

Well-Posedness of Dynamic Cosserat Plasticity*

Patrizio Neff¹ and Krzysztof Chelmiński²

¹Department of Mathematics, University of Technology Darmstadt,
64289 Darmstadt, Germany
neff@mathematik.tu-darmstadt.de

²Faculty of Mathematics and Information Science, Warsaw University of Technology,
00661 Warsaw, Poland
kchelmin@mini.pw.edu.pl

Abstract. We investigate the regularizing properties of generalized continua of micropolar type for dynamic elasto-plasticity. To this end we propose an extension of classical infinitesimal elasto-plasticity to include consistently non-dissipative micropolar effects and we show that the dynamic model allows for a unique, global in-time solution of the corresponding rate-independent initial boundary value problem of pure Dirichlet-type. The methods of choice are the Yosida approximation and a passage to the limit.

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1. Introduction

This article is a sequel to work begun in [31]. There we have established the regularizing power and well-posedness of a **geometrically linear Cosserat model** [7] in conjunction with quasistatic rate-independent elasto-plasticity. In this contribution we extend these results to cover also the dynamic case for the pure Dirichlet problem.

Classical ideal plasticity is known to be an ill-posed problem. In the quasistatic case the displacement solution is neither unique (even in the class of continuous functions) nor smooth in space (slip lines), see the counterexample on p. 77 of [25] based on [39] and [40] and the discussion of regularity in [41], [27], [23], [42], [17], [3], [16] and [4].

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In the dynamic case, existence and uniqueness have been established for the stresses and displacements, see [1] and p. 133 of [25], but the regularity in space remains open: the displacement gradient may be a measure due to the (possible) occurrence of shear bands. Therefore the need for regularizing procedures is also apparent for dynamics. We use the Cosserat approach in this respect.

For the physical relevance of the Cosserat model we refer to the introduction in [31]. Readers may also consult [12], [11] and [6] for the general elastic micropolar approach or [36], [28], [5], [8], [9], [26] and [32] for its application to elasto-plasticity. Recently, Cosserat elasto-plasticity has been studied in [22], [33]–[35], [37], [21], [14] and references therein. The first author has proposed an alternative extension of Cosserat models to finite-strain in [29] and [30]. Typical applications of Cosserat elasto-plastic models are the plasticity of polycrystalline metal where the crystal grains can in principle rotate against each other. These rotations are captured by an additional continuum field of rotations.

In the geometrically linear case micropolar models are characterized by an additional independent field of (infinitesimal) microrotations $\bar{A} \in \mathfrak{so}(3)$, coupled to the displacement u . These new degrees of freedom introduce in a natural way length scale effects into the model which are a convenient way to regularize non-wellposed situations without compromising the physical relevance of the model.

In order to be sufficiently self-contained we recapitulate briefly the static elastic micropolar model and its quasistatic elasto-plastic extension as treated in [31]. The quasistatic model is then extended in a straightforward manner to include the dynamic effects for both the standard displacement u and the new microrotation \bar{A} by writing an appropriate Lagrangian function.

Subsequently, we mathematically study the dynamic rate-independent case obtained and show, by means of the Yosida approximation and a passage to the limit, that the **rate-independent problem without hardening** admits a unique, global in-time solution for displacements and microrotations in standard Sobolev spaces under fairly mild assumptions on the data. The notation is found in the Appendix.

2. The Cosserat Model

2.1. The Infinitesimal Elastic Cosserat Model

We begin by recalling the infinitesimal Cosserat approach. First, in the purely elastic case, an infinitesimal Cosserat theory can be obtained by introducing the additive decomposition of the macroscopic displacement gradient ∇u into infinitesimal **microrotation** $\bar{A} \in \mathfrak{so}(3)$ (infinitesimal Cosserat rotation tensor) and infinitesimal **micropolar stretch tensor** (or first Cosserat deformation tensor) $\bar{\varepsilon} \in \mathbb{M}^{3 \times 3}$ with

$$\nabla u = \bar{\varepsilon} + \bar{A}, \quad (1)$$

where $\bar{\varepsilon}$ is **not necessarily symmetric**, such that (1) is in general not the decomposition of ∇u into infinitesimal continuum stretch $\text{sym}(\nabla u)$ and infinitesimal continuum rotation $\text{skew}(\nabla u)$.

In the quasistatic case the Cosserat theory is then obtained from a variational principle [35, p. 51], [38] for the infinitesimal displacement $u: [0, T] \times \bar{\Omega} \mapsto \mathbb{R}^3$ and the

independent infinitesimal microrotation \bar{A} : $\bar{\Omega} \mapsto \mathfrak{so}(3)$:

$$\begin{aligned} \mathcal{E}(u, \bar{A}) &= \int_{\Omega} \{W(\nabla u, \bar{A}, D_x \bar{A}) - \langle f, u \rangle - \langle M, \bar{A} \rangle\} dx \\ &\quad - \int_{\Gamma_S} \langle N, u \rangle ds - \int_{\Gamma_C} \langle M_c, \bar{A} \rangle ds \mapsto \min. \text{ w.r.t. } (u, \bar{A}), \end{aligned} \quad (2)$$

$$\bar{A}|_{\Gamma} = \bar{A}_d, \quad u|_{\Gamma} = u_d(t, x).$$

Here W represents the elastic energy density and $\Omega \subset \mathbb{R}^3$ will always be a bounded domain with smooth boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary where Dirichlet conditions u_d, \bar{A}_d for infinitesimal displacements and rotations, respectively, are prescribed, while $\Gamma_S \subset \partial\Omega$ is a part of the boundary where traction boundary conditions N are applied with $\Gamma \cap \Gamma_S = \emptyset$. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where external surface couples M_c are applied with $\Gamma \cap \Gamma_C = \emptyset$. The classical volume force is denoted by f and the additional volume couple by M . Variation of the action \mathcal{E} with respect to u yields the equation for linearized balance of linear momentum and variation of \mathcal{E} with respect to \bar{A} yields the linearized version of balance of angular momentum.

It remains to specify the analytic form of the energy density W . A linearized version of material frame-indifference implies the reduction $W(\nabla u, \bar{A}, D_x \bar{A}) = W(\bar{\varepsilon}, D_x \bar{A})$, and for infinitesimal displacements u and small curvature $D_x \bar{A}$ a quadratic ansatz is appropriate: $W(\bar{\varepsilon}, D_x \bar{A}) = W_{\text{mp}}^{\text{infin}}(\bar{\varepsilon}) + W_{\text{curv}}^{\text{small}}(D_x \bar{A})$ with an additive decomposition of the energy density into microstretch $\bar{\varepsilon}$ and curvature parts. In the isotropic case it is standard to assume for the stretch energy

$$W_{\text{mp}}^{\text{infin}}(\bar{\varepsilon}) = \mu \|\text{sym } \nabla u\|^2 + \mu_c \|\text{skew}(\nabla u) - \bar{A}\|^2 + \frac{\lambda}{2} \text{tr}[\text{sym}(\nabla u)]^2, \quad (3)$$

where the **Cosserat couple modulus** $\mu_c \geq 0$ [MPa] is an additional parameter, complementing the two Lamè constants $\mu, \lambda > 0$ [MPa]. For the curvature term we assume

$$W_{\text{curv}}^{\text{small}}(D_x \bar{A}) = \mu \frac{L_c^2}{2} (\alpha_5 \|\text{sym } D_x \bar{A}\|^2 + \alpha_6 \|\text{skew } D_x \bar{A}\|^2 + \alpha_7 \text{tr}[D_x \bar{A}]^2), \quad (4)$$

where $\|D_x A\|^2$ means the sum of all first partial derivatives of A squared by abuse of notation. Here, $L_c > 0$ with units of length introduces a specific **internal characteristic length** into the elastic formulation. In general, one assumes $\alpha_5 > 0, \alpha_6, \alpha_7 \geq 0$.

