



A geometrically exact membrane model for isotropic foils and fabrics

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The finite-strain-viscoelastic membrane model

The spatial deformation of a thin-walled structure $\phi_s : \omega \times (-\frac{h}{2}, \frac{h}{2}) \rightarrow \mathbb{R}^3$ is decomposed into the motion of the (initially plane) midsurface $m : \omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^2$ and of the director (initially) orthogonal to the midsurface,

$$\phi_s(x, y, z) = m(x, y) + z \varrho_m(x, y) R(x, y) \cdot e_3, \quad (1)$$

where $R = \text{polar}(F) \in \text{SO}(3)$ is the orthogonal part of the deformation gradient F with out-of plane component $R(x, y) \cdot e_3$. The variable $\varrho_m \in \mathbb{R}$ accounts for a varying thickness, see [1, 2] for details.

Basic idea: introduce an *additional* field of independently evolving viscoelastic rotations $\bar{R} \in \text{SO}(3)$. These rotations $\bar{R}$ are thought of as being physical meaningful but not exact continuum rotations R . With $R_3 \equiv \bar{R}(x, y) \cdot e_3$ denoting the corresponding out-of plane component the dimensional reduction of a three-dimensional continuum solid to a geometrically exact membrane model results in a deformation gradient of the form

$$F = (\nabla m | \varrho_m \bar{R}_3), \quad (2)$$

where $\nabla m \in \mathbb{M}^{3 \times 3}$ is the deformation gradient of the midsurface with $m_x = (m_{1,x}, m_{2,x}, m_{3,x})^T$, $m_y = (m_{1,y}, m_{2,y}, m_{3,y})^T$.

The problem: find the deformation of the midsurface $m : [0, T] \times \omega \mapsto \mathbb{R}^3$ and the independent local viscoelastic rotation $\bar{R} : [0, T] \times \omega \mapsto \text{SO}(3, \mathbb{R})$ such that

$$\int_{\omega} h W(F, \bar{R}) d\omega - \int_{\omega} \langle f_b, m \rangle d\omega - \int_{\gamma_s} \langle f_s, m \rangle ds \mapsto \min, \quad (3)$$

w.r.t. m at fixed rotation $\bar{R}$. The strain energy density $W(F, \bar{R})$ in (3) is of the form

$$W(F, \bar{R}) = \frac{\mu}{4} \|F^T \bar{R} + \bar{R}^T F - 2I\|^2 + \frac{\lambda}{8} \text{tr} \left(F^T \bar{R} + \bar{R}^T F - 2I \right)^2. \quad (4)$$

Moreover, let $W^{\text{ext}}(m)$ be the linear work of applied external forces with f_b being the resultant body forces and f_s the resultant surface traction and let $g_d : \omega \mapsto \mathbb{R}^3$ denote the prescribed Dirichlet boundary conditions for the membrane,

$$W^{\text{ext}}(m) = \int_{\omega} \langle f_b, m \rangle d\omega - \int_{\gamma_s} \langle f_s, m \rangle ds, \\ m|_{\gamma_0}(t, x, y) = g_d(t, x, y) \quad x, y \in \gamma_0 \subset \partial\omega. \quad (5)$$

