Comparison of Models for Finite Plasticity

A numerical study

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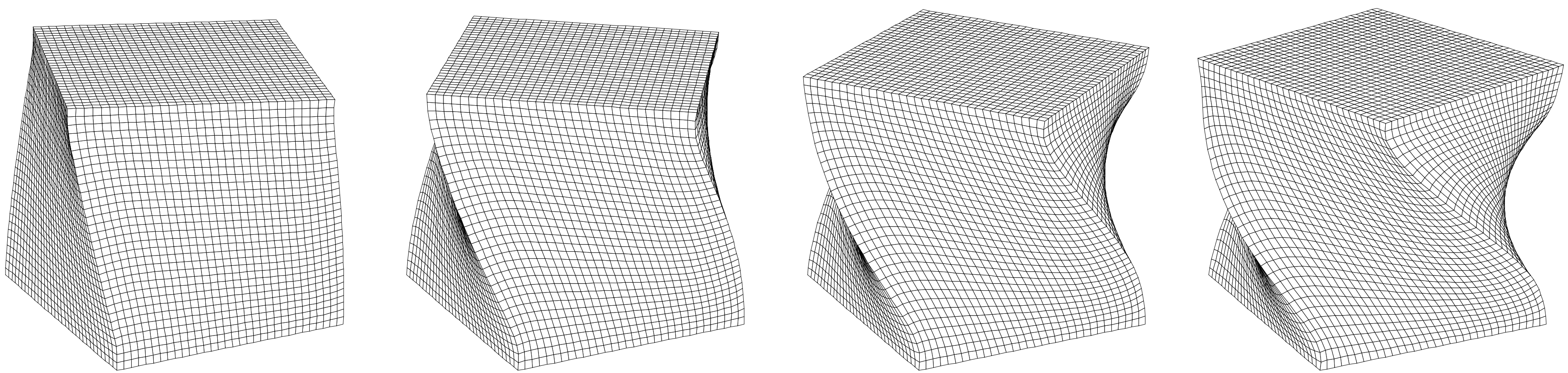
Motivation

For a wide range of applications of elasto-plasticity theory to crystalline aggregates like metals it is known that the elastic region is very small implying that elastic strains remain small.

A frame indifferent approximative model for this situation has been developed where the remaining nonlinearity (in case of small elastic strains) in the balance equation is shifted into an appended evolution equation. It has been shown that the new model in a viscous setting is locally well posed.



Here, we compare *numerically* the behavior of the new model with other well known models of finite plasticity which are based on different elastic free energies.

Models

Small elastic strains

We assume that the deformation tensor F has a multiplicative decomposition $F = F_e F_p$; we set $P = F_p^{-1}$ and $F_e = FP$. We consider the polar decomposition of the elastic part

$$F_e = \mathrm{polar}(F_e) U_e \qquad \text{with} \qquad \mathrm{polar}(F_e) \in SO(3) \qquad \text{and} \qquad U_e = \sqrt{F_e^T F_e}.$$

For small elastic strains (i. e. $|F_e^T F_e - \mathrm{id}|$ is small), also $|F_e - \mathrm{polar}(F_e)|$ is small, and

$$F_e^T \mathrm{polar}(F_e) + \mathrm{polar}(F_e)^T F_e - 2\,\mathrm{id} \;\Big(= 2(U_e - \mathrm{id})\Big) \qquad \text{is close to} \qquad F_e^T F_e - \mathrm{id}\,.$$

In the new model, $\mathrm{polar}(F_e)$ is approximated by $R \in SO(3)$ satisfying

$$\dot{R} = \nu(|\,\mathrm{skew}(F_e R^T)|)\,\mathrm{skew}(F_e R^T) R.$$

Lemma. For $R_0 \in SO(3)$ with $|R_0 - \mathrm{polar}(F_e)| < 8$, the evolution equation

$$\dot{R} = \mathrm{skew}(F_e R^T) R, \qquad R(0) = R_0$$

has a global solution $R(t)$ in $SO(3)$ satisfying $\lim\limits_{t \longrightarrow \infty} R(t) = \mathrm{polar}(F_e)$.

Material models

M1 The new model (with appended rotations R)

M2 The exact polar model (using $R = \mathrm{polar}(F_e)$)

M3 St. Venant–Kirchhoff model (straight forward finite strain extension)

M4 A Neo-Hooke model (reference model)

M5 A Hencky model with logarithmic strains (fits to exponential increment)

For comparison, we also consider a classical model with linearized plastic strain.