We observe that if $\mu_c = 0$, the infinitesimal minimization problem (3) completely decouples—the infinitesimal microrotations \bar{A} have no influence at all on the macroscopic behaviour of the infinitesimal displacements and classical infinitesimal elasticity results.¹ In the limit of zero internal length scale $L_c = 0$ and for

¹ Note that $\text{axl } \bar{A} \times \xi = \bar{A} \cdot \xi$ for all $\xi \in \mathbb{R}^3$, such that

$$\text{axl} \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \bar{A}_{ij} = \varepsilon_{ijk} \cdot \text{axl}(\bar{A})_k, \quad (5)$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\bar{A} \cdot \xi$ denotes the application of the matrix \bar{A} to the vector ξ and $a \times b$ is the usual cross-product. This induces the **canonical identification** of skew-symmetric matrices $\mathfrak{so}(3)$ with \mathbb{R}^3 .

$\mu_c > 0$,² the balance of angular momentum reads

$$D_{\bar{A}} W_{\text{mp}}(\nabla u, \bar{A}) \in \text{Sym} \Leftrightarrow D_{\bar{A}} W_{\text{mp}}(\nabla u, \bar{A}) = 0, \quad (6)$$

and implies already that infinitesimal continuum rotations and infinitesimal microrotations coincide: $\text{skew}(\nabla u) = \bar{A}$, and this in turn is equivalent to the symmetry of the infinitesimal Cauchy stress σ or the so-called **Boltzmann axiom**.

If we consider $\mu_c > 0$, it is standard to prove that the corresponding minimization problem admits a unique minimizing pair $(u, \bar{A}) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathfrak{so}(3))$. Existence results of this type have been obtained, e.g. in [10], [24], [18], [19] and [31].

2.2. Non-Dissipative Extension to Micropolar Elasto-Plasticity

Now we extend the formulation of micropolar elasticity to cover infinitesimal elasto-plasticity as well. It is clear that there exist various ways of obtaining such an extension, for an overview of the competing models we refer to the instructive survey article [15]. Incidentally, the Cosserats themselves [7, p. 5] already envisaged the application of their general theory to plasticity and fracture. Without restricting generality we base the following considerations on a simplified curvature expression by setting $\alpha_5 = \alpha_6 = 1$, $\alpha_7 = 0$.

The idea of a **non-dissipative** extension is simple. Consider the additive decomposition of the total micropolar stretch into elastic and plastic parts,

$$\bar{\varepsilon} = \bar{\varepsilon}_e + \bar{\varepsilon}_p, \quad (7)$$

and assume that microrotational effects remain purely elastic: $\bar{A}_e := \bar{A}$, i.e. no corresponding split of \bar{A} in the elastic and plastic parts is assumed. Now we formally replace $\bar{\varepsilon}$ in (3) with $\bar{\varepsilon}_e$ which yields (note that $\|D_x \bar{A}_e\|^2 = 2\|\nabla \text{axl}(\bar{A}_e)\|_{\mathbb{M}^{3 \times 3}}^2$)

$$\begin{aligned} \mathcal{E}(\bar{\varepsilon}_e, \bar{A}_e) &= \int_{\Omega} \left\{ \mu \|\text{sym } \bar{\varepsilon}_e\|^2 + \mu_c \|\text{skew}(\bar{\varepsilon}_e)\|^2 + \frac{\lambda}{2} \text{tr} [\bar{\varepsilon}_e]^2 \right. \\ &\quad \left. + \mu L_c^2 \|\nabla \text{axl}(\bar{A}_e)\|^2 \right\} dx \\ &= \int_{\Omega} \left\{ \mu \|\varepsilon - \text{sym } \bar{\varepsilon}_p\|^2 + \mu_c \|\text{skew}(\nabla u - \bar{A}_e - \bar{\varepsilon}_p)\|^2 \right. \\ &\quad \left. + \frac{\lambda}{2} \text{tr} [\varepsilon - \bar{\varepsilon}_p]^2 + \mu L_c^2 \|\nabla \text{axl}(\bar{A}_e)\|^2 \right\} dx \end{aligned} \quad (8)$$

as the **thermodynamic potential** \mathcal{E} , where $\varepsilon = \text{sym } \nabla u$ is the symmetric part of the displacement gradient. We need to supply a consistent flow rule for $\bar{\varepsilon}_p$ (note again that

² Corresponding as well to the limit of arbitrary large samples, which can be seen by a simple scaling argument.

\bar{A}_e acts solely elastically). By choosing

$$\dot{\varepsilon}_p(t) \in \mathfrak{f}(T_E), \quad T_E := -\partial_{\bar{\varepsilon}_p} W_{\text{mp}}^{\text{infn}}(\bar{\varepsilon}_e) = \partial_{\bar{\varepsilon}_e} W_{\text{mp}}^{\text{infn}}(\bar{\varepsilon}_e), \quad \bar{\varepsilon}_e = \bar{\varepsilon} - \bar{\varepsilon}_p, \quad (9)$$

$$W_{\text{mp}}^{\text{infn}}(\bar{\varepsilon}_e) = \mu \|\text{sym } \bar{\varepsilon}_e\|^2 + \mu_c \|\text{skew}(\bar{\varepsilon}_e)\|^2 + \frac{\lambda}{2} \text{tr}[\bar{\varepsilon}_e]^2,$$

with a constitutive multifunction \mathfrak{f} such that $\langle \mathfrak{f}(\Sigma), \Sigma \rangle \geq 0, \forall \Sigma \neq 0$, the **reduced dissipation inequality**

$$\frac{d}{dt} \mathcal{E}(\varepsilon, \bar{A}_e, \bar{\varepsilon}_p) \leq 0 \quad (10)$$

at fixed in time $(\nabla u, \bar{A}_e)$ is automatically satisfied, thus ensuring the second law of thermodynamics.

We assume that the multifunction \mathfrak{f} takes **trace-free symmetric values** only, i.e. $\mathfrak{f}(T_E) \in \text{Sym}(3) \cap \mathfrak{sl}(3, \mathbb{R})$. This sets the **infinitesimal plastic spin** $\text{skew}(\bar{\varepsilon}_p)$ to **zero** and restricts attention to incompressible plasticity as in classical formulations of ideal plasticity. Since then $\bar{\varepsilon}_p \in \text{Sym}(3)$ we may identify $\bar{\varepsilon}_p = \text{sym}(\bar{\varepsilon}_p) = \varepsilon_p$, formally as in classical ideal plasticity. In [31] we have shown that the ensuing quasistatic elasto-plastic model is well-posed.

