

A Numerical Solution Method for an Infinitesimal Elasto-Plastic Cosserat Model

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A micro-polar extension to infinitesimal elasticity

- ▶ We present a geometrically linear generalized continua of Cosserat micro-polar type in the elasto-plastic case.
- ▶ Starting with linear elasticity we postulate independent infinitesimal micro-rotations of the material. Thus, as a consequence of balance of angular momentum, Cauchy stresses σ are not symmetric any more.
- ▶ Cosserat regularize the mesh size dependence of localization computations where shear failure mechanisms play a dominant role.
- ▶ Models are of engineering interest: Diebels/Ehlers, Iordache/William, Dietsche/Steinmann/William, Ristinmaa/Vecchi, de Borst, ...
- ▶ We restrict Cosserat micro-rotations to the elastic response of the material. Inelasticity is formulated as in Prandtl-Reuß plasticity. The elasto-plastic Cosserat problem is well-posed (Neff/Chełmiński Appl.Math.Opti06, PRSE05).

The Elastic Static Cosserat Model

Compute displacement $\mathbf{u}$ and infinitesimal rotations $\text{id} + \bar{\mathbf{A}}$ with $\bar{\mathbf{A}}^T = -\bar{\mathbf{A}}$, such that

$$\int_{\Omega} W(D\mathbf{u}, \bar{\mathbf{A}}, D\bar{\mathbf{A}}) - \ell(\mathbf{u}) \longrightarrow \min \quad \text{w.r.t. } (\mathbf{u}, \bar{\mathbf{A}})$$

where the stored energy is

$$\begin{aligned} W(D\mathbf{u}, \bar{\mathbf{A}}, D\bar{\mathbf{A}}) = & \mu |\text{sym}(D\mathbf{u} - \bar{\mathbf{A}})|^2 + \frac{\lambda}{2} \text{tr}(\text{sym}(D\mathbf{u} - \bar{\mathbf{A}}))^2 \\ & + \mu_c |\text{skew}(D\mathbf{u} - \bar{\mathbf{A}})|^2 + \mu L_c^2 |D\bar{\mathbf{A}}|^2. \end{aligned}$$

Remark:

Coercive without Korn's first inequality!

Linear Elastic Static Cosserat Model - Equations

We want to determine

displacements

$$\mathbf{u}: \quad \bar{\Omega} \times [0, T] \longrightarrow \mathbb{R}^d,$$

infinitesimal micro-rotations

$$\bar{\mathbf{A}}: \quad \Omega \times [0, T] \longrightarrow \mathfrak{so}(d),$$

non-symmetric stresses

$$\boldsymbol{\sigma}: \quad \Omega \times [0, T] \longrightarrow \mathbf{R}^{d,d},$$

satisfying the essential boundary conditions, the equilibrium equations

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma}(\mathbf{x}, t) &= \mathbf{b}(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times [0, T], \\ \boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{n}(\mathbf{x}) &= \mathbf{t}_N(\mathbf{x}, t), & (\mathbf{x}, t) \in \Gamma_N \times [0, T], \\ -\mu_c L_c^2 \Delta \bar{\mathbf{A}}(\mathbf{x}, t) &= \mu_c (\operatorname{skew}(D\mathbf{u}(\mathbf{x}, t)) - \bar{\mathbf{A}}(\mathbf{x}, t)), & (\mathbf{x}, t) \in \Omega \times [0, T], \\ D\bar{\mathbf{A}}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) &= \mathbf{0}, & (\mathbf{x}, t) \in \Gamma_N \times [0, T], \end{aligned}$$

and the constitutive relation

$$\begin{aligned} \boldsymbol{\sigma}(\mathbf{x}, t) &= 2\mu (\operatorname{sym}(D\mathbf{u}(\mathbf{x}, t))) + \lambda \operatorname{div}(\mathbf{u})(\mathbf{x}, t) \mathbf{I} \\ &\quad + 2\mu_c (\operatorname{skew}(D\mathbf{u}(\mathbf{x}, t)) - \bar{\mathbf{A}}(\mathbf{x}, t)), \quad (\mathbf{x}, t) \in \Omega \times [0, T]. \end{aligned}$$

Infinitesimal Elastic Cosserat Model - Data

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be the reference configuration, and let $\Gamma_D \cup \Gamma_N = \partial\Omega$ be a decomposition of the boundary. We fix a time interval $[0, T]$.