M6 Perfect plasticity (small strain limit)

M7 Viscoplasticity (measuring the viscous effect)

Internal variables, strain and energy

	internal variables Z	strain $C = \mathrm{id} + 2E$	energy W
M1 new model	(P, R)	$F_e^T R + R^T F_e - \mathrm{id}$	$\mu \lvert E\rvert^2 + \frac{\lambda}{2}\operatorname{trace}(E)^2$
M2 exact polar model	P	$F_e^T \operatorname{polar}(F_e) + \operatorname{polar}(F_e)^T F_e - \mathrm{id}$	$\mu \lvert E\rvert^2 + \frac{\lambda}{2}\operatorname{trace}(E)^2$
M3 St. Venant–Kirchhoff	P	$F_e^T F_e$	$\mu \lvert E\rvert^2 + \frac{\lambda}{2}\operatorname{trace}(E)^2$
M4 Neo-Hooke	P	$F_e^T F_e$	$\frac{\mu}{2}\lvert F_e\rvert^2 + \frac{\lambda}{2}\det(C) - \frac{2\mu+\lambda}{2}\log\det(C)$
M5 Hencky	P	$F_e^T F_e$	$\frac{\mu}{4}\lvert \operatorname{dev}(\ln C)\rvert^2 + \frac{\kappa}{8}(\operatorname{trace}(\ln C))^2$
M6 Perfect plasticity	ε^p	$F + F^T - \mathrm{id} - 2\varepsilon^p$	$\mu \lvert E\rvert^2 + \frac{\lambda}{2}\operatorname{trace}(E)^2$
M7 Viscoplasticity	ε^p	$F + F^T - \mathrm{id} - 2\varepsilon^p$	$\mu \lvert E\rvert^2 + \frac{\lambda}{2}\operatorname{trace}(E)^2$

Finite strain models

For all models, we set

$$
\begin{aligned}
S &= D_F W(F,Z) \\
\Sigma &= F_e^T S P^{-T}, \\
D &= \operatorname{dev}(\operatorname{sym}(\Sigma)), \\
P^{-1}\dot{P} &= -\gamma(|D|)\,\frac{D}{|D|}, \\
\gamma(d) &\in \partial\hat{\chi}_r(d).
\end{aligned}
$$

For **M2**, **M3**, **M4**, **M5**, **M6** we use $\hat{\chi}_\infty(d) = \begin{cases} 0 & d \le K_0, \\ \infty & d > K_0. \end{cases}$

In **M1** and **M7** this is relaxed to $\hat{\chi}_r(d) = \dfrac{1}{\eta}\dfrac{1}{r+1}\max\{0, d - K_0\}^{r+1}, \qquad r \in [1,\infty).$

Quasi-static models

Let $\Omega \subset \mathbf{R}^3$ be the reference domain, let $\Gamma_D \subset \partial\Omega$ be the Dirichlet boundary part (with non-vanishing measure), let $\Gamma_N = \partial\Omega \setminus \Gamma_D$ be the Neumann boundary part, let n be the outer normal on Γ_N, and let $[0,T]$ be a time interval.

The constitutive equations are complemented by

$$
\begin{array}{rcll}
-\operatorname{div} S & = & f & \text{in} \quad [0,T]\times\Omega, \\
S\cdot n & = & g & \text{on} \quad [0,T]\times\Gamma_N, \\
F & = & \mathrm{id}+\nabla u & \text{in} \quad [0,T]\times\Omega, \\
u & = & d & \text{on} \quad [0,T]\times\Gamma_D
\end{array}
$$

for prescribed volume forces f, surface loads g, and Dirichlet boundary displacements d.

Theorem (Neff). In the pure Dirichlet case, for smooth data and $r > 6$, a time $T > 0$ exists such that the new model **M1** has a unique solution

$$(u,P,R) \in C\big([0,T],H^{2,5}(\Omega,\mathbf{R}^3))\big) \times C^1\big([0,T],H^{4,2}(\Omega,SL(3)),H^{4,2}(\Omega,SO(3))\big).$$

Variational problem

The material is completely determined by the evolution of the internal variables

$$\dot{Z} \in H(F, Z)$$

(depending on a history function H) and the stress response

$$S = S(F, Z).$$

The displacement u is determined by the boundary values $u = d$ on Γ_D and

$$\int_\Omega S(\mathrm{id} + \nabla u, Z) \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds$$

for all v with $v = 0$ on Γ_D.