2.3. Infinitesimal Dynamic Elasto-Plastic Cosserat Model

The dynamic infinitesimal-strain system with non-dissipative Cosserat effects can be obtained by augmenting the previous strain energy with suitable inertia terms for both the displacement u and the elastic microrotation \bar{A}_e . Without loss of generality we assume henceforth the density $\varrho(x) \equiv 1$. We assume the **Lagrangian** expression

$$\begin{aligned} & \int_0^T \left\{ \int_{\Omega} \frac{1}{2} \|u_t\|^2 + 2 \|\text{axl } \bar{A}_e\|^2 + \langle f, u \rangle + \langle M, \bar{A}_e \rangle \right. \\ & \quad \left. - \left(\mu \|\varepsilon - \varepsilon_p\|^2 + \mu_c \|\text{skew}(\nabla u - \bar{A}_e)\|^2 + \frac{\lambda}{2} \text{tr}[\varepsilon]^2 + \mu L_c^2 \|\nabla \text{axl}(\bar{A}_e)\|^2 \right) dx \right. \\ & \quad \left. + \int_{\Gamma_s} \langle N, u \rangle ds + \int_{\Gamma_c} \langle M_c, \bar{A}_e \rangle ds \right\} d\tau \mapsto \text{stat. w.r.t. } (u, \bar{A}_e) \text{ at fixed } \varepsilon_p, \quad (11) \end{aligned}$$

together with the flow rule

$$\dot{\varepsilon}_p(t) \in \mathfrak{f}(T_E), \quad T_E = 2\mu(\varepsilon - \varepsilon_p), \quad (12)$$

and suitable initial and boundary values. In order to remain close to classical elasto-plasticity as far as the boundary and initial conditions for the infinitesimal microrotation \bar{A} are concerned one can assume the following coupling conditions:

$$\bar{A}(x, 0) = \text{skew } \nabla u(x, 0) = \text{skew } \nabla u^0(x),$$

$$\bar{A}_t(x, 0) = \text{skew } \nabla u_t(x, 0) = \text{skew } \nabla u^1(x),$$

ensuring symmetry of the initial Cauchy stresses σ . In the following we do, however, consider the more general situation of decoupled boundary and initial conditions.

The **corresponding system of dynamic partial differential equations** coupled with the flow rule is given by (use that $\|\bar{A}_e\|_{\mathbb{M}^{3 \times 3}}^2 = 2\|\text{axl}(\bar{A}_e)\|_{\mathbb{R}^3}^2$ for $\bar{A}_e \in \mathfrak{so}(3)$)

$$\begin{aligned}
\text{Div } \sigma &= u_{tt} - f, & x \in \Omega, \\
\sigma &= 2\mu(\varepsilon - \varepsilon_p) + 2\mu_c(\text{skew}(\nabla u) - \bar{A}_e) + \lambda \text{tr}[\varepsilon] \cdot \mathbb{1}, \\
\text{axl}(\bar{A}_{e,t}) - \mu \frac{L_c^2}{2} \Delta \text{axl}(\bar{A}_e) &= \mu_c \text{axl}(\text{skew}(\nabla u) - \bar{A}_e) + \frac{1}{2} \text{axl}(\text{skew}(M)), \\
\dot{\varepsilon}_p(t) &\in \mathfrak{f}(T_E), & T_E &= 2\mu(\varepsilon - \varepsilon_p), \\
u|_{\Gamma}(t, x) &= u_d(t, x), & \bar{A}_e|_{\Gamma} &= \bar{A}_d(t, x)|_{\Gamma}, \\
u(x, 0) &= u^0(x), & \dot{u}(x, 0) &= u^1(x), \\
\bar{A}_e(x, 0) &= A^0(x), & \dot{\bar{A}}_e(x, 0) &= A^1(x), \\
\varepsilon_p(0) &= \varepsilon_p^0, \\
\sigma \cdot \bar{n}|_{\Gamma_S}(t, x) &= N, & \sigma \cdot \bar{n}|_{\partial\Omega \setminus (\Gamma \cup \Gamma_S)}(t, x) &= 0, \\
\mu L_c^2 \nabla \text{axl}(\bar{A}_e) \cdot \bar{n}|_{\Gamma_C}(t, x) &= \text{axl}(\text{skew}(M_c)), \\
\mu L_c^2 \nabla \text{axl}(\bar{A}_e) \cdot \bar{n}|_{\partial\Omega \setminus (\Gamma \cup \Gamma_C)}(t, x) &= 0, \\
\text{tr}[\varepsilon_p(0)] &= 0, & \varepsilon_p(0) &\in \text{Sym}(3).
\end{aligned} \tag{13}$$

3. Mathematical Analysis of the Dynamic Model

For brevity of notation, in this part we write A instead of \bar{A}_e and l_c instead of the positive constant $\mu(L_c^2/2)$. Moreover, we study pure Dirichlet boundary conditions, i.e. $\Gamma = \partial\Omega$. Second time derivatives are written as \ddot{u} . Thus we consider the well-posedness of the following non-linear initial boundary-value problem:

$$\begin{aligned}
\ddot{u} - \text{Div } \sigma &= f, \\
\sigma &= 2\mu(\varepsilon - \varepsilon_p) + 2\mu_c(\text{skew}(\nabla u) - A) + \lambda \text{tr}[\varepsilon] \cdot \mathbb{1}, \\
\text{axl}(\ddot{A}) - l_c \Delta \text{axl}(A) &= \mu_c \text{axl}(\text{skew}(\nabla u) - A) + g, \\
\dot{\varepsilon}_p &\in \mathfrak{f}(T_E), & T_E &= 2\mu(\varepsilon - \varepsilon_p), \\
u|_{\partial\Omega} &= u_d, & A|_{\partial\Omega} &= A_d, \\
u(0) &= u^0, & \dot{u}(0) &= u^1, \quad A(0) = A^0, & \dot{A}(0) &= A^1, & \varepsilon_p(0) &= \varepsilon_p^0,
\end{aligned} \tag{14}$$

where f, g are the given volume force and volume couple, u_d, A_d are the given boundary data and $u^0, u^1, A^0, A^1, \varepsilon_p^0$ are the given initial data. Moreover, we assume that the inelastic constitutive multifunction $\mathfrak{f}: D(\mathfrak{f}) \subset \text{Sym}(3) \rightarrow \mathcal{P}(\text{Sym}(3))$ is a **maximal monotone** mapping [2, Definition 1, p. 140] satisfying $0 \in \mathfrak{f}(0)$. Here, for any set X the symbol $\mathcal{P}(X)$ denotes the family of all subsets of X . The monotonicity assumption for \mathfrak{f} yields that the considered model is thermodynamical admissible. Note that the flow function corresponding to classical ideal plasticity possesses the same properties.