Given data:

$$\begin{array}{ll} \text{displacement vector} & \mathbf{u}_D: \Gamma_D \times [0, T] \longrightarrow \mathbb{R}^d, \\ \text{suitable infinitesimal micro-rotations} & \bar{\mathbf{A}}_D: \Gamma_D \times [0, T] \longrightarrow \mathfrak{so}(d), \end{array}$$

where $\mathfrak{so}(d) = \{\boldsymbol{\tau} \in \mathbb{R}^{d,d} : \boldsymbol{\tau}^T = -\boldsymbol{\tau}\}$ is the Lie algebra of skew-symmetric matrices, and a load functional

$$\ell(t, \mathbf{v}) = \int_{\Omega} \mathbf{b}(t) \cdot \mathbf{v} d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N(t) \cdot \mathbf{v} d\mathbf{a}$$

depending on traction force densities $\mathbf{t}_N$ and body force densities $\mathbf{b}$.

The material is described by a linear elastic response depending on the Lamé constants $\lambda, \mu > 0$ and Cosserat constants $L_c > 0$ and $\mu_c \geq 0$.

Infinitesimal Elasto-Plastic Cosserat Model - Flow rule

Inelastic material behavior is modeled by a convex function

$$\phi: \text{Sym}(d) \longrightarrow \mathbb{R}$$

determining the convex set of admissible symmetric stresses

$\mathbf{K} = \{\boldsymbol{\tau} \in \text{Sym}(d): \phi(\boldsymbol{\tau}) \leq 0\}$. We assume that ϕ is smooth for $\boldsymbol{\tau} \neq \mathbf{0}$, and we assume $\phi(\mathbf{0}) < 0$. In a first approach we choose the von Mises yield criterion $\phi(\boldsymbol{\tau}) = |\text{dev}(\boldsymbol{\tau})| - K_0$ for a given yield stress $K_0 > 0$. Here $\text{dev}: \text{Sym}(d) \longrightarrow \text{Sym}(d)$ with $\text{dev } \boldsymbol{\tau} = \boldsymbol{\tau} - \frac{1}{d} \text{tr } \boldsymbol{\tau}$.

We have $P_{\mathbf{K}}(\boldsymbol{\theta}) = \boldsymbol{\theta} - \max\{0, \gamma\} \frac{\text{dev}(\boldsymbol{\theta})}{|\text{dev}(\boldsymbol{\theta})|}$.

We use the following realization $\mathbf{C}(\boldsymbol{\theta}) \in \partial P_{\mathbf{K}}(\boldsymbol{\theta})$, where $P_{\mathbf{K}}(\boldsymbol{\theta})$ is the multi-valued derivative of the projection defined by $\mathbf{C}(\boldsymbol{\theta}) = \text{id}$ for $|\text{dev}(\boldsymbol{\theta})| \leq K_0$ and

$$\mathbf{C}(\boldsymbol{\theta}) = \frac{1}{d} \mathbf{I} \otimes \mathbf{I} + \frac{K_0}{|\text{dev}(\boldsymbol{\theta})|} \left(\left(\text{id} - \frac{1}{d} \mathbf{I} \otimes \mathbf{I} \right) - \frac{\text{dev}(\boldsymbol{\theta})}{|\text{dev}(\boldsymbol{\theta})|} \otimes \frac{\text{dev}(\boldsymbol{\theta})}{|\text{dev}(\boldsymbol{\theta})|} \right) \text{ for } |\text{dev}(\boldsymbol{\theta})| > K_0.$$

Infinitesimal Elasto-Plastic Cosserat Model - Equations

We want to determine
displacements

$$\mathbf{u}: \quad \bar{\Omega} \times [0, T] \longrightarrow \mathbb{R}^d,$$

infinitesimal micro-rotations

$$\bar{\mathbf{A}}: \quad \Omega \times [0, T] \longrightarrow \mathfrak{so}(d),$$

non-symmetric stresses

$$\boldsymbol{\sigma}: \quad \Omega \times [0, T] \longrightarrow \mathbf{R}^{d,d},$$

symmetric plastic strains

$$\varepsilon_p: \quad \Omega \times [0, T] \longrightarrow \text{Sym}(d) \text{ with } \varepsilon_p(0) = \mathbf{0},$$

