Introduction. Non-commutative generalizations of (commutative) valuation rings play a role in several areas in geometry and algebra (see for example [2], [3], [6], [7]). These rings are right (left) chain rings which means that their lattice of right (left) ideals is linearly ordered.

A ring $R$ is said to be invariant or a duo-ring if $aR = Ra$ holds for all $a$ in $R$. If $aR \subseteq Ra$ or $Ra \subseteq aR$ holds for all $a$ in $R$, then we say $R$ is semi-invariant.

In this paper we will be concerned mainly with the existence of certain chain rings, with questions related with the invariance and semi-invariance of these rings and we will prove as one of the results (Th. 3.6):

A semi-invariant chain ring is invariant provided it satisfies d.c.c. for prime ideals.

This result indicates the importance of the structure of the lattice of the prime ideals of a chain ring.

Examples of non-invariant chain rings without zero divisors and $n$ ($n \geq 3$) prime ideals will be given.

1. Notations, Definitions and Preliminary Results. All rings considered have a unit element. We write $J(R)$ for the Jacobson radical and $U(R)$ for the set of units for a ring $R$. If $R/P$ is a prime ring for a two-sided ideal $P = R$ we say $P$ is a prime ideal; if $R/P$ has no zero divisors $P$ is called a completely prime ideal.

An element $x$ in $R$ is called a left (right) zero divisor if an element $t + 0$ exists with $xt = 0$ ($tx = 0$). We write $M^r$ ($M^l$) for the set $\{r \in R | Mr = 0\}$ ($\{r \in R | rM = 0\}$) for a subset $M$ of $R$.

Definition 1.1. a) A ring $R$ is called left (right) invariant if all left (right) ideals are two-sided, or equivalently if $aR \subseteq Ra$ ($Ra \subseteq aR$) holds for all $a$ in $R$.

b) A ring $R$ is called semi-invariant if for any $a$ in $R$ either $aR \subseteq Ra$ or $Ra \subseteq aR$ holds.

c) We say that a ring is invariant or a duo-ring if $R$ is right and left invariant.

We finally need a set of definitions generalizing valuation rings.

Definition 1.2. a) If the lattice of right (left) ideals of a ring $R$ is linearly ordered we say $R$ is a right (left) chain ring. If $R$ is a right and left chain ring we say $R$ is a chain ring.
b) An invariant chain ring without zero divisors is called a valuation ring \([17]\).

c) A chain ring in which every non unit is a left and right zero divisor is called a Hjelmslev ring (H-ring, \([2]\)).

**Proposition 1.3.** A chain ring with exactly one prime ideal is an invariant H-ring \([11]\).

The principal ideals of an invariant ring form a semigroup under multiplication. The following theorem gives information about this semigroup in a special case.

**Proposition 1.4** \([11]\). The semigroup \(\Pi\) of principal ideals of a chain ring \(R\) with exactly one prime ideal is order isomorphic to one of the following linearly ordered semigroups:

The semigroup \(\Pi_1\) of real numbers in the interval \([0, 1]\) with the usual order and \(\alpha \circ \beta = \min(\alpha + \beta, 1)\) as operation,

or the semigroup \(\Pi_2\) of real numbers in the interval \([0, 1]\) and one element \(\infty\) with the usual order and

\[
\alpha \circ \beta = \alpha + \beta \quad \text{for} \quad \alpha + \beta \leq 1
\]

\[
= \infty \quad \text{for} \quad \alpha + \beta > 1
\]

as operation.

**Proposition 1.5.** A right noetherian right chain ring is right invariant \([1]\).

**Proposition 1.6.** Let \(R\) be a chain ring with two prime ideals \(J(R)\) and \((0)\) only. If there exists a two-sided ideal \(I\) with \((0) \subseteq I \subseteq J(R)\) then \(R\) is a duo-ring.

Proof. Proposition 1.3 shows that \(R/I\) is a duo-ring. This implies \(aR = Ra\) for all \(a \in R \setminus I\). Let \(L\) be the intersection of all two-sided ideals \(\neq (0)\) of \(R\). If \(L = (0)\) we are done. Consider the case \(L \neq (0)\). As \(R\) is prime and \(L\) is not a prime ideal, there are \(x, y \notin L\) such that \(xy \in L\) but \(xRy \neq (0)\). As \(xR = Rx\) and \(yR = Ry\) by Prop. 1.3, it follows that \(L = xRy = xyR = Rxy = xyR \cdot J\) holds. Nakayama’s Lemma asserts \(L = (0)\), a contradiction.

We will now give examples of chain rings which will be used on the one hand to illustrate our results and on the other hand to motivate questions.

**3. Examples.** That there exist non-invariant chain rings was probably first shown by Radó in \([6]\). The ring constructed there is left, but not right invariant and has infinitely many prime ideals. Other examples were given by Mathiak in \([3]\).

We begin by restating some of the results in \([3]\).