Discretization

Incremental problem

For a time series $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ with time increments $\tau_n = t_n - t_{n-1}$, the incremental problem is completely determined by a time discretization

$$Z_n = H_n(\tau_n, F_n, Z_{n-1})$$

of the evolution of the internal variables.
This defines the incremental stress response

$$S = S(F_n, H_n(\tau_n, F_n, Z_{n-1})).$$

The displacement u_n is determined by the boundary values $u_n = d(t_n)$ on Γ_D and

$$\int_\Omega S\big(\mathrm{id} + \nabla u_n, H_n(\tau_n, \mathrm{id} + \nabla u_n, Z_{n-1})\big) \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds$$

for all v with $v = 0$ on Γ_D, depending on the history variables Z_{n-1}.

Exponential update in finite plasticity

The evolution problem in $t \in [t_{n-1}, t_n]$

$$\dot{P}(t) = -\gamma(t) P(t) \frac{D(t)}{|D(t)|} \quad \text{with} \quad \begin{array}{rcl} P(t_{n-1}) & = & P_{n-1} \\ D(t) & = & \operatorname{dev}(\operatorname{sym}(P^T(t) F^T(t) S(t) P^{-T}(t))), \\ \gamma(t) & \in & \partial\hat{\chi}_r(|D(t)|) \end{array}$$

is discretized by the linear evolution equation

$$\dot{P}(t) = -\gamma_n P(t) \frac{D_n}{|D_n|} \quad \text{with} \quad \begin{array}{rcl} P(t_{n-1}) & = & P_{n-1} \\ D_n & = & \operatorname{dev}(\operatorname{sym}(P_n^T F_n^T S_n P_n^{-T})), \\ \gamma_n & \in & \partial\hat{\chi}_r(|D_n|); \end{array}$$

this gives (depending on γ_n and D_n)

$$P_n = P_{n-1} \exp\Big(-\tau_n \gamma_n \frac{D_n}{|D_n|}\Big)$$

Note that this procedure preserves the constraint $\det(P_n) = 1$.

History increment for M2

For given F_n and P_{n-1}, compute $D_n^{\text{trial}} = D(F_n, P_{n-1})$. We consider two cases:

for $|D_n^{\text{trial}}| \leq K_0$ we set $P_n = H_n(F_n, P_{n-1}) = P_{n-1}$;

for $|D_n^{\text{trial}}| > K_0$ we compute (P_n, γ_n) such that

$$\begin{pmatrix} P_n - P_{n-1} \exp\Big(-\tau_n \gamma_n \, \dfrac{D(F_n,P_n)}{|D(F_n,P_n)|}\Big) \\ |D(F_n,P_n)| - K_0 \end{pmatrix} = 0$$

with

$$\begin{aligned} D(F,P) \;\; &= \;\; \operatorname{dev}\Big(P^T F^T \operatorname{polar}(FP)\big(\mu\,(P^T F^T \operatorname{polar}(FP) + \operatorname{polar}(FP)^T FP - 2\,\mathrm{id}) \\ &\qquad\qquad + \frac{\lambda}{2} \operatorname{trace}(P^T F^T \operatorname{polar}(FP) + \operatorname{polar}(FP)^T FP - 2\,\mathrm{id})\big)\Big). \end{aligned}$$

History increment for M1

For given F_n and $Z_{n-1} = (P_{n-1}, R_{n-1})$, compute $D_n^{\mathrm{trial}} = D(F_n, P_{n-1}, R_{n-1})$.

For $|D_n^{\mathrm{trial}}| > K_0$ we compute $(P_n, R_n, \gamma_n, \nu_n)$ such that

$$\begin{pmatrix} P_n - P_{n-1} \exp\Big(-\tau_n \gamma_n \, \dfrac{D(F_n, P_n, R_n)}{|D(F_n, P_n, R_n)|}\Big) \\ R_n - \exp\Big(\tau_n \nu_n \; \mathrm{skew}(F_n P_n R_n^T)\Big) R_{n-1} \\ \gamma_n - \dfrac{1}{\eta} \, \big(|D(F_n, P_n, R_n)| - K_0\big)^r \\ \nu_n - \dfrac{1}{\eta} \, \big(1 + |\,\mathrm{skew}(F_n P_n R_n^T)|\big)^r \end{pmatrix} = 0$$

with

$$\begin{aligned} D(F, P, R) \quad = \quad & \mathrm{dev}\Big(\big(\mathrm{id} + \tfrac{1}{2}(P^T F^T R + R^T F P - 2\,\mathrm{id})\big) \\ & \qquad \big(\mu(P^T F^T R + R^T F P - 2\,\mathrm{id}) + \frac{\lambda}{2}\mathrm{trace}(P^T F^T R + R^T F P - 2\,\mathrm{id})\big)\Big). \end{aligned}$$