and a plastic multiplier

$$\Lambda: \quad \Omega \times [0, T] \longrightarrow \mathbb{R},$$

satisfying the essential boundary conditions and the equilibrium equations

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma}(\mathbf{x}, t) &= \mathbf{b}(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times [0, T], \\ \boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{n}(\mathbf{x}) &= \mathbf{t}_N(\mathbf{x}, t), & (\mathbf{x}, t) \in \Gamma_N \times [0, T], \\ -\mu L_c^2 \Delta \bar{\mathbf{A}}(\mathbf{x}, t) &= \mu_c (\operatorname{skew}(D\mathbf{u}(\mathbf{x}, t)) - \bar{\mathbf{A}}(\mathbf{x}, t)), & (\mathbf{x}, t) \in \Omega \times [0, T], \\ D\bar{\mathbf{A}}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) &= \mathbf{0}, & (\mathbf{x}, t) \in \Gamma_N \times [0, T], \end{aligned}$$

the constitutive relation

$$\begin{aligned} \boldsymbol{\sigma}(\mathbf{x}, t) &= 2\mu (\operatorname{sym}(D\mathbf{u}(\mathbf{x}, t)) - \varepsilon_p(\mathbf{x}, t)) + \lambda \operatorname{div}(\mathbf{u})(\mathbf{x}, t) \mathbf{I} \\ &\quad + 2\mu_c (\operatorname{skew}(D\mathbf{u}(\mathbf{x}, t)) - \bar{\mathbf{A}}(\mathbf{x}, t)), \quad (\mathbf{x}, t) \in \Omega \times [0, T], \end{aligned}$$

Infinitesimal Elasto-Plastic Cosserat Model - Equations

the complementary conditions for the yield criterion

$$\Lambda(\mathbf{x}, t) \phi(T_E(\mathbf{x}, t)) = 0, \quad \Lambda(\mathbf{x}, t) \geq 0, \quad \phi(T_E(\mathbf{x}, t)) \leq 0, \quad (\mathbf{x}, t) \in \Omega \times [0, T].$$

and the flow rule

$$\frac{d}{dt} \varepsilon_p(\mathbf{x}, t) = \Lambda(\mathbf{x}, t) D\phi(T_E(\mathbf{x}, t)), \quad (\mathbf{x}, t) \in \Omega \times [0, T],$$

depending on $T_E(\mathbf{x}, t) = 2\mu(\text{sym}(D\mathbf{u}(\mathbf{x}, t)) - \varepsilon_p(\mathbf{x}, t))$.

For given material history $\varepsilon_p(t)$ at fixed time t , the displacement and the micro-rotations are determined by minimizing the total elastic energy

$$\mathcal{I}(\mathbf{u}, \bar{\mathbf{A}}, \varepsilon_p) = \mathcal{E}(\varepsilon(\mathbf{u}), \bar{\mathbf{A}}, \varepsilon_p) - \ell(t, \mathbf{u}),$$

$$\begin{aligned} \text{with } \mathcal{E}(\varepsilon, \bar{\mathbf{A}}, \varepsilon_p) = & \mu \int_{\Omega} |\text{sym}(\varepsilon) - \varepsilon_p|^2 d\mathbf{x} + \frac{\lambda}{2} \int_{\Omega} \text{tr}(\varepsilon)^2 d\mathbf{x} \\ & + \mu_c \int_{\Omega} |\text{skew}(\varepsilon) - \bar{\mathbf{A}}|^2 d\mathbf{x} + \mu L_c^2 \int_{\Omega} |D\bar{\mathbf{A}}|^2 d\mathbf{x}. \end{aligned}$$

Infinitesimal Model - subdifferential formulation

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma} &= \mathbf{b} , \\ -\mu L_c^2 \Delta \bar{\mathbf{A}} &= \mu_c (\operatorname{skew}(D\mathbf{u}) - \bar{\mathbf{A}}) , \\ \boldsymbol{\sigma} &= 2\mu (\operatorname{sym}(D\mathbf{u}) - \boldsymbol{\varepsilon}_p) + \lambda \operatorname{div}(\mathbf{u}) \mathbf{I} + 2\mu_c (\operatorname{skew}(D\mathbf{u}) - \bar{\mathbf{A}}) , \\ \frac{d}{dt} \boldsymbol{\varepsilon}_p &\in \partial \chi(T_E) , \\ T_E &= 2\mu (\operatorname{sym}(D\mathbf{u}) - \boldsymbol{\varepsilon}_p) \end{aligned}$$

where χ is the indicator function of the elastic domain $\mathbf{K}$.

