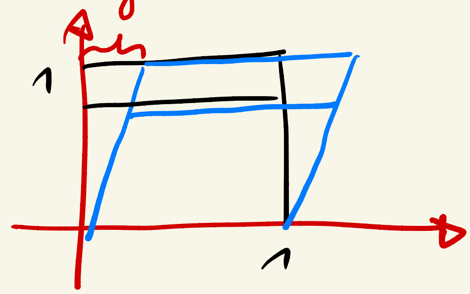
Shear - pure and simple

classical setting:

$$D\varphi = F_\gamma = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

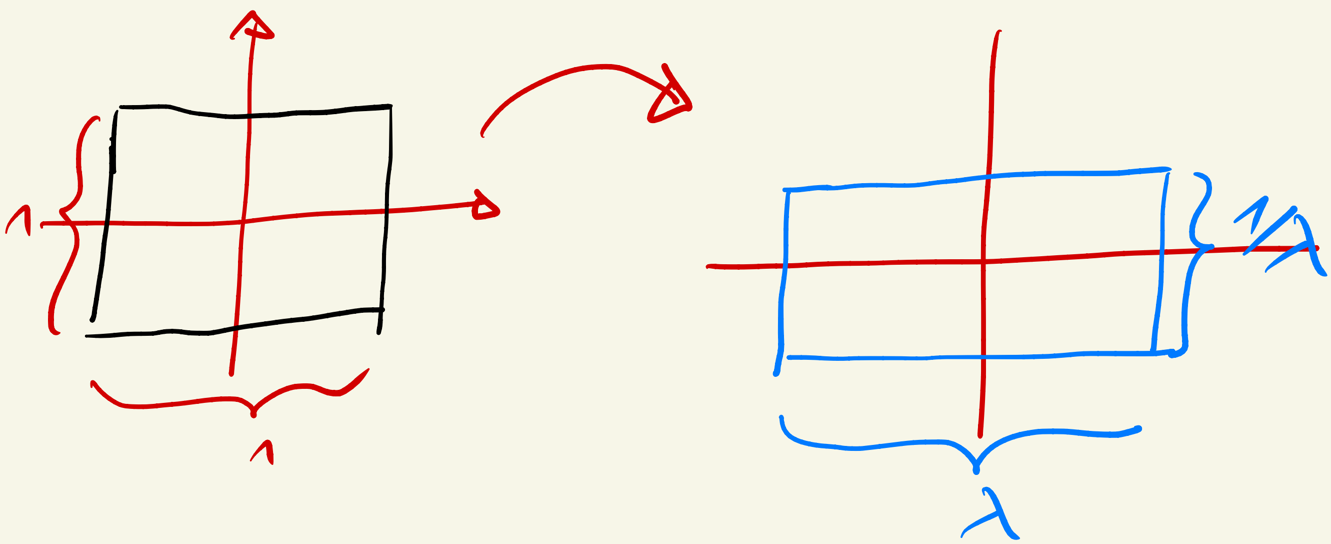
γ - amount of shear



simple shear / simple glide

versus pure shear stretch

$\lambda_1 = \lambda, \ \lambda_2 = 1/\lambda, \ \lambda_3 = 1$



- both are volume-preserving
- both leave third direction invariant

Same question in isotropic linear elasticity?

$$\sigma_{lin} = 2\mu\,\varepsilon + \lambda\,\mathrm{tr}(\varepsilon)\cdot \mathbb{1}$$

$$\sigma_{lin} = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 2\mu\,\varepsilon = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \varepsilon = \begin{pmatrix} 0 & s/2\mu & 0 \\ s/2\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