Finite element setting

For finite elements V_h with nodal points $\mathcal{P}_h$ set $V_h(d) = \{v \mid v(P) = d(P),\ P \in \mathcal{P}_h \cap \Gamma_D\}$.
For given material history Z_{n-1}, find $u_n \in V_h(d_n)$ such that

$$\int_\Omega S\big(\mathrm{id} + \nabla u_n, H_n(\tau_n, \mathrm{id} + \nabla u_n, Z_{n-1})\big) \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds, \qquad v \in V_h(0).$$

This is solved by a Newton method: for given $u_n^m \in V_h(d_n)$ find $w \in V_h(0)$ such that

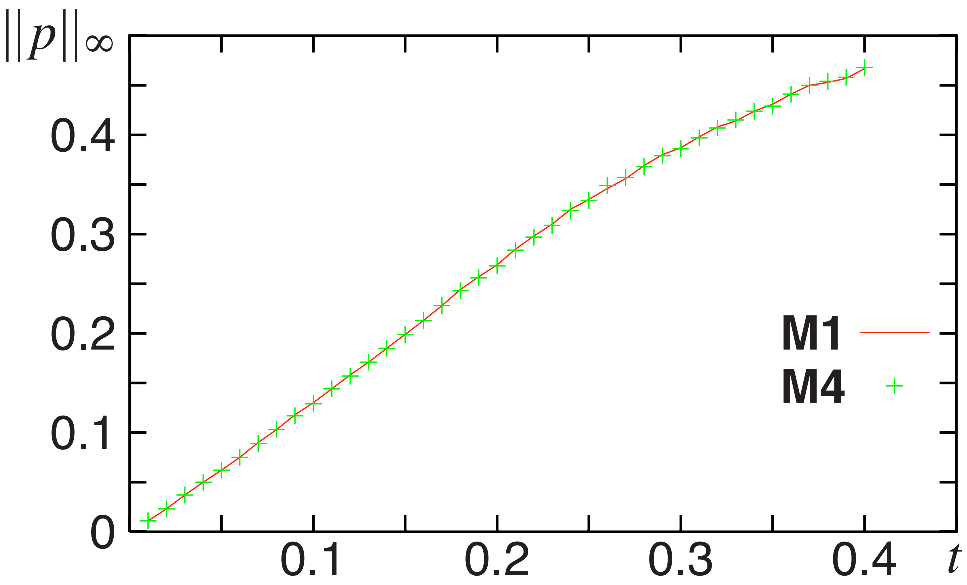
$$\int_\Omega C_n^m \nabla w \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds - \int_\Omega S\big(\mathrm{id} + \nabla u_n^m, Z(\tau_n, \mathrm{id} + \nabla u_n^m, Z_{n-1})\big) \cdot \nabla v \, dx$$

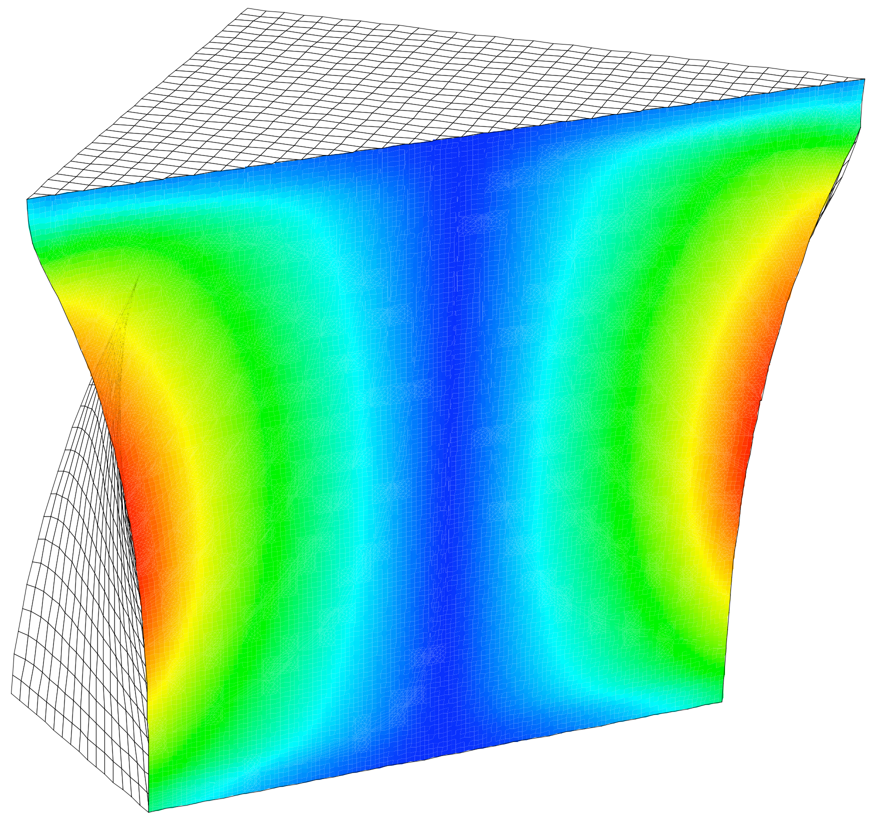
for $v \in V_h(0)$ with the approximated consistent tangent operator $C_n^m \approx D_F S = D_F^2 W$