Discretization in space

Let h be a mesh size parameter, and let $\mathbf{V}_h \subset C^{0,1}(\Omega, \mathbb{R}^d)$ and $W_h \subset C^{0,1}(\Omega, \mathfrak{sl}(d))$ be finite element spaces, and set

$$\mathbf{V}_h(\mathbf{u}_D) = \{\mathbf{v} \in \mathbf{V}_h : \mathbf{v}(\mathbf{x}) = \mathbf{u}_D(\mathbf{x}) \text{ for } \mathbf{x} \in D_h\},$$

$$W_h(A_D) = \{B \in W_h : B(\mathbf{x}) = A_D(\mathbf{x}) \text{ for } \mathbf{x} \in D'_h\}$$

where $D_h, D'_h \subset \Gamma_D$ are the sets of all nodal points on Γ_D for $\mathbf{u}$ and A .

Let $\Xi_h \subset \Omega$ be quadrature points and let ω_ξ be corresponding quadrature weights such that

$$\int_{\Omega} \mathbf{v} \cdot \mathbf{w} d\mathbf{x} = \sum_{\xi \in \Xi_h} \omega_\xi \mathbf{v}(\xi) \cdot \mathbf{w}(\xi), \quad \mathbf{v}, \mathbf{w} \in \mathbf{V}_h.$$

We set

$$\mathbf{\Lambda} = \{\mathbf{\Lambda} : \Xi_h \longrightarrow \mathbb{R}\},$$

$$\mathbf{\Sigma}_h = \{\boldsymbol{\tau} : \Xi_h \longrightarrow \mathbf{R}^{d,d}\},$$

$$\text{and } \mathbf{E}_h^p = \{\boldsymbol{\tau} : \Xi_h \longrightarrow \mathfrak{sl}(d) \cap \text{Sym}(d)\},$$

where $\mathfrak{sl}(d) = \{\boldsymbol{\tau} \in \mathbb{R}^{d,d} : \text{tr}(\boldsymbol{\tau}) = 0\}$ is the Lie algebra of trace-free matrices.

Discretization in space

Determine

displacements $\mathbf{u}$:	$[0, T] \longrightarrow \mathbf{V}_h,$
stresses $\boldsymbol{\sigma}$:	$[0, T] \longrightarrow \boldsymbol{\Sigma}_h,$
micro-rotations $\bar{\mathbf{A}}$:	$[0, T] \longrightarrow \mathbf{W}_h,$
plastic strains $\boldsymbol{\varepsilon}_p$:	$[0, T] \longrightarrow \mathbf{E}_h^p,$
and a plastic multiplier Λ :	$[0, T] \longrightarrow \Lambda$

satisfying

- ▶ the equilibrium equations,

$$\int_{\Omega} \boldsymbol{\sigma} : D\mathbf{v} \, d\mathbf{x} = \ell(\cdot, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}),$$

$$\mu_c L_c^2 \int_{\Omega} D\bar{\mathbf{A}} \cdot D\bar{\mathbf{B}} \, d\mathbf{x} = \mu_c \int_{\Omega} (\text{skew}(D\mathbf{u}) - \bar{\mathbf{A}}) : \bar{\mathbf{B}} \, d\mathbf{x}, \quad \bar{\mathbf{B}} \in \mathbf{W}_h(\mathbf{0})$$

- ▶ the essential boundary conditions,
- ▶ the constitutive relation,
- ▶ the complementary conditions (Kuhn-Tucker),
- ▶ and the flow rule.

Discretization in time

The model of incremental infinitesimal plasticity is obtained by a decomposition $0 = t_0 < t_1 < \dots < t_N = T$ of the time interval and backward Euler scheme. For $n = 1, 2, 3, \dots$ the next increment depends on the material history described by ε_p^{n-1} , the new load $\ell^n[\mathbf{v}] = \ell(t_n, \mathbf{v})$, and the new Dirichlet boundary values $\mathbf{u}_D^n = \mathbf{u}_D(t_n)$ and $\bar{A}_D^n = \bar{A}_D(t_n)$.