To prove that system (14) possesses global in time L^2 -solutions we approximate the flow function \mathbf{f} by single-valued, global Lipschitz functions \mathbf{f}_η , called in the literature the **Yosida approximation** (see for example Theorem 2, p. 144 of [2]). Thus, we first consider system (14) with \mathbf{f}_η instead of \mathbf{f} and try to pass to the limit $\eta \rightarrow 0^+$. Following this idea, for all $\eta > 0$ we study the approximated initial boundary-value problem in the form

$$\begin{aligned}
\ddot{u}^\eta - \text{Div } \sigma^\eta &= f, \\
\sigma^\eta &= 2\mu(\varepsilon^\eta - \varepsilon_p^\eta) + 2\mu_c(\text{skew}(\nabla u^\eta) - A^\eta) + \lambda \text{tr}[\varepsilon^\eta] \cdot \mathbb{1}, \\
\text{axl}(\ddot{A}^\eta) - l_c \Delta \text{axl}(A^\eta) &= -\mu_c \text{axl}(A^\eta) + \mu_c \text{axl}(\text{skew}(\nabla u^\eta)) + g, \\
\dot{\varepsilon}_p^\eta &= \mathbf{f}_\eta(T_E^\eta), \quad T_E^\eta = 2\mu(\varepsilon^\eta - \varepsilon_p^\eta), \\
u_{|\partial\Omega}^\eta &= u_d, \quad A_{|\partial\Omega}^\eta = A_d, \\
u^\eta(0) &= u^0, \quad \dot{u}^\eta(0) = u^1, \quad A^\eta(0) = A^0, \quad \dot{A}^\eta(0) = A^1, \quad \varepsilon_p^\eta(0) = \varepsilon_p^0.
\end{aligned} \tag{15}$$

The system (15) contains only global Lipschitz non-linearities, hence using the standard fixed-point method we obtain the following existence and uniqueness result:

Theorem 3.1 (Global Existence and Uniqueness for Approximated Problem). *Let us assume that the given data possess the following regularity: for all times $T > 0$,*

$$\begin{aligned}
f &\in C^1([0, T], L^2(\Omega, \mathbb{R}^3)), \quad g \in C^1([0, T], L^2(\Omega, \mathfrak{so}(3))), \\
u_d &\in C^1([0, T], H^{1/2}(\partial\Omega, \mathbb{R}^3)), \quad A_d \in C^1([0, T], H^{3/2}(\partial\Omega, \mathfrak{so}(3)))
\end{aligned}$$

and the initial data satisfy

$$\begin{aligned}
u^0, u^1 &\in H^1(\Omega, \mathbb{R}^3), \quad A^0 \in H^2(\Omega, \mathfrak{so}(3)), \quad A^1 \in H^1(\Omega, \mathfrak{so}(3)), \\
\varepsilon_p^0 &\in L^2(\Omega, \text{Sym}(3)).
\end{aligned}$$

Moreover, suppose that the following compatibility condition holds:

$$\begin{aligned}
u^0(x) &= u_d(x, 0), \quad u^1(x) = \dot{u}_d(x, 0), \\
A^0(x) &= A_d(x, 0), \quad A^1(x) = \dot{A}_d(x, 0) \quad \text{for } x \in \partial\Omega.
\end{aligned}$$

Then the approximated problem has a global in time unique solution $(u^\eta, \varepsilon_p^\eta, A^\eta)$ with the regularity

$$\begin{aligned}
u^\eta &\in C^1([0, T], H^1(\Omega, \mathbb{R}^3)), \quad \ddot{u}^\eta \in C([0, T], L^2(\Omega, \mathbb{R}^3)), \\
\varepsilon_p^\eta &\in C^1([0, T], L^2(\Omega, \text{Sym}(3))), \\
A^\eta &\in C([0, T], H^2(\Omega, \mathfrak{so}(3))), \quad \ddot{A}^\eta \in C([0, T], L^2(\Omega, \mathfrak{so}(3))).
\end{aligned}$$

Proof. The proof is a standard application of the Banach Fixed-Point Theorem and can therefore be omitted. Note that the H^2 regularity of the microrotations follows from the

regularity of the initial data, the smoothness of $\partial\Omega$ and the linearity of the hyperbolic system of equations (compare with Theorem 5, p. 389 of [13]. For similar results the reader may consult [25]. \square

Next, we are going to obtain some estimates for the approximated sequence $(u^\eta, \varepsilon_p^\eta, A^\eta)$. To do this we use the **energy** associated with the dynamic problem (not the Lagrangian) which is defined by

$$\begin{aligned} \mathcal{E}(u, \varepsilon, \varepsilon_p, A)(t) := & \int_{\Omega} \left(\frac{1}{2} \|\dot{u}\|^2 + 2 \|\text{axl}(\dot{A})\|^2 + \mu \|\varepsilon - \varepsilon_p\|^2 + \frac{\lambda}{2} \text{tr}[\varepsilon]^2 \right. \\ & \left. + \mu_c \|\text{skew}(\nabla u) - A\|^2 + 2l_c \|\nabla \text{axl}(A)\|^2 \right) dx. \end{aligned}$$

For $\lambda > 0$ (strictly positive Poisson ratio, satisfied for all metals) the energy function is **elastically coercive** by which we mean that

$$\mathcal{E}(u, \varepsilon, \varepsilon_p, A) + C_d \geq C_E (\|u\|_{H^1(\Omega \times (0, T))}^2 + \|A\|_{H^1(\Omega \times (0, T))}^2), \quad (16)$$

where the constant C_E does not depend on u and A and the constant C_d depends on boundary data of u and A only. The proof of this important property is based on the fact that the operators curl and Div together control the total gradient, see p. 36 of [20], i.e. the inequality

$$\begin{aligned} \exists C > 0, \quad \forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^3), \\ \int_{\Omega} \|\text{curl} \varphi(x)\|_{\mathbb{R}^3}^2 + (\text{Div} \varphi(x))^2 dx \geq C \|\varphi\|_{H^{1,2}(\Omega, \mathbb{R}^3)}^2, \end{aligned} \quad (17)$$

holds for smooth functions with compact support $C_0^\infty(\Omega, \mathbb{R}^3)$. Observe that $(\text{Div} u)^2 = \text{tr}[\varepsilon]^2$ and $\|\text{curl} u\|_{\mathbb{R}^3}^2 = 4 \|\text{axl} \text{skew}(\nabla u)\|^2$.

We denote by v_d the time derivative of u_d and by B_d the time derivative of A_d . In contrast to the quasistatic case we first prove energy estimates for the time derivatives and from this result we conclude the boundedness of the energy on finite time intervals.

Theorem 3.2 (Energy Estimate for Time Derivatives). *Suppose that the given data possess more time regularity than in the last theorem and satisfy additionally: for all times $T > 0$,*

$$\ddot{v}_d \in L^2((0, T); H^{1/2}(\partial\Omega, \mathbb{R}^3)), \quad \ddot{B}_d \in L^2((0, T); H^{1/2}(\partial\Omega, \text{so}(3))). \quad (18)$$

Moreover, assume that the initial data $u^0, u^1, A^0, A^1, \varepsilon_p^0$ have the regularity required in Theorem 3.1 and assume that the initial value of the reduced Eshelby tensor $T_E(0) = 2\mu (\frac{1}{2}(\nabla u^0 + \nabla^T u^0) - \varepsilon_p^0)$ belongs to the domain of the maximal monotone operator \mathfrak{f} . Then there exists a positive constant $C(T)$, independent of η , such that

$$\mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) \leq C(T) \quad \text{for all } t \in [0, T].$$