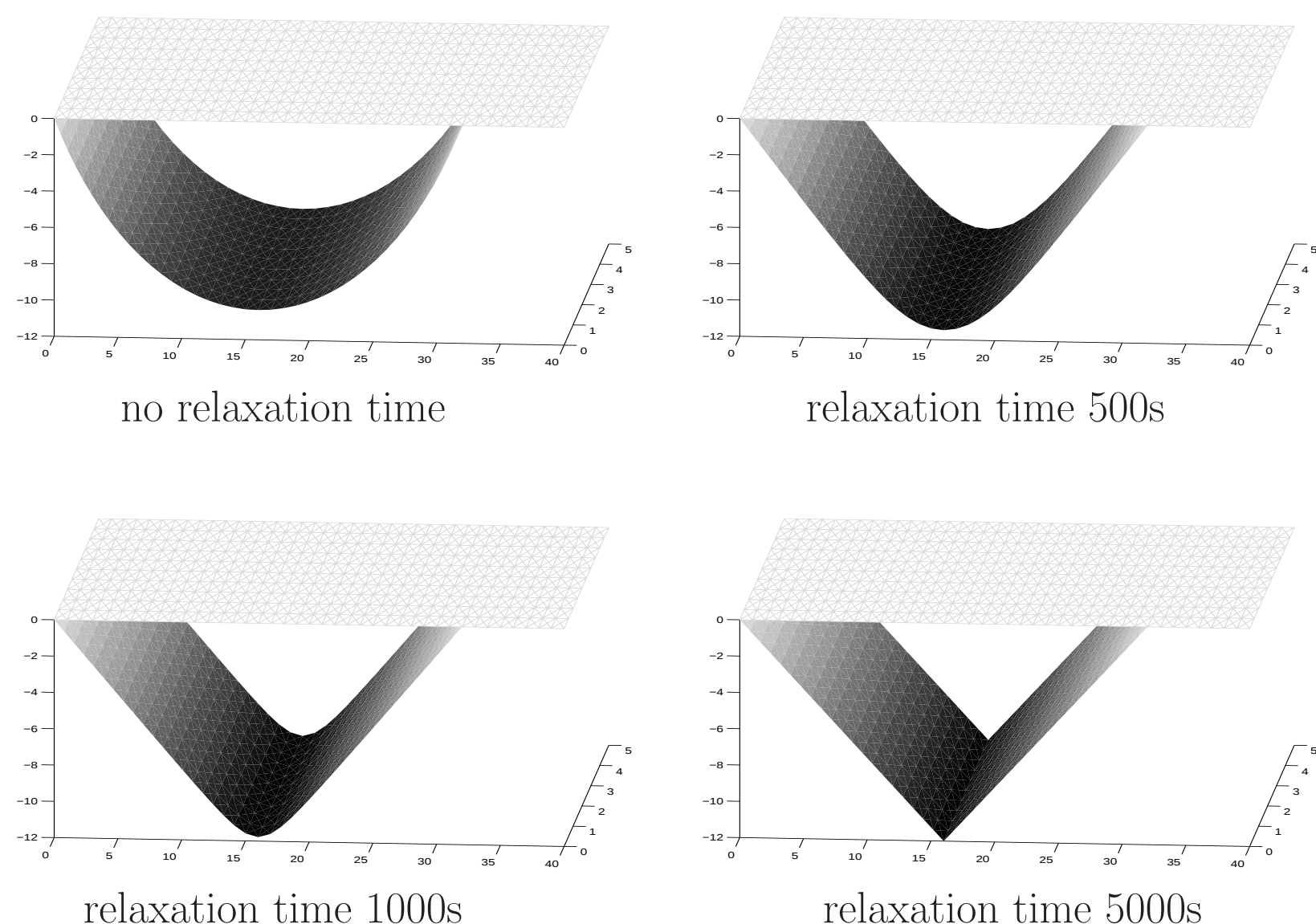
The field of local viscoelastic rotation follows an **evolution equation**

$$\frac{d}{dt} \bar{R}(t) = \nu^+ \cdot \text{skew}(B) \cdot \bar{R}(t) \quad \text{with} \quad \nu^+ := \frac{1}{\eta} \nu^+(F, \bar{R}), \quad \text{and} \quad B = F \bar{R}^T. \quad (6)$$

Here $\nu^+ \in \mathbb{R}^+$ represents a scalar valued function introducing an *artificial viscosity* and η plays the role of an *artificial relaxation time* (with units [sec]). The evolution equation (6) and parameter ν^+ are introduced into the model to preserve ellipticity of the force balance. Physically, one may imagine the viscoelastic rotation $\bar{R}$ as *shadowing* the exact continuum rotation in a viscous sense.