We compute the displacement vector $\mathbf{u}^n \in \mathbf{V}_h(\mathbf{u}_D^n)$, the stresses $\boldsymbol{\sigma}^n \in \boldsymbol{\Sigma}_h$, the micro-rotations $\bar{A} \in W_h(\bar{A}_D^n)$, the plastic strains $\varepsilon_p^n \in \mathbf{E}_h^n$, and the plastic multiplier $\Lambda^n \in \boldsymbol{\Lambda}$ satisfying additionally the discrete flow-rule:

$$\frac{1}{t_n - t_{n-1}} \left(\varepsilon_p^n(\boldsymbol{\xi}) - \varepsilon_p^{n-1}(\boldsymbol{\xi}) \right) = \Lambda^n(\boldsymbol{\xi}) D\phi(T_E^n(\boldsymbol{\xi})) , \quad \boldsymbol{\xi} \in \Xi_h ,$$

depending on $T_E^n(\boldsymbol{\xi}) = 2\mu(\text{sym}(D\mathbf{u}^n(\boldsymbol{\xi})) - \varepsilon_p^n(\boldsymbol{\xi}))$.

Since the problem is rate-independent, we define $\gamma^n = (t_n - t_{n-1})\Lambda^n \in \boldsymbol{\Lambda}$.

Fully discrete elasto-plastic problem

Together, we can state the fully discrete elasto-plastic Cosserat problem.

For given $\varepsilon_p^{n-1} \in \mathbf{E}_h^p$ find $\sigma^n, T_E^n \in \Sigma_h$, $\mathbf{u}^n \in \mathbf{V}_h(\mathbf{u}_D^n)$, $\bar{A} \in W_h(\bar{A}_D^n)$ and $\gamma^n \in \Lambda$ such that

$$T_E^n(\xi) = 2\mu \left(\text{sym}(D\mathbf{u}^n(\xi)) - \varepsilon_p^{n-1}(\xi) - \gamma^n(\xi) D\phi(T_E^n(\xi)) \right), \quad \xi \in \Xi_h,$$

$$\phi(T_E^n(\xi)) \leq 0, \quad \gamma^n(\xi) \phi(T_E^n(\xi)) = 0, \quad \gamma^n(\xi) \geq 0, \quad \xi \in \Xi_h,$$

$$\sigma^n(\xi) = T_E^n(\xi) + \lambda \text{div}(\mathbf{u}^n)(\xi) \mathbf{I} + 2\mu_c (\text{skew}(D\mathbf{u}^n(\xi)) - \bar{A}^n(\xi)), \quad \xi \in \Xi_h,$$

$$\int_{\Omega} \sigma^n : D\mathbf{v} \, d\mathbf{x} = \ell^n[\mathbf{v}], \quad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}),$$

$$\mu L_c^2 \int_{\Omega} D\bar{A}^n \cdot D\bar{B} \, d\mathbf{x} = \mu_c \int_{\Omega} (\text{skew}(D\mathbf{u}^n) - \bar{A}^n) : \bar{B} \, d\mathbf{x}, \quad \bar{B} \in W_h(\mathbf{0}).$$

Discrete formulation of the Elasto-Plastic Model

Lemma:

The fully discrete elasto-plastic problem is equivalent to the following nonlinear variational problem. For given ε_p^{n-1} find $(\mathbf{u}^n, \bar{\mathbf{A}}^n) \in \mathbf{V}_h(\mathbf{u}_D^n) \times W_h(\bar{\mathbf{A}}_D^n)$ such that

$$\begin{aligned} \int_{\Omega} P_K(2\mu(\text{sym}(D\mathbf{u}^n) - \varepsilon_p^{n-1})) : D\mathbf{v} \, d\mathbf{x} + \lambda \int_{\Omega} \text{div}(\mathbf{u}^n) \text{div}(\mathbf{v}) \, d\mathbf{x} \\ + 2\mu_c \int_{\Omega} (\text{skew}(D\mathbf{u}^n) - \bar{\mathbf{A}}^n) : D\mathbf{v} \, d\mathbf{x} = \ell^n[\mathbf{v}], \quad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}), \end{aligned}$$

$$\mu L_c^2 \int_{\Omega} D\bar{\mathbf{A}}^n \cdot D\bar{\mathbf{B}} \, d\mathbf{x} = \mu_c \int_{\Omega} (\text{skew}(D\mathbf{u}^n) - \bar{\mathbf{A}}^n) : \bar{\mathbf{B}} \, d\mathbf{x}, \quad \bar{\mathbf{B}} \in W_h(\mathbf{0}).$$