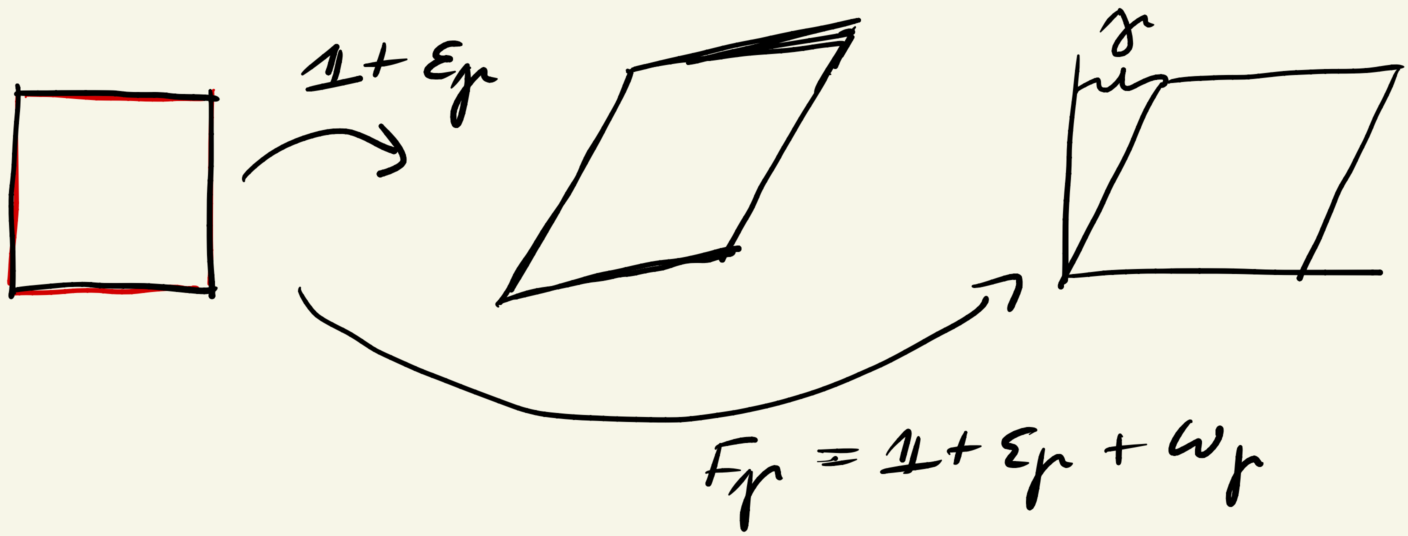
Connection to simple shear

$$F_\gamma = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1} + \begin{pmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & 0 & 0 \\ & & 0 \end{pmatrix}$$

- F_γ is infinitesimally volume preserving

$$\mathrm{tr}(\mathrm{sym}(F_\gamma - \mathbb{1})) = 0$$

- the deformation F_γ is planar, i.e. F_γ has eigenvalue 1 to eigenvector e_3
- the deformation F_γ is ground parallel, i.e. e_1 and e_3 are eigenvectors

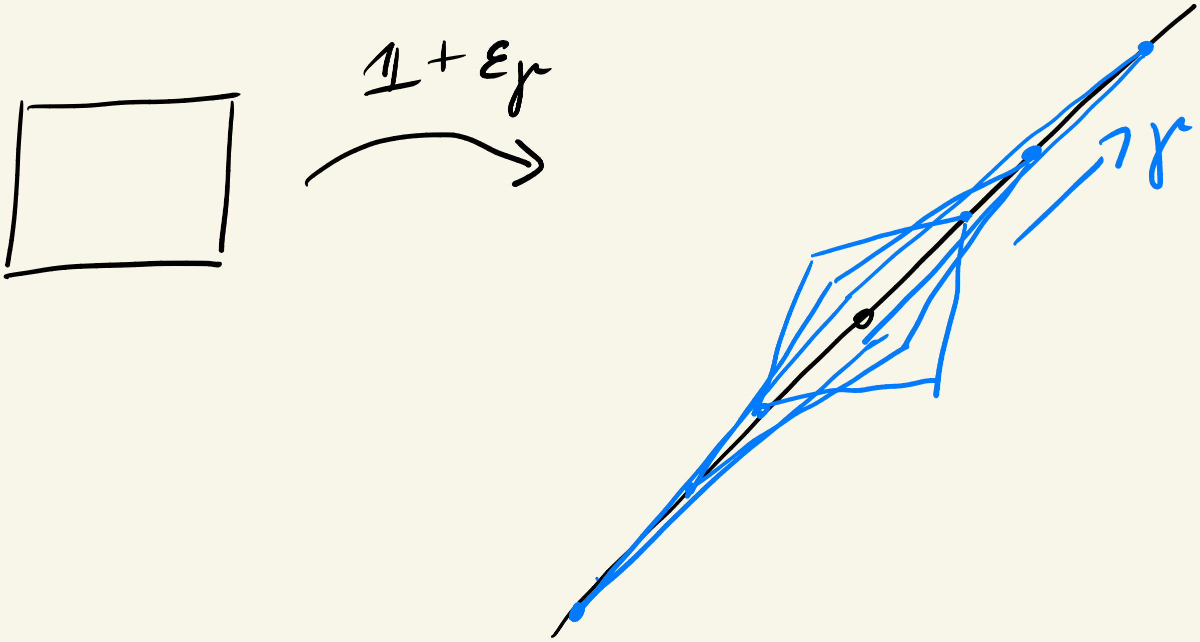


Observe that

$$\mathbb{1} + \varepsilon_\gamma = \begin{pmatrix} 1 & \gamma/2 & 0 \\ \gamma/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is <u>not</u> volume preserving