$$\begin{aligned} C_n^m \nabla w &= \frac{1}{\delta}\Big(S\big(\mathrm{id} + \nabla u_n^m + \delta \nabla w, Z(\tau_n, \mathrm{id} + \nabla u_n^m + \delta \nabla w, Z_{n-1})\big) \\ &\qquad - S\big(\mathrm{id} + \nabla u_n^m, Z(\tau_n, \mathrm{id} + \nabla u_n^m, Z_{n-1})\big)\Big) \qquad (\delta > 0). \end{aligned}$$

Numerical Comparison

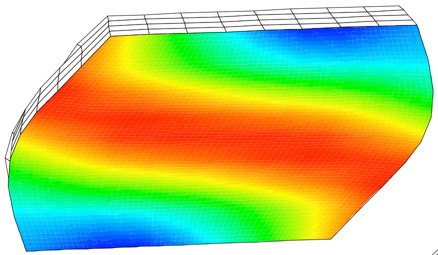
Torsion test



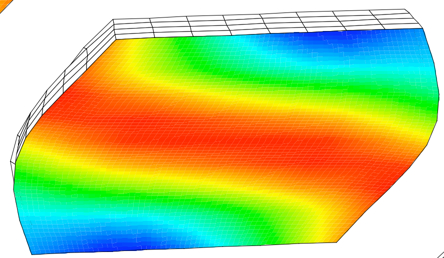


Evolution the equivalent plastic strain $p(t) = \sqrt{\frac{2}{3}} \int_0^t \gamma \, ds$ for **M1** and **M4**