Variational Formulation of the discrete problem

Lemma:

Any minimizer $(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V}_h(\mathbf{u}_D^n) \times W_h(\bar{A}_D^n)$ of the functional

$$\mathcal{I}_{\text{incr}}^n(\mathbf{u}, \bar{A}) = \mathcal{E}_{\text{incr}}(\varepsilon(\mathbf{u}), \bar{A}, \varepsilon_p^{n-1}) - \ell_n[\mathbf{u}]$$

solves the nonlinear variational update problem. Here $\mathcal{E}_{\text{incr}}$ denotes the free energy of the incremental update problem defined by

$$\begin{aligned} \mathcal{E}_{\text{incr}}(D\mathbf{u}, \bar{A}, \varepsilon_p) &= \frac{1}{2\mu} \int_{\Omega} \psi_{\mathbf{K}}(2\mu(\text{sym}(D\mathbf{u}) - \varepsilon_p)) \, d\mathbf{x} + \frac{\lambda}{2} \int_{\Omega} \text{tr}(D\mathbf{u})^2 \, d\mathbf{x} \\ &\quad + \mu_c \int_{\Omega} |\text{skew}(D\mathbf{u}) - \bar{A}|^2 \, d\mathbf{x} + \mu L_c^2 \int_{\Omega} |D\bar{A}|^2 \, d\mathbf{x}, \end{aligned}$$

$$\psi_{\mathbf{K}}(\boldsymbol{\theta}) = \begin{cases} \frac{1}{2} |\boldsymbol{\theta}|^2 & |\text{dev}(\boldsymbol{\theta})| \leq K_0, \\ \frac{1}{2} \left(\frac{1}{d} \text{tr}(\boldsymbol{\theta})^2 + 2K_0 |\text{dev}(\boldsymbol{\theta})| - K_0^2 \right) & |\text{dev}(\boldsymbol{\theta})| > K_0. \end{cases}$$

If $\varepsilon_p^{n-1} = 0$ and $\mu_c = 0 \rightarrow$ classical Hencky-Problem.

The Dual Formulation

Lemma:

Let $\varepsilon_p^{n-1} \in L_2(\Omega, \text{Sym}(3))$ with $\text{tr}(\varepsilon_p^{n-1}) = 0$ be given.

Then, $(\sigma^n, \bar{A}^n) \in L_2(\Omega, \mathbf{R}^{3,3}) \times W$ is uniquely determined by minimizing the dual functional

$$\begin{aligned} \mathcal{D}_{\text{incr}}^n(\sigma, \bar{A}) &= \int_{\Omega} \sigma : \varepsilon_p^{n-1} d\mathbf{x} + \frac{1}{4\mu} \int_{\Omega} |\text{dev}(\text{sym}(\sigma))|^2 d\mathbf{x} \\ &+ \frac{1}{6(2\mu + 3\lambda)} \int_{\Omega} \text{tr}(\sigma)^2 d\mathbf{x} + \frac{1}{4\mu_c} \int_{\Omega} |\text{skew}(\sigma)|^2 d\mathbf{x} + \mu L_c^2 \int_{\Omega} |D\bar{A}|^2 d\mathbf{x} . \end{aligned}$$

subject to the plastic constraint $\phi(\text{sym}(\sigma^n)) \leq 0$ and the equilibrium constraint

$$\begin{aligned} \int_{\Omega} \sigma^n : D\mathbf{v} d\mathbf{x} &= \ell^n[\mathbf{v}], \quad \mathbf{v} \in \mathbf{V} \\ 2\mu L_c^2 \int_{\Omega} D\bar{A}^n \cdot D\bar{B} d\mathbf{x} &= \int_{\Omega} \text{skew}(\sigma^n) : \bar{B} d\mathbf{x}, \quad \bar{B} \in W . \end{aligned}$$

The Dual Formulation

For the limit case $\mu_c = 0$, we have to assume that the convex set

$$\mathbf{K}^n = \left\{ \boldsymbol{\tau} \in L^2(\Omega, \text{Sym}(3)) : \boldsymbol{\tau} \in \mathbf{K} \text{ a. e. in } \Omega, \int_{\Omega} \boldsymbol{\tau} : D\mathbf{v} \, d\mathbf{x} = \ell^n[\mathbf{v}], \mathbf{v} \in \mathbf{V} \right\}$$

is not empty (weak safe-load assumption).