since

$$\det \begin{pmatrix} 1 & \gamma/2 & 0 \\ \gamma/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 -$$



Back to nonlinear elasticity : Hencky 1928

consider a simple example

$$\sigma = 2\mu \log V + \lambda \operatorname{tr}(\log V) \cdot \mathbb{1}$$

- not hyperelastic, but isotropic
- $V = \sqrt{FF^T}$ left stretch tensor

solve

$$\sigma = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2\mu \log V \iff$$

$$\log V = \begin{pmatrix} 0 & s/2\mu & 0 \\ s/2\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \iff \boxed{V = \exp\begin{pmatrix} 0 & s/2\mu & 0 \\ s/2\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

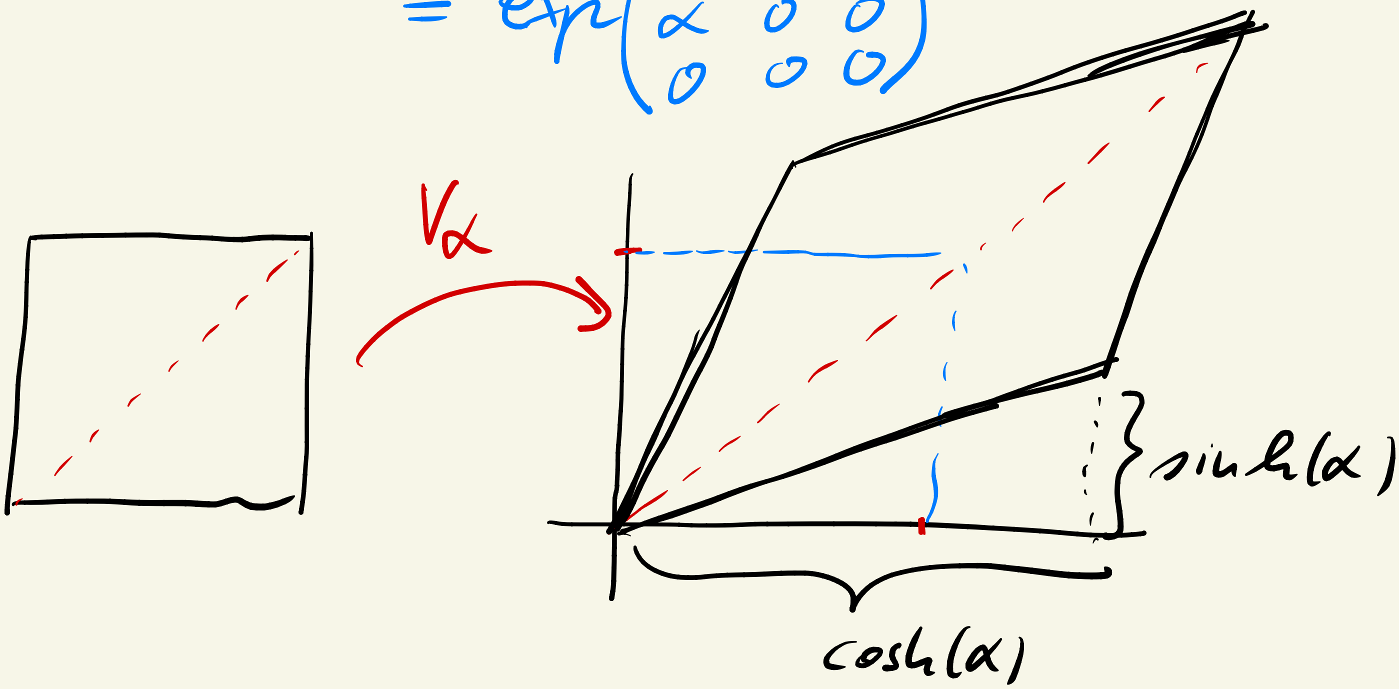
$$S_2 = 2\mu \log U + \lambda \operatorname{tr}(\log U)\cdot \mathbb{1}$$