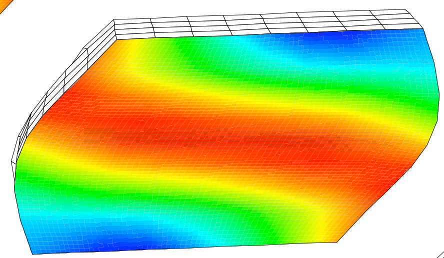
Shear and compression test



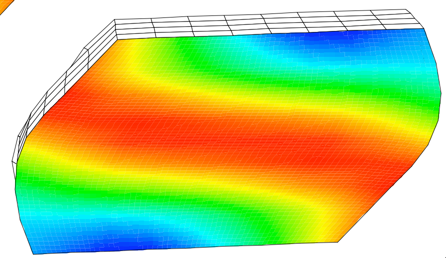
M2



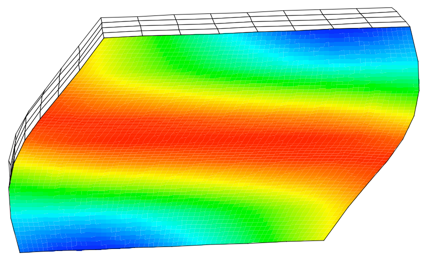
M3



M4



M5



M6

Distribution of the equivalent plastic strain
for a deformation of 25%

Shear and compression test

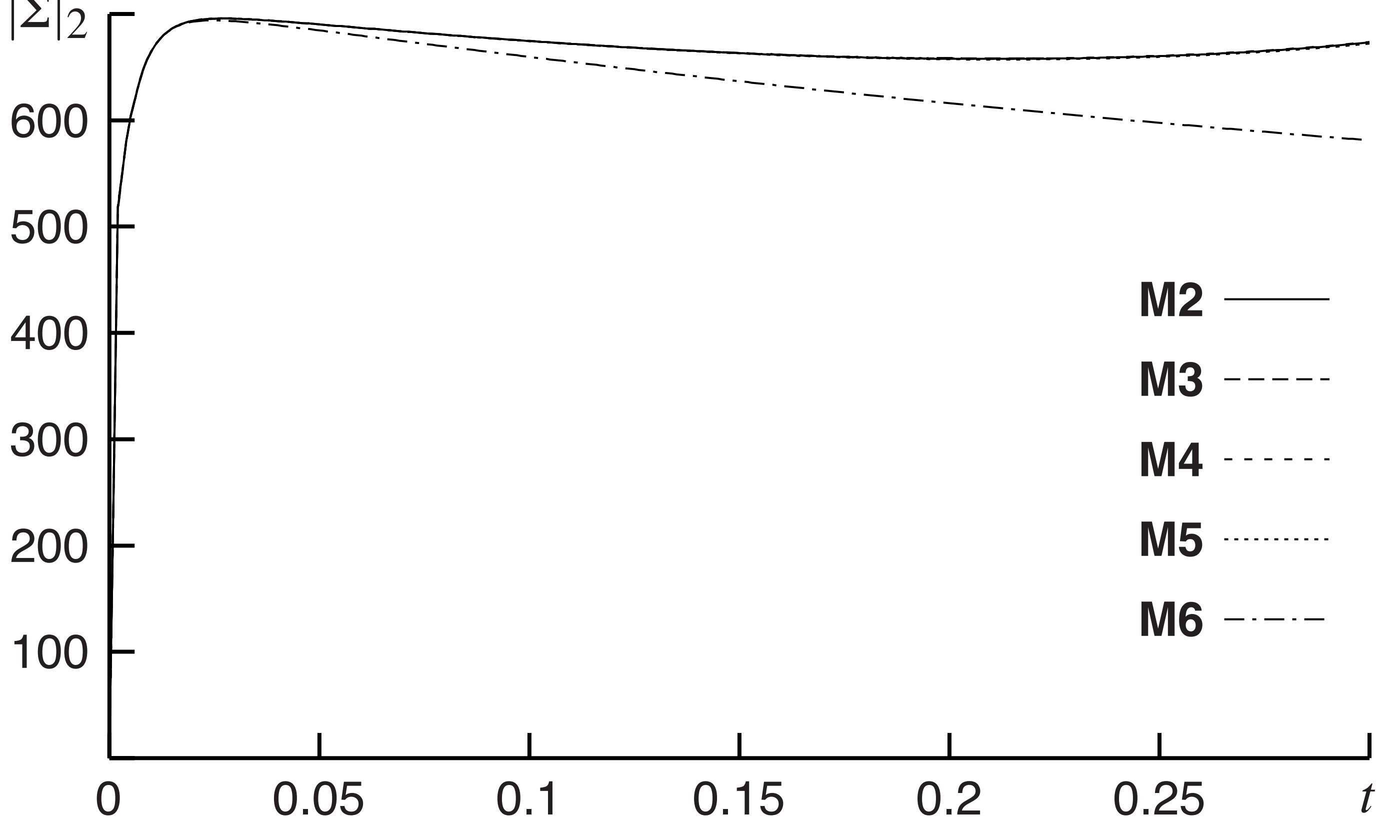
$\|p(t)\|_{\infty,\Omega}$	**M2**	**M3**	**M4**	**M5**	**M6**
$\|p(0.02)\|_{\infty,\Omega}$	0.060	0.060	0.060	0.060	0.053
$\|p(0.1)\|_{\infty,\Omega}$	0.289	0.289	0.289	0.289	0.251
$\|p(0.3)\|_{\infty,\Omega}$	0.878		0.877	0.879	0.662

Results for the finite strain models **M2**, **M3**, **M4**, **M5** (and for comparison perfect plasticity **M6**) for deformations of 2%, 10% and 30% (for **M3** and for $t > 0.2$ the numerical algorithm is converged only on coarser meshes).

As a cross check for nonlinear elasticity, we present the stress invariants for a deformation of 0.1%; the stress response is purely elastic in this case (where $F_p \equiv 0$).

	M2	**M3**	**M4**	**M5**	**M6**
$\Sigma_I(x_m)$	15.7	15.8	15.7	15.7	15.6
$\Sigma_{II}(x_m)$	-7.8	-7.7	-7.8	-7.8	-7.8
$\Sigma_{III}(x_m)$	-263.6	-263.1	-263.9	-264.2	-264.0

Shear and compression test



The evolution of the maximal stress $|\Sigma|_2 = \max\{\Sigma_I(x_m), \Sigma_{II}(x_m), \Sigma_{III}(x_m)\}$ for the models **M2**, **M3**, **M4**, **M5**, and **M6**. Here, $x_m = (0.5, 0.5, 0.5)$ denotes the midpoint of Ω, and Σ_I, Σ_{II}, Σ_{III} are the eigenvalues of the stress tensor Σ (resp. $\mathrm{sym}(\Sigma)$ for **M2**).