Proof. For $h > 0$ we denote by $(u_h^\eta(t), \varepsilon_h^\eta(t), \varepsilon_{p,h}^\eta(t), A_h^\eta(t))$ the shifted functions $(u^\eta(t+h), \varepsilon^\eta(t+h), \varepsilon_p^\eta(t+h), A^\eta(t+h))$ and calculate the energy evaluated on the

differences $(u_h^\eta - u^\eta, \dots)$. Then for the time derivative we have

$$\begin{aligned}
& \dot{\mathcal{E}}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta)(t) \\
&= \int_{\Omega} \langle \dot{u}_h^\eta - \dot{u}^\eta, \dot{u}_h^\eta - \dot{u}^\eta \rangle \, dx \\
&\quad + 4 \int_{\Omega} \langle \text{axl}(\dot{A}_h^\eta - \dot{A}^\eta), \text{axl}(\ddot{A}_h^\eta - \ddot{A}^\eta) \rangle \, dx \\
&\quad + \int_{\Omega} 2\mu \langle \varepsilon_h^\eta - \varepsilon^\eta - \varepsilon_{p,h}^\eta + \varepsilon_p^\eta, \dot{\varepsilon}_h^\eta - \dot{\varepsilon}^\eta - \dot{\varepsilon}_{p,h}^\eta + \dot{\varepsilon}_p^\eta \rangle \, dx \\
&\quad + 2\mu_c \int_{\Omega} \langle \text{skew}(\nabla u_h^\eta - \nabla u^\eta) - A_h^\eta + A^\eta, \text{skew}(\nabla \dot{u}_h^\eta - \nabla \dot{u}^\eta) - \dot{A}_h^\eta + \dot{A}^\eta \rangle \, dx \\
&\quad + \lambda \int_{\Omega} \text{tr}[\varepsilon_h^\eta - \varepsilon^\eta] \text{tr}[\dot{\varepsilon}_h^\eta - \dot{\varepsilon}^\eta] \, dx \\
&\quad + 4l_c \int_{\Omega} \langle \nabla \text{axl}(A_h^\eta - A^\eta), \nabla \text{axl}(\dot{A}_h^\eta - \dot{A}^\eta) \rangle \, dx \\
&= - \int_{\Omega} \langle T_{E,h}^\eta - T_E^\eta, \dot{\varepsilon}_{p,h}^\eta - \dot{\varepsilon}_p^\eta \rangle \, dx + \int_{\Omega} \langle \sigma_h^\eta - \sigma^\eta, \nabla \dot{u}_h^\eta - \nabla \dot{u}^\eta \rangle \, dx \\
&\quad + \int_{\Omega} \langle \dot{u}_h^\eta - \dot{u}^\eta, \dot{u}_h^\eta - \dot{u}^\eta \rangle \, dx + 4 \int_{\Omega} \langle \text{axl}(\dot{A}_h^\eta - \dot{A}^\eta), \text{axl}(\ddot{A}_h^\eta - \ddot{A}^\eta) \rangle \, dx \\
&\quad - 4\mu_c \int_{\Omega} \langle \text{axl} \text{skew}(\nabla u_h^\eta - \nabla u^\eta) - \text{axl}(A_h^\eta - A^\eta), \text{axl}(\dot{A}_h^\eta - \dot{A}^\eta) \rangle \, dx \\
&\quad + 4l_c \int_{\Omega} \langle \nabla \text{axl}(A_h^\eta - A^\eta), \nabla \text{axl}(\dot{A}_h^\eta - \dot{A}^\eta) \rangle \, dx, \tag{19}
\end{aligned}$$

where $T_{E,h}^\eta(t) = T_E^\eta(t+h)$ and $\sigma_h^\eta(t) = \sigma^\eta(t+h)$. Using the monotonicity of \mathbf{f}_η we have that the first integral on the right-hand side of (19) is non-positive. Next, we integrate by parts in the second and in the last integral and use the equation of motion and the equation for microrotations. Hence, we conclude that

$$\begin{aligned}
& \dot{\mathcal{E}}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta)(t) \\
&\leq \int_{\Omega} \langle f_h - f, \dot{u}_h^\eta - \dot{u}^\eta \rangle \, dx + 4 \int_{\Omega} \langle g_h - g, \text{axl} \dot{A}_h^\eta - \text{axl} \dot{A}^\eta \rangle \, dx \\
&\quad + \int_{\partial\Omega} \langle (\sigma_h^\eta - \sigma^\eta) \cdot n, \dot{u}_{d,h} - \dot{u}_d \rangle \, ds \\
&\quad + 4l_c \int_{\partial\Omega} \langle \nabla \text{axl}(A_h^\eta - A^\eta) \cdot n, \text{axl}(\dot{A}_{d,h} - \dot{A}_d) \rangle \, ds, \tag{20}
\end{aligned}$$

where $f_h(t) = f(t+h)$, $g_h(t) = g(t+h)$, $u_{d,h}(t) = u_d(t+h)$ and $A_{d,h}(t) = A_d(t+h)$.

Next, we integrate (20) in time and obtain

$$\begin{aligned}
& \mathcal{E}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta)(t) \\
& \leq \mathcal{E}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta)(0) \\
& \quad + \int_0^t \int_\Omega \langle f_h - f, \dot{u}_h^\eta - \dot{u}^\eta \rangle \, dx \, d\tau + 4 \int_0^t \int_\Omega \langle g_h - g, \text{axl } \dot{A}_h^\eta - \text{axl } \dot{A}^\eta \rangle \, dx \, d\tau \\
& \quad + \int_0^t \int_{\partial\Omega} \langle (\sigma_h^\eta - \sigma^\eta) \cdot n, \dot{u}_{d,h} - \dot{u}_d \rangle \, ds \, d\tau \\
& \quad + 4l_c \int_0^t \int_{\partial\Omega} \langle \nabla \text{axl}(A_h^\eta - A^\eta) \cdot n, \text{axl}(\dot{A}_{d,h} - \dot{A}_d) \rangle \, ds \, d\tau.
\end{aligned}$$

In the last two integrals on the right-hand side of (21) we shift the difference operator onto the given data. Next, we divide by h^2 and pass to the limit $h \rightarrow 0^+$. Hence, we arrive at the following inequality:

$$\begin{aligned}
& \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) \\
& \leq \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(0) + \int_0^t \int_\Omega \langle \dot{f}, \ddot{u}^\eta \rangle \, dx \, d\tau + 4 \int_0^t \int_\Omega \langle \dot{g}, \text{axl } \ddot{A}^\eta \rangle \, dx \, d\tau \\
& \quad - \int_0^t \int_{\partial\Omega} \langle \sigma^\eta \cdot n, \ddot{v}_d \rangle \, ds \, d\tau - \int_{\partial\Omega} \langle \sigma^\eta(0) \cdot n, \dot{v}_d(0) \rangle \, ds \\
& \quad + \int_{\partial\Omega} \langle \sigma^\eta(t) \cdot n, \dot{v}_d(t) \rangle \, ds \\
& \quad - 4l_c \int_0^t \int_{\partial\Omega} \langle \nabla \text{axl}(A^\eta) \cdot n, \text{axl}(\ddot{B}_d) \rangle \, ds \, d\tau \\
& \quad - 4l_c \int_{\partial\Omega} \langle \nabla \text{axl}(A^\eta)(0) \cdot n, \text{axl}(\dot{B}_d)(0) \rangle \, ds \\
& \quad + 4l_c \int_{\partial\Omega} \langle \nabla \text{axl}(A^\eta)(t) \cdot n, \text{axl}(\dot{B}_d)(t) \rangle \, ds. \tag{21}
\end{aligned}$$

