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Chain Rings and Prime Ideals

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Introduction. Non-commutative generalizations of (commutative) valuation rings play a role in several areas in geometry and algebra (see for example [2], [3], [6], [7]). These rings are right (left) *chain rings* which means that their lattice of right (left) ideals is linearly ordered.

A ring R is said to be *invariant* or a *duo-ring* if $aR = Ra$ holds for all a in R. If $aR \subseteqq Ra$ or $Ra \subseteqq aR$ holds for all a in R, then we say R is *semi-invariant*.

In this paper we will be concerned mainly with the existence of certain chain rings, with questions related with the invariance and semi-invariance of these rings and we will prove as one of the results (Th. 3.6):

A semi-invariant chain ring is invariant provided it satisfies d.c.c. for prime ideals.

This result indicates the importance of the structure of the lattice of the prime ideals of a chain ring.

Examples of non-invariant chain rings without zero divisors and n ($n \geqq 3$) prime ideals will be given.

1. Notations, Definitions and Preliminary Results. All rings considered have a unit element. We write $J(R)$ for the *Jacobson radical* and $U(R)$ for the set of *units* for a ring R. If R/P is a prime ring for a two-sided ideal $P \neq R$ we say P is a prime ideal; if R/P has no zero divisors P is called a completely prime ideal.

An element x in R is called a left (right) zero divisor if an element $t \neq 0$ exists with $xt = 0$ $(tx = 0)$. We write M^r (M^l) for the set $\{r \in R \mid Mr = 0\}$ $(\{r \in R \mid rM = 0\})$ for a subset M of R.

Definition 1.1. a) *A ring R is called left (right) invariant if all left (right) ideals are two-sided, or equivalently if $aR \subseteqq Ra$ $(Ra \subseteqq aR)$ holds for all a in R.*

b) *A ring R is called semi-invariant if for any a in R either $aR \subseteqq Ra$ or $Ra \subseteqq aR$ holds.*

c) *We say that a ring is invariant or a duo-ring if R is right and left invariant.*

We finally need a set of definitions generalizing valuation rings.

Definition 1.2. a) *If the lattice of right (left) ideals of a ring R is linearly ordered we say R is a right (left) chain ring. If R is a right and left chain ring we say R is a chain ring.*

b) *An invariant chain ring without zero divisors is called a valuation ring* ([7]).

c) *A chain ring in which every non unit is a left and right zero divisor is called a Hjelmslev ring* (*H-ring*, [2]).

Proposition 1.3. *A chain ring with exactly one prime ideal is an invariant H-ring* ([11]).

The principal ideals of an invariant ring form a semigroup under multiplication. The following theorem gives information about this semigroup in a special case.

Proposition 1.4 [11]. *The semigroup Π of principal ideals of a chain ring R with exactly one prime ideal is order isomorphic to one of the following linearly ordered semigroups:*

The semigroup Γ_1 of real numbers in the interval $[0, 1]$ with the usual order and $\alpha \circ \beta = \min\{\alpha + \beta, 1\}$ as operation,

or the semigroup Γ_2 of real numbers in the interval $[0, 1]$ and one element ∞ with the usual order and

$$\begin{aligned} \alpha \circ \beta &= \alpha + \beta \quad \text{for} \quad \alpha + \beta \leqq 1 \\ &= \infty \quad\quad\quad \text{for} \quad \alpha + \beta > 1 \end{aligned}$$

as operation.

Proposition 1.5. *A right noetherian right chain ring is right invariant* ([1]).

Proposition 1.6. *Let R be a chain ring with two prime ideals $J(R)$ and (0) only. If there exists a two-sided ideal I with $(0) \subsetneqq I \subsetneqq J(R)$ then R is a duo-ring.*

Proof. Proposition 1.3 shows that R/I is a duo-ring. This implies $aR = Ra$ for all $a \in R \backslash I$. Let L be the intersection of all two-sided ideals $\neq (0)$ of R. If $L = (0)$ we are done. Consider the case $L \neq (0)$. As R is prime and L is not a prime ideal, there are $x, y \notin L$ such that $xy \in L$ but $xRy \neq (0)$. As $xR = Rx$ and $yR = Ry$ by Prop. 1.3, it follows that $L = xRy = xyR = Rxy = xyR \cdot J$ holds. Nakayama's Lemma asserts $L = (0)$, a contradiction.