Shear and compression test - the effect of viscoplastic relaxation

	M6		M7	
$\|\|p\|\|_{\infty,\Omega}$	0.027	0.057	0.027	0.057
$\|\|\mathrm{dev}(\sigma)\|\|_{\infty,\Omega}$	450	450	450.5	450.5
$\Sigma_I(x_m)$	-665.2	-692.7	-665.8	-693.3
$\Sigma_{II}(x_m)$	-175.4	-185.3	-175.4	-185.4
$\Sigma_{III}(x_m)$	-68.5	-106.3	-68.4	-106.3

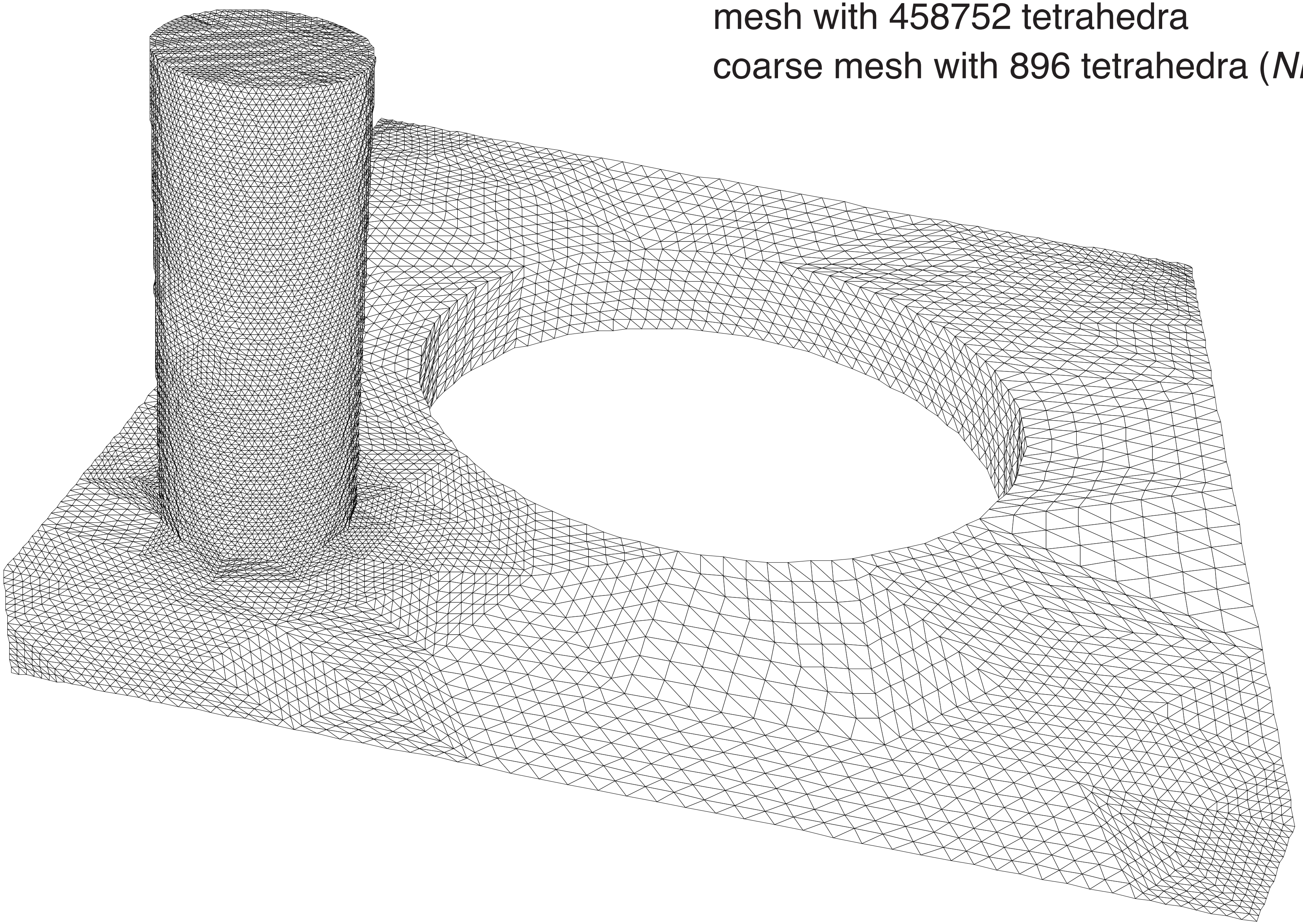
Results for the perfect plasticity model **M6** and the viscoplastic regularization **M7** for a deformation of 1% and 2%.