The boundedness of the initial energy for time derivatives follows from the assumption $T_E(0) \in \mathcal{D}(\mathbf{f})$. This implies that the sequence $\{\mathbf{f}_\eta(T_E(0))\}$ is bounded in $L^2(\Omega, \text{Sym}(3))$. Next, we estimate all integral terms from the right-hand side of (21):

$$\begin{aligned}
\left| \int_0^t \int_\Omega \langle \dot{f}, \ddot{u}^\eta \rangle \, dx \, d\tau \right| & \leq \int_0^t \|\dot{f}\|_{L^2} \|\ddot{u}^\eta\|_{L^2} \, d\tau \\
& \leq \frac{1}{2} \int_0^t \|\dot{f}\|_{L^2}^2 \, d\tau + \int_0^t \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(\tau) \, d\tau. \tag{22}
\end{aligned}$$

In the same manner we obtain

$$\left| \int_0^t \int_\Omega \langle \dot{g}, \text{axl } \ddot{A}^\eta \rangle \, dx \, d\tau \right| \leq \frac{1}{2} \int_0^t \|\dot{g}\|_{L^2}^2 \, d\tau + \frac{1}{4} \int_0^t \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(\tau) \, d\tau. \tag{23}$$

To estimate the appearing boundary integrals we use the trace theorem in the space $L^2(\text{Div})$:

$$\begin{aligned}
\left| \int_0^t \int_{\partial\Omega} \langle \sigma^\eta \cdot n, \ddot{v}_d \rangle \, ds \, d\tau \right| &\leq \int_0^t \|\sigma^\eta \cdot n\|_{H^{-1/2}} \|\ddot{v}_d\|_{H^{1/2}} \, d\tau \\
&\leq C \int_0^t (\|\sigma^\eta\|_{L^2} + \|\text{Div} \sigma^\eta\|_{L^2}) \|\ddot{v}_d\|_{H^{1/2}} \, d\tau \\
&\leq C \int_0^t (\|\dot{\sigma}^\eta\|_{L^2} + \|\sigma^\eta(0)\|_{L^2}) \|\ddot{v}_d\|_{H^{1/2}} \, d\tau \\
&\quad + C \int_0^t (\|f\|_{L^2} + \|\ddot{u}^\eta\|_{L^2}) \|\ddot{v}_d\|_{H^{1/2}} \, d\tau \\
&\leq C \int_0^t \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(\tau) \, d\tau + C_1, \tag{24}
\end{aligned}$$

where the positive constant C does not depend on η and the constant C_1 depends on given data only. For the next boundary term we obtain

$$\begin{aligned}
\left| \int_{\partial\Omega} \langle \sigma^\eta(t) \cdot n, \dot{v}_d(t) \rangle \, ds \right| &\leq \|\sigma^\eta \cdot n\|_{H^{-1/2}} \|\dot{v}_d\|_{H^{1/2}} \\
&\leq C (\|\sigma^\eta\|_{L^2} + \|\text{Div} \sigma^\eta\|_{L^2}) \|\dot{v}_d\|_{H^{1/2}} \\
&\leq C (\|\dot{\sigma}^\eta\|_{L^2} + \|\sigma^\eta(0)\|_{L^2} + \|f\|_{L^2} + \|\ddot{u}^\eta\|_{L^2}) \|\dot{v}_d\|_{H^{1/2}} \\
&\leq C \mathcal{E}^{1/2}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) + D, \tag{25}
\end{aligned}$$

where the positive constants C, D depend on given data only. The boundary integrals containing microrotations are estimated using the same idea. Note that by the evolution equation for microrotations, using the H^2 -regularity, we have $\text{Div} \nabla \text{axl}(A^\eta) = l_c^{-1} \text{axl}(\dot{A}^\eta) - \mu_c l_c^{-1} \text{axl}(\text{skew}(\nabla u) - A) - l_c^{-1} g$ and the first two terms on the right-hand side of the last equality appear in the energy function. Hence, we conclude that

$$\begin{aligned}
\left| \int_0^t \int_{\partial\Omega} \langle \nabla \text{axl}(A^\eta) \cdot n, \text{axl}(\dot{B}_d) \rangle \, ds \, d\tau \right| + \left| \int_{\partial\Omega} \langle \nabla \text{axl}(A^\eta)(t) \cdot n, \text{axl}(\dot{B}_d)(t) \rangle \, ds \right| \\
\leq C \int_0^t \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(\tau) \, d\tau + C_1 \mathcal{E}^{1/2}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) + C_2, \tag{26}
\end{aligned}$$

where all positive constants C, C_1, C_2 do not depend on η . Inserting (22), (23), (24), (25) and (26) into (21) we finally arrive at the following inequality:

$$\begin{aligned}
&\mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) \\
&\leq C_1 \mathcal{E}^{1/2}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) + C_2 \int_0^t \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(\tau) \, d\tau + C_3,
\end{aligned}$$

where C_1, C_2, C_3 do not depend on η . This inequality immediately implies that

$$\mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(t) \leq D_1 \int_0^t \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta)(\tau) \, d\tau + D_2,$$

where again the constants D_1, D_2 do not depend on η . Gronwall's inequality completes the proof. \square

The energy estimate for time derivatives yields that the sequence $(\dot{\sigma}^\eta, \nabla \dot{A}^\eta, \ddot{u}^\eta, \ddot{A}^\eta)$ is $L^\infty(L^2)$ -bounded. This implies that the sequence $(\sigma^\eta, \nabla A^\eta, \dot{u}^\eta, \dot{A}^\eta)$ is also $L^\infty(L^2)$ -bounded. Note that for example the equality $\sigma^\eta(t) = \int_0^t \dot{\sigma}^\eta(\tau) d\tau + \sigma^\eta(0)$ implies that $\|\sigma^\eta\|_{L^2} \leq \int_0^t \|\dot{\sigma}^\eta\|_{L^2} + \|\sigma^\eta(0)\|_{L^2}$. Moreover, by the coercivity of the energy (16) we have that the sequences $\{\dot{\varepsilon}^\eta\}$ and $\{\dot{\varepsilon}_p^\eta\}$ are $L^\infty(L^2)$ -bounded. Hence, for a subsequence (again denoted using the superscript η) we have: for all $T > 0$,

$$\begin{aligned} \sigma^\eta &\overset{*}{\rightharpoonup} \sigma && \text{in } L^\infty((0, T), L^2(\Omega, \text{Sym}(3))), \\ \dot{\sigma}^\eta &\overset{*}{\rightharpoonup} \dot{\sigma} && \text{in } L^\infty((0, T), L^2(\Omega, \text{Sym}(3))), \\ A^\eta &\overset{*}{\rightharpoonup} A && \text{in } L^\infty((0, T), H^1(\Omega, \mathfrak{so}(3))), \\ \dot{A}^\eta &\overset{*}{\rightharpoonup} \dot{A} && \text{in } L^\infty((0, T), L^2(\Omega, \mathfrak{so}(3))), \\ u^\eta &\overset{*}{\rightharpoonup} u && \text{in } L^\infty((0, T), H^1(\Omega, \mathbb{R}^3)), \\ \ddot{u}^\eta &\overset{*}{\rightharpoonup} \ddot{u} && \text{in } L^\infty((0, T), L^2(\Omega, \mathbb{R}^3)), \\ \varepsilon^\eta &\overset{*}{\rightharpoonup} \varepsilon && \text{in } L^\infty((0, T), L^2(\Omega, \text{Sym}(3))), \\ \dot{\varepsilon}^\eta &\overset{*}{\rightharpoonup} \dot{\varepsilon} && \text{in } L^\infty((0, T), L^2(\Omega, \text{Sym}(3))), \\ \varepsilon_p^\eta &\overset{*}{\rightharpoonup} \varepsilon_p && \text{in } L^\infty((0, T), L^2(\Omega, \text{Sym}(3))), \\ \dot{\varepsilon}_p^\eta &\overset{*}{\rightharpoonup} \dot{\varepsilon}_p && \text{in } L^\infty((0, T), L^2(\Omega, \text{Sym}(3))) \end{aligned}$$