Lemma:

If $\boldsymbol{\eta}^n \in \mathbf{K}^n$ exists, we have

$$\| \text{skew}(\boldsymbol{\sigma}^n) \|_{L^2(\Omega)}^2 \leq 4\mu_c \mathcal{D}_{\text{incr}}^n(\boldsymbol{\eta}^n, \mathbf{0})$$

and

$$\| D\bar{A}^n \|_{L^2(\Omega)}^2 \leq \frac{C_1^2}{\mu^2 L_c^4} \mu_c \mathcal{D}_{\text{incr}}^n(\boldsymbol{\eta}^n, \mathbf{0}) .$$

Theorem:

If $\boldsymbol{\eta}^n \in \mathbf{K}^n$ exists, we have

$$\lim_{\mu_c \rightarrow 0} \| \text{sym}(\boldsymbol{\sigma}_{\mu_c}^n) - \boldsymbol{\sigma}_0^n \|_{L^2(\Omega)} = 0 .$$

The FEM convergence

Theorem:

For pure dirichlet data on $\mathbf{u}$ we have.

$$\|\mathbf{v}\|_{H^1(\Omega)}^2 \leq \bar{C} \left(\|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)}^2 + \|\operatorname{curl} \mathbf{v}\|_{L^2(\Omega)}^2 \right), \quad \mathbf{v} \in \mathbf{V}.$$

Thus,

$$\|(\mathbf{u} - \mathbf{u}_h, \bar{A} - \bar{A}_h)\|_{\mathbf{V} \times W} \leq \frac{C}{\mu_C} \inf_{(\mathbf{v}_h, \bar{B}_h) \in \mathbf{V}_h \times W_h} \|(\mathbf{u} - \mathbf{v}_h, \bar{A} - \bar{B}_h)\|_{\mathbf{V} \times W}.$$

C is independent of $\mu_C \in (0, \mu]$.

Remark :

For $\mu_C > 0$ we observe $(\mathbf{u}_{\mu_C}, \bar{A}_{\mu_C}) \in H^2(\Omega)$ (Neff/Knees), thus

$$\|(\mathbf{u}_{\mu_C} - \mathbf{u}_{\mu_C}^h, \bar{A}_{\mu_C} - \bar{A}_{\mu_C}^h)\|_{H^1(\Omega)} \leq \frac{C}{\mu_C} h \|(\mathbf{u}_{\mu_C}, \bar{A}_{\mu_C})\|_{H^2(\Omega)}$$

for isoparametric Q1 elements.

Balance μ_C with h !

Numerical Solution Algorithm

We formulate a semi-smooth Newton method for the nonlinear variational problem

$$(\mathbf{u}^n, \bar{\mathbf{A}}^n) \in \mathbf{V}_h(\mathbf{u}_D^n) \times W_h(\bar{\mathbf{A}}_D^n): \quad F^n(\mathbf{u}^n, \bar{\mathbf{A}}^n) = 0$$

in every time step n , where F^n is the first variation of $\mathcal{I}_{\text{incr}}^n$ defined by

$$F^n(\mathbf{u}, \bar{\mathbf{A}})[\mathbf{v}, \bar{\mathbf{B}}] = D\mathcal{I}_{\text{incr}}^n(\mathbf{u}, \bar{\mathbf{A}})[\mathbf{v}, \bar{\mathbf{B}}], \quad (\mathbf{v}, \bar{\mathbf{B}}) \in \mathbf{V}_h(\mathbf{0}) \times W_h(\mathbf{0}).$$

The functional F^n is semi-smooth, and the second variation $\partial^2 \mathcal{I}_{\text{incr}}^n = \partial F^n$ is multi-valued. Thus, the corresponding semi-smooth Newton method can be formally written as

$$0 \in F^n(\mathbf{u}^{n,k}, \bar{\mathbf{A}}^{n,k}) + \partial F^n(\mathbf{u}^{n,k}, \bar{\mathbf{A}}^{n,k})[\mathbf{u}^{n,k+1} - \mathbf{u}^{n,k}, \bar{\mathbf{A}}^{n,k+1} - \bar{\mathbf{A}}^{n,k}].$$