We will now give examples of chain rings which will be used on the one hand to illustrate our results and on the other hand to motivate questions.

3. Examples. That there exist non-invariant chain rings was probably first shown by Radó in [6]. The ring constructed there is left, but not right invariant and has infinitely many prime ideals. Other examples were given by Mathiak in [3].

We begin by restating some of the results in [3].

Lemma 2.1. *Let R be a chain ring without zero divisors, $D = Q(R)$ its skew field of quotients. Then there is a one-to-one correspondence between the set of rings T between R and D and the set of completely prime ideals of R given by*

$$P \to R_S, \text{ where } S = R \backslash P \text{ is an Ore-system,}$$
$$T \to P = R \backslash S \text{ with } S = \{s \in R \mid s \text{ is a unit in } T\}.$$

R_S is the ring of quotients of R with respect to the Ore-system S.

The proof consists of checking the correspondence given in the statement of the Lemma and will be omitted (see [3]).

A linearly ordered group G is associated with any valuation ring R. The positive cone G^+ of G is order isomorphic to the semigroup of principal ideals of R (here $aR \geqq bR$ if and only if $aR \subseteqq bR$ holds). Using generalized power series rings ([4]) it is clear that given any linearly ordered group G there exists a valuation ring R with G as its associated linearly ordered group.

A valuation ring R satisfies the hypothesis of Lemma 2.1. Every overring of R is therefore a localization at a completely prime ideal of R. We say a prime ideal P of R is invariant if $\alpha P \alpha^{-1} = P$ holds for all α in $D = Q(R)$, the skew field of quotients of R. The following result shows the importance of this notion:

Lemma 2.2. *Let R be a valuation ring with $D = Q(R)$ as its skew field of fractions, P a prime ideal in R.*

a) *The ring of quotients R_S with $S = R \backslash P$ is a semi-invariant chain ring.*

b) *R_S is again a valuation ring if and only if P is an invariant prime ideal.*

We use Lemma 2.1 to prove part a). Let α be an element in R_S. Consider $\alpha R_S \alpha^{-1} \supseteqq \alpha R \alpha^{-1} = R$. This means that both rings R_S and $\alpha R_S \alpha^{-1}$ are overrings of R. Using Lemma 2.1 we have either $\alpha R_S \alpha^{-1} \supseteqq R_S$ or $R_S \supseteqq \alpha R_S \alpha^{-1}$ and the semi-invariance follows. If P is an invariant prime ideal it follows from [7], p. 15, that R_S is again a valuation ring. If conversely R_S is a valuation ring we have $P = P \cdot R_S$ and the invariance of R_S implies $\alpha P \alpha^{-1} \subseteqq P$ for all α in D and P is invariant.

There exists a converse of part a) of Lemma 2.2 in the following sense:

Lemma 2.3. *Every semi-invariant chain ring T without zero divisors is of the form R_S for a valuation ring R and an Ore-system $S = R \backslash P$ for a prime ideal P of R.*

Proof. Consider $R = \bigcap \alpha T \alpha^{-1}$ where the intersection is taken over all $\alpha \neq 0$ in $Q(T) = D$, the skew field of fractions of T. We will show that R is a chain ring with $Q(R) = D$. The invariance of R follows by definition; and since $R \subseteqq T \subseteqq Q(T)$ is obvious, $T = R_S$ follows from Lemma 2.1. In order to prove that R is a chain ring with $Q(R) = D$ let y be an element in D, $y \notin R$. We have $y \notin \alpha T \alpha^{-1}$ for some α in D and $y^{-1} \in \alpha T \alpha^{-1}$ follows. This implies that y^{-1} is contained in every ring $\beta T \beta^{-1}$ that contains $\alpha T \alpha^{-1}$ or is contained in it. But these are the only two possibilities since T is semi-invariant and $y^{-1} \in R$ follows. We proved that R is a chain ring with $Q(R) = Q(T)$ and the Lemma follows. It is now easy to construct semi-invariant chain rings which are not invariant.