and the limit functions satisfy

$$\begin{aligned} \ddot{u} - \text{Div } \sigma &= f, \\ \sigma &= 2\mu(\varepsilon - \varepsilon_p) + 2\mu_c(\text{skew}(\nabla u) - A) + \lambda \text{tr}[\varepsilon] \cdot \mathbb{1}, \\ \text{axl}(\ddot{A}) - l_c \Delta \text{axl}(A) &= \mu_c \text{axl}(\text{skew}(\nabla u) - A) + g, \\ \dot{\varepsilon}_p &= \mathring{f}_0 = \text{weak-lim } \mathring{f}_\eta(T_E^\eta), \quad T_E = 2\mu(\varepsilon - \varepsilon_p), \\ u|_{\partial\Omega} &= u_d, \quad A|_{\partial\Omega} = A_d, \\ u(0) &= u^0, \quad \dot{u}(0) = u^1, \quad A(0) = A^0, \quad \dot{A}(0) = A^1, \quad \varepsilon_p(0) = \varepsilon_p^0. \end{aligned} \tag{27}$$

To finish the existence theory for our system we need only prove that

$$\mathring{f}_0(t, x) \in \mathring{f}(T_E(t, x)) \quad \text{a.e. in } (0, T) \times \Omega. \tag{28}$$

To do this we follow the standard idea which is based on the following property: the graph of a maximal monotone operator is weakly-strongly closed. Thus, we are going to improve the weak convergence of the sequence $\{T_E^\eta\}$.

Theorem 3.3 (Strong Convergence of Stresses). *Let us assume that the given data satisfy all requirements of Theorem 3.2. Then $\mathcal{E}(u^\eta - u^v, \varepsilon^\eta - \varepsilon^v, \varepsilon_p^\eta - \varepsilon_p^v, A^\eta - A^v)(t) \rightarrow 0$ for $\eta, v \rightarrow 0^+$ uniformly on bounded time intervals.*

Proof. We use the standard energy method and calculate the time derivative of the energy evaluated on the differences of two approximation steps. Hence, we obtain

$$\begin{aligned}
& \dot{\mathcal{E}}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) \\
&= \int_{\Omega} \langle \dot{u}^\eta - \dot{u}^\nu, \ddot{u}^\eta - \ddot{u}^\nu \rangle \, dx + 4 \int_{\Omega} \langle \text{axl}(\dot{A}^\eta - \dot{A}^\nu), \text{axl}(\ddot{A}^\eta - \ddot{A}^\nu) \rangle \, dx \\
&\quad + 2\mu \int_{\Omega} \langle \varepsilon^\eta - \varepsilon^\nu - \varepsilon_p^\eta + \varepsilon_p^\nu, \dot{\varepsilon}^\eta - \dot{\varepsilon}^\nu - \dot{\varepsilon}_p^\eta + \dot{\varepsilon}_p^\nu \rangle \, dx \\
&\quad + \lambda \int_{\Omega} \text{tr}[\varepsilon^\eta - \varepsilon^\nu] \text{tr}[\dot{\varepsilon}^\eta - \dot{\varepsilon}^\nu] \, dx \\
&\quad + 4l_c \int_{\Omega} \langle \nabla \text{axl}(A^\eta - A^\nu), \nabla \text{axl}(\dot{A}^\eta - \dot{A}^\nu) \rangle \, dx \\
&\quad + 2\mu_c \int_{\Omega} \langle \text{skew}(\nabla u^\eta - \nabla u^\nu) - A^\eta + A^\nu, \text{skew}(\nabla \dot{u}^\eta - \nabla \dot{u}^\nu) - \dot{A}^\eta + \dot{A}^\nu \rangle \, dx.
\end{aligned}$$

Using that the given data for both approximation steps are the same we conclude that

$$\dot{\mathcal{E}}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) = - \int_{\Omega} \langle T_E^\eta - T_E^\nu, \mathfrak{f}_\eta(T_E^\eta) - \mathfrak{f}_\nu(T_E^\nu) \rangle \, dx. \quad (29)$$

Next, to estimate the right-hand side of (29), we use the standard procedure from the theory of maximal monotone operators (compare with the proof of Theorem 1 on p. 147 of [2]). This yields that

$$\dot{\mathcal{E}}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu)(t) \leq (\eta + \nu)C(T),$$

where the positive constant $C(T)$ does not depend on η and ν . The last inequality immediately completes the proof. \square

Theorem 3.3 implies that the sequence of stresses $\{T_E^\eta\}$ is a Cauchy sequence in the space $L^\infty((0, T); L^2(\Omega; \text{Sym}(3)))$. Hence, $\{T_E^\eta\}$ converges strongly to T_E . Moreover, by the definition of the Yosida approximation we have $\mathfrak{f}_\eta(T_E^\eta) \in \mathfrak{f}(J_\eta(T_E^\eta))$, where $J_\eta(T_E^\eta) = T_E^\eta - \eta \mathfrak{f}_\eta(T_E^\eta)$ is the resolvent operator. We see that J_η is a global Lipschitz operator and therefore the sequence $\{J_\eta(T_E^\eta)\}$ converges strongly to T_E . Consequently, the sequence $(J_\eta(T_E^\eta), \mathfrak{f}_\eta(T_E^\eta))$ is contained in the graph of the maximal monotone operator \mathfrak{f} and converges strongly-weakly to (T_E, \mathfrak{f}_0) . Hence, the maximality of \mathfrak{f} yields that \mathfrak{f}_0 belongs to the set $\mathfrak{f}(T_E)$ and the limit functions $(u, \varepsilon, \varepsilon_p, A)$ satisfy (14). This finishes the existence part (of Theorem 3.5). \square

Next, we study the uniqueness of solutions for system (14).

