta : \omega \mapsto \mathbb{R}^2$

$$\int_{\omega} h \left(\mu \|\operatorname{sym} \nabla(v_1, v_2)\|^2 + \underbrace{\kappa \frac{\mu}{2} \|\nabla v_3 - \theta\|^2}_{\text{transverse shear energy}} + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr} [\operatorname{sym} \nabla(v_1, v_2)]^2 \right) + \frac{h^3}{12} \left(\mu \|\operatorname{sym} \nabla \theta\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr} [\operatorname{sym} \nabla \theta]^2 \right) d\omega \mapsto \min . \text{ w.r.t. } (v, \theta)$$

$v|_{\gamma_0} = u^d(x, y, 0) \quad \text{simply supported}$

$-\theta|_{\gamma_0} = (u_{1,z}^d, u_{2,z}^d)^T$

Open problems

- Γ -limit for $\mu_c = \infty$ is perhaps not the existing limit $\mu_c \rightarrow \infty$ of the Γ -limit formula? (work in progress Neff/Chelmiński)
- Optimal value of shear correction term $0 < \kappa \leq 1$?
- Other boundary conditions for $\bar{R}$ on Dirichlet boundary yield other Γ -limit result? Not present in classical theories.
- What is the boundary condition for 3D-Cosserat problem on Dirichlet boundary for the microrotations?
- Extension to (more realistic) non-quadratic bulk energy in $\bar{U}$, e.g. $W_{\text{mp}}(\bar{U}) \rightarrow \infty$ for $\det[\bar{U}] \rightarrow 0$?

Conclusion

- Γ -limit underestimates the transverse shear stresses (**Reuss-bound**), modulus $2\mu \frac{\mu_c}{\mu + \mu_c}$.
- Rigorous Γ -limit is **not coercive** for $\mu_c = 0$. But $\mu_c = 0$ is the consistent value.
- Formal limit (ansatz) perhaps overestimates the transverse shear stresses (**Voigt-bound**), modulus $\frac{(\mu + \mu_c)}{2}$.
- Transverse shear correction factor $0 < \kappa \leq 1$ interpolates between both extremes. Stable computations need $\kappa > 0$!
- For $\mu_c = 0$ and $p > 2$ classical **Reissner-Mindlin** is the linearization of the formal **Cosserat** plate model.

Extended Korn's first inequality

Let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL^+(3, \mathbb{R}))$ be given with $\det[F_p(x)] \geq c > 0$. Suppose **dislocation density** $\text{Curl } F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3 \times 3})$. Then

$$\exists c^+ > 0 \quad \forall \phi \in H^1(\Omega), \varphi_\Gamma = 0 :$$

$$\|\nabla \phi F_p^{-1}(x) + F_p^{-T}(x) \nabla \phi^T\|_{L^2(\Omega)}^2 \geq c^+ \|\phi\|_{H^{1,2}(\Omega)}^2 .$$

Proof: [Neff (PRSE02)], recent improvement: $F_p \in C^0(\overline{\Omega})$. But work in progress suggests $F_p \in L^\infty(\Omega)$, $\text{Curl } F_p \in L^{3+\delta}(\Omega)$ suffices. Counterexample for $F_p \in L^\infty(\Omega)$.

Reminder: **Classical Korn's first inequality:**

$$\exists c^+ > 0 \quad \forall \phi \in H^1(\Omega), \varphi_\Gamma = 0 : \quad \|\nabla \phi + \nabla \phi^T\|_{L^2(\Omega)}^2 \geq c^+ \|\phi\|_{H^{1,2}(\Omega)}^2 .$$

Sketch of proof for Korn's inequality

Gårding's inequality: F_p invertible and smooth, exists $\lambda > 0$:

$$\|\nabla \phi F_p^{-1} + F_p^{-T} \nabla \phi^T\|_{L^2(\Omega)}^2 + \lambda \|\phi\|_{L^2(\Omega)}^2 \geq c^+ \|\phi\|_{H^{1,2}(\Omega)}^2.$$

Show $\|\nabla \phi F_p^{-1} + F_p^{-T} \nabla \phi^T\|_{L^2(\Omega)}^2 = 0 \Rightarrow \phi = 0$, then use compactness. Let $\phi \in C_0^\infty(\Omega, \Gamma) \cap H^{1,2}(\Omega)$.