We choose a linearly ordered group G that has a non-invariant isolated subgroup H for the generalized power series ring construction mentioned before. The result is a valuation ring R with a non-invariant prime ideal R corresponding to H. Localizing R at P yields a semi-invariant chain ring which is not invariant.

Lemma 2.4. *A semi-invariant chain ring R without zero divisors and d.c.c. for prime ideals is invariant.*

Proof. Assume $xR \subsetneqq Rx$ for some x in R. This implies $R \subsetneqq x^{-1}Rx$. Since $x^{-1}Rx$ is isomorphic to R we can repeat this process and obtain a strictly ascending chain of overrings of R. This chain corresponds (Lemma 2.1) to a strictly descending chain of completely prime ideals of R and we reached a contradiction.

Even though we gave a quite general construction of non-invariant chain rings it is not clear at this point if every chain ring has to be semi-invariant. To show that this is not the case we will now construct (generalizing an idea of Stephenson, [9]) chain rings without zero divisors with exactly n ($n \geqq 3$) prime ideals which are not invariant, and hence not semi-invariant (Lemma 2.4). Let G be any abelian linearly ordered group of rank $n-2$ and H an isomorphic copy of G. We form the group ring A of the direct sum $G \oplus H$ over a (commutative) field F. The quotient field K of A contains two isomorphic valuation ring V_1 and V_2 of rank $n-2$ such that $V_1 \nsubseteq V_2 \nsubseteq V_1$ holds. Let σ be the automorphism of K obtained by extending the isomorphism between V_1 and V_2 to K.

Next we form the skew power series ring $W = K[[x, \sigma]]$ with elements $\sum_{i=0}^{\infty} x^i k_i$, $k_i \in K$, and $kx = xk^\sigma$ determining the multiplication.

Finally consider the subring

$$R = \left\{ v + \sum_{i=1}^{\infty} x^i k_i \in W \,\middle|\, v \in V_1,\ k_i \in K \right\}$$

of W consisting of those elements whose constant term is in V_1. (We assume σ maps V_1 onto V_2.) Then it follows that R is a chain ring without zero divisors, and since $V_1 \nsubseteq V_2 \nsubseteq V_1$ holds we have $Rx \nsubseteq xR \nsubseteq Rx$, i.e. R is not semi-invariant.

In order to determine the prime ideals in R let $(0) = Q_0 \subsetneqq Q_1 \subsetneqq \cdots \subsetneqq Q_{n-2} \subsetneqq$ $\subsetneqq Q_{n-1} = V_1$ be the chain of prime ideals of V_1 and define

$$P_j = \left\{ v + \sum_{i=1}^{\infty} x^i k_i \in R \,\middle|\, v \in Q_j \right\} \quad \text{for } j = 0, 1, \ldots, n-2 .$$

Every P_j is a completely prime ideal $\neq (0)$ in R.

If any prime ideal P in R is given we define

$$Q = \left\{ r \in V_1 \,\middle|\, r + \sum_{i=1}^{\infty} x^i k_i \in P \right\} .$$

If $Q \neq 0$ and $a = r + \cdots$ is an element in P with $r \neq 0$ in V_1, then $ax(r^\sigma)^{-1} = x(1 + \cdots)$ is in P. But $(1 + \cdots)$ is a unit and $P_0 \subseteqq P$ follows. The ideal $Q = P \cap V_1$ is a prime ideal in V_1 and $P = P_j$ for some $j \in \{1, \ldots, n-2\}$ follows.

We are left with the case $Q = (0)$. This implies $P \subseteqq P_0$. But $\bigcap P_0^n = (0)$ implies $P = P_0$ or $P = (0)$.

Problems. 1. We don't know if there exists a chain ring without zero divisor with exactly two prime ideals which is non-invariant.

2. We could not decide if non-invariant H-rings with exactly two prime ideals exist. This kind of existence problem was encountered by Osofsky in [5], where rings are classified whose cyclic modules have cyclic injective hulls, and appears as an unsolved problem in geometry [11].