$$S_2 = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow U = \begin{pmatrix} \cosh(\alpha) & \cdots & \\ & & \\ & & \end{pmatrix}$$

same procedure for Lagrangian stress tensors,
here S_2 - 2nd Piola-Kirchhoff stress

$$V_\alpha = \underbrace{\begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) & 0 \\ \sinh(\alpha) & \cosh(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{"left" "finite pure shear stretch"}}, \quad \alpha = \frac{s}{2\mu}$$

$$= \exp\begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\det V_\alpha = \cosh(\alpha)^2 - \sinh(\alpha)^2 = 1$$

$$=$$

$$\log \det V = \operatorname{tr}(\log V) \iff$$

$$\det V = e^{\operatorname{tr}(\log V)} = e^{\operatorname{tr}\begin{pmatrix} 0 & s/2\mu & 0 \\ s/2\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} = 1$$

Observation: Class of finite pure shear stretches is a <u>multiplicative group</u>: the group of hyperbolic rotations:

$$\exp\begin{pmatrix}0 & \alpha\\ \alpha & 0\end{pmatrix}\cdot\exp\begin{pmatrix}0 & \beta\\ \beta & 0\end{pmatrix} = \exp\begin{pmatrix}0 & \alpha+\beta\\ \alpha+\beta & 0\end{pmatrix}$$

since

$$\begin{pmatrix}0 & \alpha\\ \alpha & 0\end{pmatrix},\ \begin{pmatrix}0 & \beta\\ \beta & 0\end{pmatrix} \quad \text{commute.}$$

(1978) → Hencky strain

Claude Vallée (1978) was perhaps the first to calculate the stretch corresponding to Cauchy shear stress for the quadratic Hencky model:

$$= \mu\,\|\log\sqrt{FF^T}\|^2 + \frac{\lambda}{2}(\log\det F)^2$$

$$W_H(F) = \mu\,\|\log V\|^2 + \frac{\lambda}{2}\,\mathrm{tr}(\log V)^2$$

Richter formula: τ - Kirchhoff stress

$$\tau = D_{\log V}\,\hat{W}(\log V)$$

$$\tau = 2\mu\,\log V + \lambda\,\mathrm{tr}(\log V)\cdot \mathbb{1}$$

$$\sigma = \frac{1}{\det V}\cdot\tau = \begin{pmatrix}0 & s & 0\\ s & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

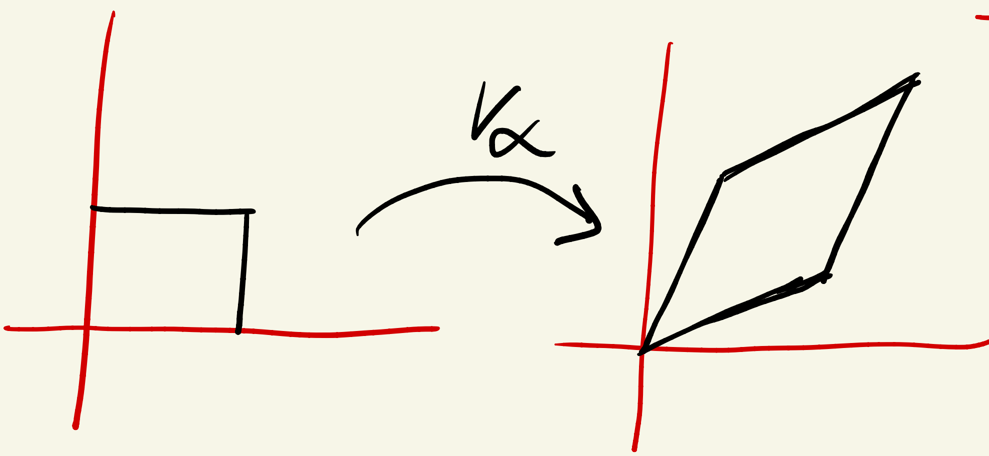
Solution is $V = V_\alpha$, $\det V_\alpha = 1$ ...