Example: rectangular sheet

A hard sheet is loaded by dead load and subjected to in-plane displacement of one side. The figures show the initial and the deformed state after different time periods of relaxation:



Discretization of the model

We consider a fully implicit time discretized version of model (3). Let $(m^{n-1}, \bar{R}^{n-1})$ be the given solution for the deformation of the midsurface and the rotations at time t_{n-1} . Now, compute the new solution $(m^n, \bar{R}^n) \in \mathbb{V}$ at time t_n such that

$$\int_{\omega} h W(F^n, \bar{R}^n) d\omega - W^{\text{ext},n}(m^n) \mapsto \min, \quad (7)$$

w.r.t. m^n at fixed $\bar{R}^n$. The current deformation gradient $F^n = F(t_n)$ is

$$F^n = (\nabla m^n | \varrho_m^n \bar{R}_3^n), \quad (8)$$

and the current boundary conditions are

$$m|_{\gamma_0}^n(t_n, x, y) = g_d(t_n, x, y), \quad x, y \in \gamma_0 \subset \partial\omega. \quad (9)$$

The **evolution equation** for the rotations is mapped by a **local exponential update**. This implies that $\bar{R}^n = \bar{R}^n(\nabla m^n)$ solves the following highly nonlinear problem

$$\bar{R}^n = \exp \left(\Delta t \nu_n^+ \text{skew} \left(F^n \bar{R}^{n,T} \right) \right) \cdot \bar{R}^{n-1} \quad \text{with} \quad \nu_n^+ = \frac{1}{\eta} \left(1 + \|\text{skew} F^n \bar{R}^{n,T}\| \right)^2. \quad (10)$$

By the properties of logarithmic and exponential mapping it can be shown that (10) converges to (6) for the limit $\Delta t \rightarrow 0$, see [1].

The **finite element discretization** of problem (7) considers discrete subspaces $\mathbb{V}_h$ of the continuous solution spaces $\mathbb{V}$ for the membrane's deformation. We employ

$$\mathbb{V}_h = \mathcal{P}_1^o(\mathcal{T})^3 \times \mathcal{P}_0(\mathcal{T})^{3 \times 3}, \quad (11)$$

where $\mathcal{P}_k(\mathcal{T})$ denotes the linear space of $\mathcal{T}$ -piecewise polynomials of degree $\leq k$, and, $\mathcal{P}_k^o(\mathcal{T})$ are the continuous discrete functions in $\mathcal{P}_k(\mathcal{T})$ with homogeneous boundary values. Thus, the **discrete problem** reads: find the deformation of the midsurface of the membrane and the independent local viscoelastic rotation $(m_h, \bar{R}_h) : [0, T] \times \mathbb{V}_h$ such that,