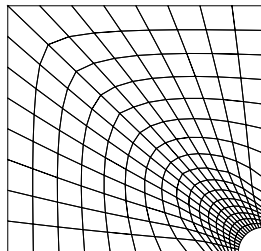
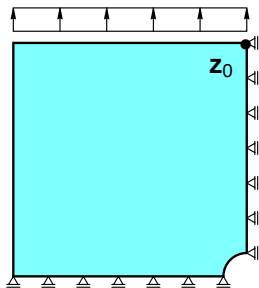
We consider the special case of the von Mises flow rule. The Newton increment is realized using the consistent linearization $\mathbf{C}(\boldsymbol{\theta})$.

Benchmark problem for parameter study of μ_c

Let $\Omega = (0, 10) \times (0, 10) \setminus B_1(10, 0)$. We use Q1 discretization and present results for 198147 unknowns on uniform refinement level 4. We have chosen the parameters $K_0 = 450$, $\lambda \approx 1.1 \cdot 10^6$, $\mu \approx 8.0 \cdot 10^5$ and $L_c \approx 0.02$.

And apply traction force by Neumann boundary condition according to:

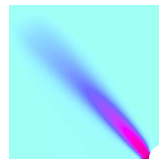
$$\ell(t, \mathbf{v}) = 100t \int_0^{10} \mathbf{v}(x_1, 10) dx_1.$$



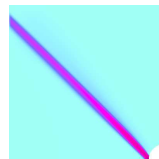
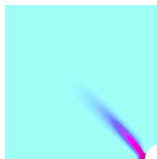
Geometry, boundary conditions and coarse mesh.

Numerical Experiment with M++

Cosserat Model ($\mu_c = \mu$) : Effective plastic strain



Prandtl-Reuß ($\mu_c = 0$) : Effective plastic strain



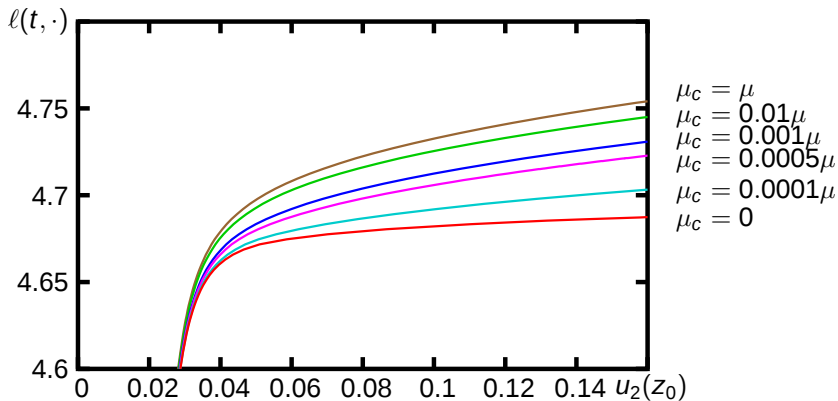
$t = 4.00$

$t = 4.40$

$t = 4.69$

no difference for $t < 4.50$

Numerical Experiment with M++



Load-displacement curve: displacement $\mathbf{u}$ is evaluated at special point $\mathbf{z}_0$.

Summary and Outlook

- ▶ The Elasto-Plastic Cosserat Model with pure Dirichlet data is well-posed. Solution exists globally in time.
- ▶ This Elasto-Plastic Cosserat Model is a regularization for classical perfect plasticity (shear failure mechanisms).
- ▶ Complete finite element analysis (depending on μ_c) is available: to appear in Math. Meth. Mod. Appl. Sci.
- ▶ Compared to gradient plasticity: no change of plasticity formulation needed! Classical variational concepts carry over.
- ▶ Future work will be the analysis and implementation of geometrically nonlinear elasto-plastic Cosserat Models.
- ▶ And a comparison with implementations of higher gradient plasticity models.
- ▶ μ_c measures rotational inhomogeneity
- ▶ no Cosserat effects in uniaxial tests (unlike hardening or viscosity)