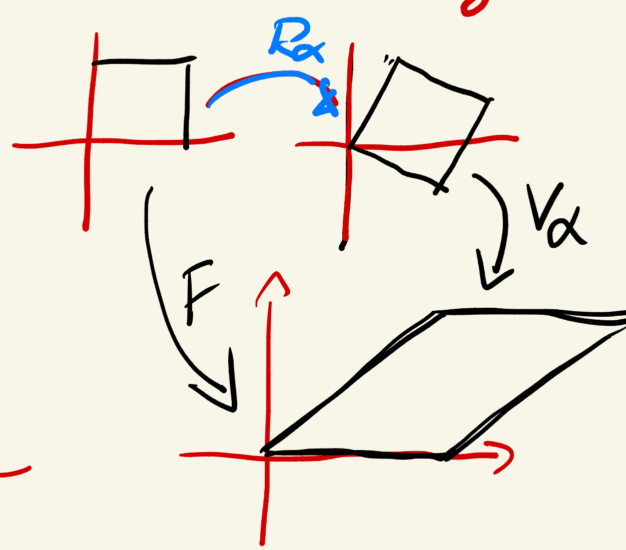
The same result can be obtained for the exponential Hencky energy

$$W_{eH}(F) = \mu\, e^{\|\log V\|^2} + \frac{\lambda}{2}\, e^{\operatorname{tr}^2(\log V)}$$

$$\sigma_{eH} = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow V = V_\alpha$$

Define a corresponding deformation by "rotating down", similar to linear elasticity:





Solve

$$\sqrt{FF^T} = V_\alpha, \qquad F = V_\alpha \cdot R_\alpha$$

Rotation R_α is yet free!

Definition: $F = VR$, $V \in \mathrm{Sym}^{++}(3)$ and $R \in SO(3)$ is called an *idealized finite shear deformation* if

1. V is volume preserving
2. V is planar: $Ve_3 = 1 \cdot e_3$
3. R is such that F is ground-parallel, i.e. e_1 and e_3 are eigenvectors of F.

Then it holds:

$$F = VR \quad \text{with} \quad V = V_\alpha = \exp\begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$F = \frac{1}{\sqrt{\cosh(2\alpha)}} \begin{pmatrix} 1 & \sinh(2\alpha) & 0 \\ 0 & \cosh(2\alpha) & 0 \\ 0 & 0 & \sqrt{\cosh(2\alpha)} \end{pmatrix}$$

$$R = R_\alpha = \frac{1}{\sqrt{\cosh(2\alpha)}} \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) & 0 \\ -\sinh(\alpha) & \cosh(\alpha) & 0 \\ 0 & 0 & \sqrt{\cosh(2\alpha)} \end{pmatrix}$$

linearization consistency

$$(\underline{1}+\varepsilon+\dots)(\underline{1}+\omega+\dots) = 1+\varepsilon+\omega+\varepsilon\omega+\dots$$

$$F_\alpha = \overbrace{V_\alpha}^{} \cdot R_\alpha \longrightarrow F_\gamma = \underline{1} + \varepsilon_\gamma + \omega_\gamma$$

Constitutive question: For which isotropic Cauchy stress strain law is it true that

$$b = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \iff V = \exp\begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (1)$$

Theorem: $W(F) = W_{tc}(F) + h(\det F)$

with $W_{tc}(F) = W_{tc}(F^{-1})$, isotropic, and

$h'(1) = 0$. Then (1) holds.

Theorem: $W(F) = W_{iso}(F) + h(\det F)$ ...

Theorem: $W(F) = \sum w(\lambda_i) + h(\det F)$

$\nearrow$ tens./comp. symmetric

Example: $W(F) = \frac{\mu}{4}\|B - B^{-1}\|^2$

Without additional structural assumptions, Cauchy pure shear stress will not be connected to finite pure shear stretch!

$$\sigma = \begin{pmatrix} 0 & S & 0 \\ S & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma = \beta_0 \, \mathbb{1} + \beta_1 \cdot B + \beta_{-1} \cdot B^{-1}$$

$$\beta_k = \beta_k\left(I_1(B), I_2(B), I_3(B) \right)$$

lin. elasticity $\xrightarrow{\exp}$ nonlinear elasticity

$$F_\gamma = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

σ isotropic

$$\sigma(F_\gamma) \neq \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad ↯$$

$$\sigma_{lin}(F_\gamma - \mathbb{1}) = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad ✓$$

$$\boxed{\sigma_{lin}(\varepsilon) = -\sigma_{lin}(-\varepsilon)}$$

$$\mu \|\varepsilon\|^2 + \frac{\lambda}{2} \operatorname{tr}(\varepsilon)^2$$