	M2		M1	
$\|\|p\|\|_{\infty,\Omega}$	0.289	0.878	0.289	0.875
$\|\|\mathrm{dev}(\Sigma)\|\|_{\infty,\Omega}$	450	450	450.5	450.5
$\Sigma_I(x_m)$	-674.6	-673.1	-682.8	-697.4
$\Sigma_{II}(x_m)$	-151.3	-141.8	-158.4	-166.3
$\Sigma_{III}(x_m)$	-99.2	-104.1	-107.3	-127.0

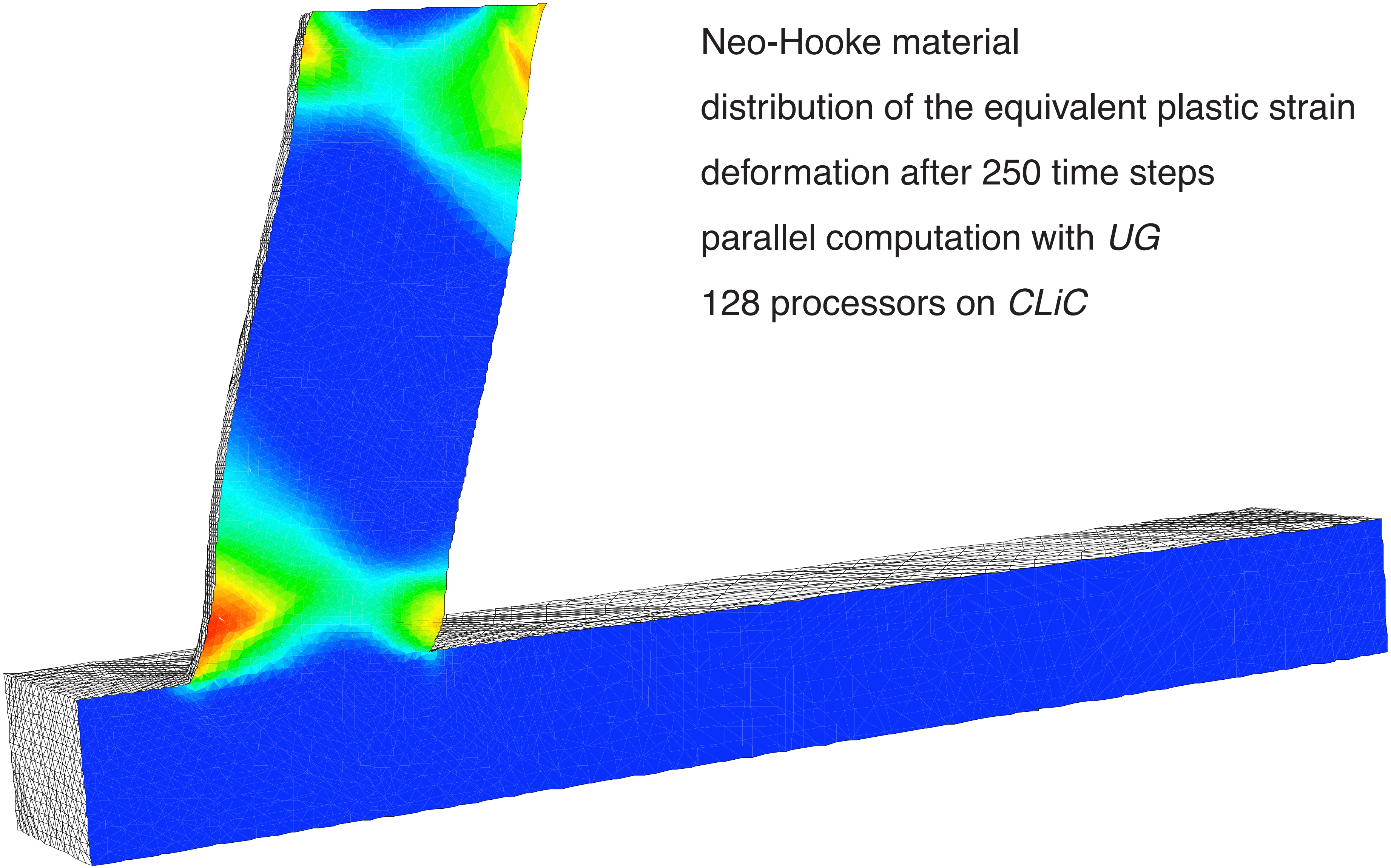
Results for the model **M2** and the viscoplastic regularization **M1** ($t = 0.1$ and 0.3).

Advanced application

mesh with 458752 tetrahedra
coarse mesh with 896 tetrahedra (*NETGEN*)



Advanced application



Neo-Hooke material

distribution of the equivalent plastic strain

deformation after 250 time steps

parallel computation with *UG*

128 processors on *CLiC*