$$\nabla \phi F_p^{-1} = A \in L^2(\Omega, \mathfrak{so}(3)) \Rightarrow \nabla \phi = A F_p \Rightarrow$$

$$0 = \text{Curl } \nabla \phi = L_{F_p} \cdot \nabla A + A \text{ Curl } F_p, L_{F_p} \text{ invertible, if } \det[F_p] \neq 0 \Rightarrow$$

$$\nabla A = -L_{F_p}^{-1} \cdot [A \text{ Curl } F_p] \Rightarrow A \in W^{1,2}(\Omega)$$

$$\phi|_\Gamma = 0 \Rightarrow 0 = \nabla \phi \cdot \tau_i = A(F_p \cdot \tau_i), i = 1, 2 \Rightarrow \text{rank}(A)|_\Gamma \leq 1 \Rightarrow A|_\Gamma = 0$$

$$\nabla A = -L_{F_p}^{-1} \cdot [A \text{ Curl } F_p], A|_\Gamma = 0, \text{ PDE 1. Ord., linear} \Rightarrow A \equiv 0$$

$$A \equiv 0 \Rightarrow \nabla \phi = A F_p = 0 \Rightarrow \nabla \phi = 0 \Rightarrow \phi = 0.$$

Extended Korn's inequality for shells

Theorem [N. M3AS07]

Let $\omega \subset \mathbb{R}^2$ be a smooth domain and $\gamma_0 \subset \partial\omega$ has full one-dimensional Hausdorff-measure. Moreover $F_p, F_p^{-1} \in W^{1,2+\delta}(\overline{\omega}, GL(3))$. Then

$$\exists c^+ > 0 \quad \forall m \in H^1(\omega), m|_{\gamma_0} = 0 :$$

$$\|(\nabla m|0) F_p^{-1}(x) + F_p^{-T}(x) (\nabla m|0)^T\|_{L^2(\omega)}^2 \geq c^+ \|m\|_{H^1(\omega)}^2 .$$

Apply to energy

$$\begin{aligned} & \frac{1}{4} \|R^T (\nabla m|R_3) + (\nabla m|R_3)^T R - 2 \mathbb{1}\|^2 \\ & \sim \|\text{sym}((R_1|R_2)^T \nabla m - \mathbb{1}_2)\|^2 + \left(\langle m_x, R_3 \rangle^2 + \langle m_y, R_3 \rangle^2 \right) \\ & \sim \|R^T (\nabla m|0) + (\nabla m|0)^T R\|^2 + \text{l.o.t} \end{aligned}$$

Rigidity estimates

Qualitative Version with relation to well known rigidity statements (John, Sverak, Müller, Friesecke)

Theorem [N./Münch ESAIM:COCV07]

$$\forall \mathbf{R} \in C^1(\mathbb{R}^3, \mathrm{SO}(3)) : \quad \|\mathrm{Curl} \mathbf{R}\|^2 \geq \frac{1}{2} \|\nabla \mathbf{R}\|^2.$$

No integration! Special case: $\mathbf{R} = \nabla \varphi$ and $\mathrm{Curl} \nabla \varphi = 0$ yields $\varphi(x) = \mathbf{R}.x + b$.

Known "linear" result: $\mathbf{R} = \mathbb{1} + \mathbf{A} + \dots$, $\mathbf{A}^T + \mathbf{A} = 0$

$$\forall \mathbf{A} \in C^1(\mathbb{R}^3, \mathfrak{so}(3)) : \quad \|\mathrm{Curl} \mathbf{A}\|^2 \geq \frac{1}{2} \|\nabla \mathbf{A}\|^2.$$

Is implicitly used in the proof of Korn's first inequality:

$$\|\nabla u^T + \nabla u\|_{\Omega}^2 = 0 \Rightarrow \nabla u = \mathbf{A} \in \mathfrak{so}(3) \Rightarrow 0 = \mathrm{Curl} \nabla u = \mathrm{Curl} \mathbf{A} \Rightarrow \nabla \mathbf{A} = 0.$$

Kill constant skew-symmetric matrices $\mathbf{A}$ with boundary conditions!