Theorem 3.4 (Uniqueness of Solutions). *Let us assume that the given data $f, u_d, A_d, \varepsilon_p^0$ satisfy all requirements of Theorem 3.2. Then the system (14) possesses a unique global in time solution $(u, \varepsilon, \varepsilon_p, A)$.*

Proof. The proof is based on the energy method. Assume that $(u^1, \varepsilon^1, \varepsilon_p^1, A^1)$ and $(u^2, \varepsilon^2, \varepsilon_p^2, A^2)$ are two solutions of (14) for the same given data. Then for the energy function evaluated on differences of these solutions we have

$$\begin{aligned}
& \dot{\mathcal{E}}(u^1 - u^2, \varepsilon^1 - \varepsilon^2, \varepsilon_p^1 - \varepsilon_p^2, A^1 - A^2)(t) \\
&= \int_{\Omega} \langle \dot{u}^1 - \dot{u}^2, \ddot{u}^1 - \ddot{u}^2 \rangle \, dx \\
&\quad + 4 \int_{\Omega} \langle \text{axl}(\dot{A}^1 - \dot{A}^2), \text{axl}(\ddot{A}^1 - \ddot{A}^2) \rangle \, dx \\
&\quad + 2\mu \int_{\Omega} \langle \varepsilon^1 - \varepsilon^2 - \varepsilon_p^1 + \varepsilon_p^2, \dot{\varepsilon}^1 - \dot{\varepsilon}^2 - \dot{\varepsilon}_p^1 + \dot{\varepsilon}_p^2 \rangle \, dx \\
&\quad + \lambda \int_{\Omega} \text{tr}[\varepsilon^\eta - \varepsilon^\nu] \text{tr}[\dot{\varepsilon}^\eta - \dot{\varepsilon}^\nu] \, dx \\
&\quad + 4l_c \int_{\Omega} \langle \nabla \text{axl}(A^1 - A^2), \nabla \text{axl}(\dot{A}^1 - \dot{A}^2) \rangle \, dx \\
&\quad + 2\mu_c \int_{\Omega} \langle \text{skew}(\nabla u^1 - \nabla u^2) - A^1 + A^2, \text{skew}(\nabla \dot{u}^1 - \nabla \dot{u}^2) - \dot{A}^1 + \dot{A}^2 \rangle \, dx \\
&= - \int_{\Omega} \langle T_E^1 - T_E^2, \dot{\varepsilon}_p^1 - \dot{\varepsilon}_p^2 \rangle \, dx \leq 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \mathcal{E}(u^1 - u^2, \varepsilon^1 - \varepsilon^2, \varepsilon_p^1 - \varepsilon_p^2, A^1 - A^2)(t) \\
&\leq \mathcal{E}(u^1 - u^2, \varepsilon^1 - \varepsilon^2, \varepsilon_p^1 - \varepsilon_p^2, A^1 - A^2)(0) = 0
\end{aligned}$$

and the statement is a consequence of the coerciveness of the energy function. \square

At the end of this section we formulate the existence and uniqueness theorem, which we have proved:

Theorem 3.5 (Existence for the Dynamical Model). *Suppose that the given data f , g , u_d , A_d satisfy: for all times $T > 0$,*

$$\begin{aligned}
& f \in C^1([0, T], L^2(\Omega, \mathbb{R}^3)), \quad g \in C^1([0, T], L^2(\Omega, \mathbb{R}^3)), \\
& u_d \in C^2([0, T], H^{1/2}(\partial\Omega, \mathbb{R}^3)), \quad \ddot{v}_d \in L^2((0, T); H^{1/2}(\partial\Omega, \mathbb{R}^3)), \\
& A_d \in C^2([0, T], H^{\frac{3}{2}}(\partial\Omega, \mathfrak{so}(3))), \quad \ddot{B}_d \in L^2((0, T); H^{1/2}(\partial\Omega, \mathfrak{so}(3))),
\end{aligned}$$

where $v_d = \dot{u}_d$ and $B_d = \dot{A}_d$. Moreover, assume that the initial data have the regularity

$$\begin{aligned}
& u^0, u^1 \in H^1(\Omega, \mathbb{R}^3), \quad A^0 \in H^2(\Omega, \mathfrak{so}(3)), \\
& A^1 \in H^1(\Omega, \mathfrak{so}(3)), \quad \varepsilon_p^0 \in L^2(\Omega, \text{Sym}(3))
\end{aligned}$$

and satisfy the compatibility condition

$$\begin{aligned} u^0(x) &= u_d(x, 0), & u^1(x) &= \dot{u}_d(x, 0), \\ A^0(x) &= A_d(x, 0), & A^1(x) &= \dot{A}_d(x, 0) \quad \text{for } x \in \partial\Omega. \end{aligned}$$

Additionally, suppose that the initial data is chosen such that the initial value of the reduced Eshelby tensor $T_E(0) = 2\mu (\frac{1}{2}(\nabla u^0 + \nabla^T u^0) - \varepsilon_p^0)$ belongs to the domain of the maximal monotone operator \mathfrak{f} . Then the system (14) possesses a unique global in time solution $(u, \varepsilon, \varepsilon_p, A)$ with the regularity: for all times $T > 0$,

$$\begin{aligned} u &\in H^{1,\infty}((0, T), H^1(\Omega, \mathbb{R}^3)), & \ddot{u} &\in L^\infty((0, T), L^2(\Omega, \mathbb{R}^3)), \\ A &\in L^\infty((0, T), H^2(\Omega, \mathfrak{so}(3))), & \ddot{A} &\in L^\infty((0, T), L^2(\Omega, \mathfrak{so}(3))), \\ \varepsilon, \varepsilon_p &\in H^{1,\infty}((0, T), L^2(\Omega, \text{Sym}(3))). \end{aligned}$$

Appendix. Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing two-dimensional Hausdorff measure. We denote by $\mathbb{M}^{3 \times 3}$ the set of real 3×3 second-order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3 \times 3}$ is given by $\langle X, Y \rangle_{\mathbb{M}^{3 \times 3}} = \text{tr}[XY^T]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{M}^{3 \times 3}}$ (we use these symbols indifferently for tensors and vectors). The identity tensor on $\mathbb{M}^{3 \times 3}$ will be denoted by $\mathbb{1}$, so that $\text{tr}[X] = \langle X, \mathbb{1} \rangle$. We let Sym and Psym denote the symmetric and positive definite symmetric tensors, respectively. We adopt the usual abbreviations of Lie-algebra theory, i.e. $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3 \times 3} | X^T = -X\}$ are skew symmetric second-order tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3 \times 3} | \text{tr}[X] = 0\}$ are traceless tensors. We set $\text{sym}(X) = \frac{1}{2}(X^T + X)$ and $\text{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \text{sym}(X) + \text{skew}(X)$. For $X \in \mathbb{M}^{3 \times 3}$ we set for the deviatoric part $\text{dev } X = X - \frac{1}{3}\text{tr}[X]\mathbb{1} \in \mathfrak{sl}(3)$. For a second-order tensor X we let $X \cdot e_i$ be the application of the tensor X to the column vector e_i and we define the third-order tensor $\mathfrak{h} = D_x X(x) = (\nabla(X(x) \cdot e_1), \nabla(X(x) \cdot e_2), \nabla(X(x) \cdot e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in (\mathbb{M}^{3 \times 3})^3$. For \mathfrak{h} we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$ together with $\text{sym}(\mathfrak{h}) := (\text{sym } \mathfrak{h}^1, \text{sym } \mathfrak{h}^2, \text{sym } \mathfrak{h}^3)$ and $\text{tr}[\mathfrak{h}] := (\text{tr}[\mathfrak{h}^1], \text{tr}[\mathfrak{h}^2], \text{tr}[\mathfrak{h}^3]) \in \mathbb{R}^3$. The first and second differentials of a scalar-valued function $W(F)$ are written $D_F W(F) \cdot H$ and $D_F^2 W(F) \cdot (H, H)$, respectively. Sometimes we also use $\partial_X W(X)$ to denote the first derivative of W with respect to X . We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega)$, $H^{1,2}(\Omega)$, $H_\circ^{1,2}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions.

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