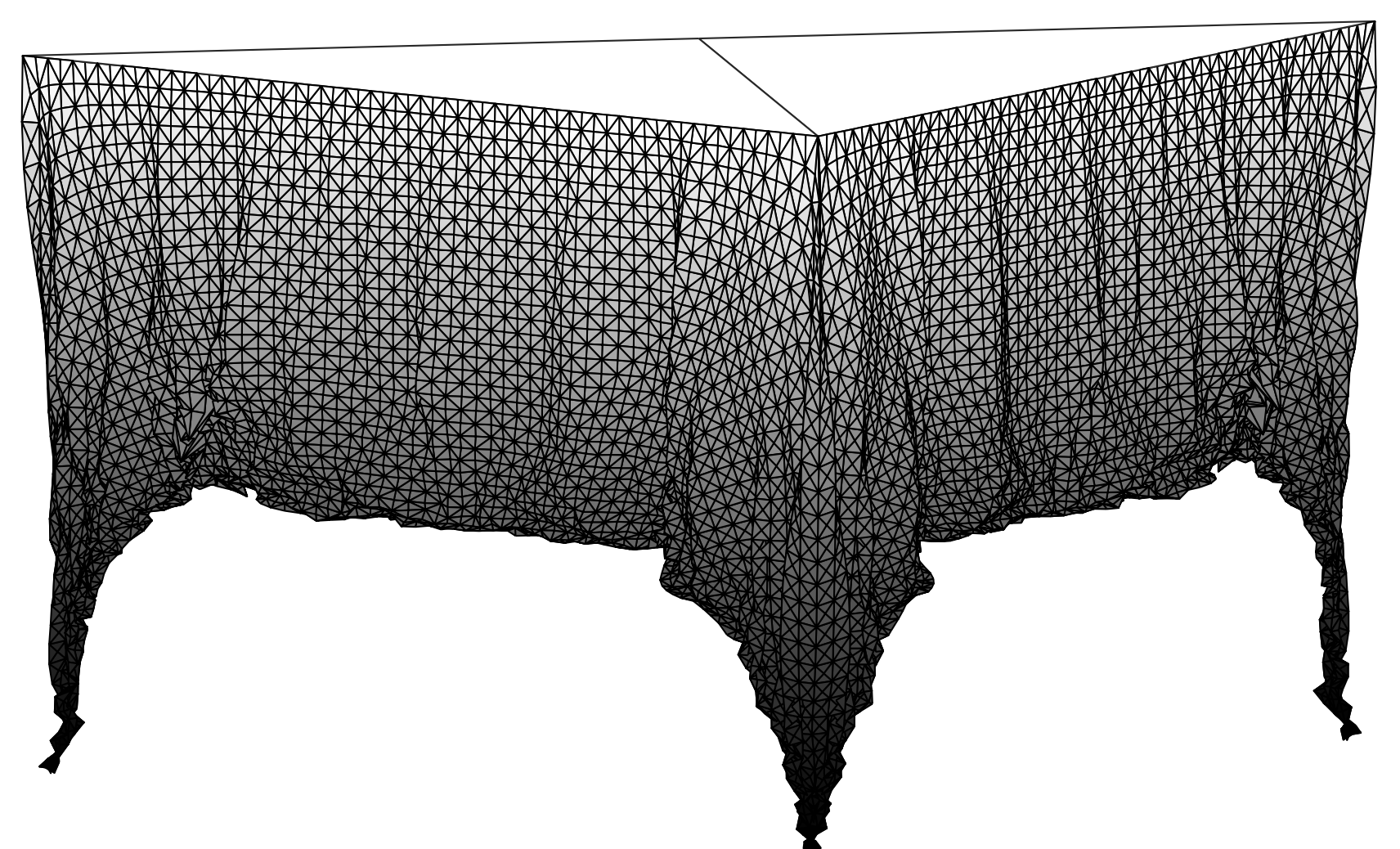
$$\int_{\omega} h W(F(m_h), \bar{R}_h) d\omega - W^{\text{ext}}(m_h, \bar{R}_h) \mapsto \min, \quad (12)$$

w.r.t. m_h at fixed rotation $\bar{R}_h$ such that R_h satisfies (10).

Example: wrinkling of a thin foil

We apply our model to the problem of a 2×2 m elastic foil under pressure load. The foil is 1mm thick and lays on a square obstacle (like a cloth on a table) and only the unsupported part of it can deform. A pressure of $p_0 = 0.75$ MPa acts from above.

Wrinkling of a soft foil (relaxed state):



- [1] K. Weinberg and P. Neff: *A geometrically exact thin membrane model - investigation of large deformations and wrinkling*, IJNME, to appear.
- [2] P. Neff: *A geometrically exact viscoplastic membrane-shell with viscoelastic transverse shear resistance avoiding degeneracy in the thin-shell limit. Part I: The viscoelastic membrane-plate.*, ZAMP, **56** (2005), 148–182.