**Lemma 2.1.** Let \(R\) be a chain ring without zero divisors, \(D = Q(R)\) its skew field of quotients. Then there is a one-to-one correspondence between the set of rings \(T\) between \(R\) and \(D\) and the set of completely prime ideals of \(R\) given by

\[
P \mapsto R_S, \quad \text{where} \quad S = R \setminus P \quad \text{is an Ore-system},
\]

\[
T \mapsto P = R \setminus S \quad \text{with} \quad S = \{s \in R \mid s \text{ is a unit in } T\).
\]

\(R_S\) is the ring of quotients of \(R\) with respect to the Ore-system \(S\).
The proof consists of checking the correspondence given in the statement of the Lemma and will be omitted (see [3]).

A linearly ordered group $G$ is associated with any valuation ring $R$. The positive cone $G^+$ of $G$ is order isomorphic to the semigroup of principal ideals of $R$ (here $aR \supseteq bR$ if and only if $aR \subseteq bR$ holds). Using generalized power series rings ([4]) it is clear that given any linearly ordered group $G$ there exists a valuation ring $R$ with $G$ as its associated linearly ordered group.

A valuation ring $R$ satisfies the hypothesis of Lemma 2.1. Every overring of $R$ is therefore a localization at a completely prime ideal of $R$. We say a prime ideal $P$ of $R$ is invariant if $\alpha P \alpha^{-1} \subseteq P$ holds for all $\alpha$ in $D = Q(R)$, the skew field of quotients of $R$. The following result shows the importance of this notion:

**Lemma 2.2.** Let $R$ be a valuation ring with $D = Q(R)$ as its skew field of fractions, $P$ a prime ideal in $R$.

a) The ring of quotients $R_S$ with $S = R \setminus P$ is a semi-invariant chain ring.

b) $R_S$ is again a valuation ring if and only if $P$ is an invariant prime ideal.

We use Lemma 2.1 to prove part a). Let $\alpha$ be an element in $R_S$. Consider $\alpha R_S \alpha^{-1} \supseteq \supseteq \alpha R \alpha^{-1} = R$. This means that both rings $R_S$ and $\alpha R_S \alpha^{-1}$ are overrings of $R$. Using Lemma 2.1 we have either $\alpha R_S \alpha^{-1} \supseteq R_S$ or $R_S \supseteq \alpha R_S \alpha^{-1}$ and the semi-invariance follows. If $P$ is an invariant prime ideal it follows from [7], p. 15, that $R_S$ is again a valuation ring. If conversely $R_S$ is a valuation ring we have $P = P \cdot R_S$ and the invariance of $R_S$ implies $\alpha P \alpha^{-1} \subseteq P$ for all $\alpha$ in $D$ and $P$ is invariant.

There exists a converse of part a) of Lemma 2.2 in the following sense:

**Lemma 2.3.** Every semi-invariant chain ring $T$ without zero divisors is of the form $R_S$ for a valuation ring $R$ and an Ore-system $S = R \setminus P$ for a prime ideal $P$ of $R$.

**Proof.** Consider $R = \bigcap_{\alpha \in D} \alpha T \alpha^{-1}$ where the intersection is taken over all $\alpha \neq 0$ in $Q(T) = D$, the skew field of fractions of $T$. We will show that $R$ is a chain ring with $Q(R) = D$. The invariance of $R$ follows by definition; and since $R \subseteq T \subseteq Q(T)$ is obvious, $T = R_S$ follows from Lemma 2.1. In order to prove that $R$ is a chain ring with $Q(R) = D$ let $y$ be an element in $D$, $y \notin R$. We have $y \notin \alpha T \alpha^{-1}$ for some $\alpha$ in $D$ and $y^{-1} \in \alpha T \alpha^{-1}$ follows. This implies that $y^{-1}$ is contained in every ring $\beta T \beta^{-1}$ that contains $\alpha T \alpha^{-1}$ or is contained in it. But these are the only two possibilities since $T$ is semi-invariant and $y^{-1} \in R$ follows. We proved that $R$ is a chain ring with $Q(R) = Q(T)$ and the Lemma follows. It is now easy to construct semi-invariant chain rings which are not invariant.

We choose a linearly ordered group $G$ that has a non-invariant isolated subgroup $H$ for the generalized power series ring construction mentioned before. The result is a valuation ring $R$ with a non-invariant prime ideal $R$ corresponding to $H$. Localizing $R$ at $P$ yields a semi-invariant chain ring which is not invariant.

**Lemma 2.4.** A semi-invariant chain ring $R$ without zero divisors and d.c.c. for prime ideals is invariant.
Proof. Assume \( xR \subseteq Rx \) for some \( x \) in \( R \). This implies \( R \cong x^{-1}Rx \). Since \( x^{-1}Rx \) is isomorphic to \( R \) we can repeat this process and obtain a strictly ascending chain of overrings of \( R \). This chain corresponds (Lemma 2.1) to a strictly descending chain of completely prime ideals of \( R \) and we reached a contradiction.

Even though we gave a quite general construction of non-invariant chain rings it is not clear at this point if every chain ring has to be semi-invariant. To show that this is not the case we will now construct (generalizing an idea of Stephenson, [9]) chain rings without zero divisors with exactly \( n \) \((n \geq 3)\) prime ideals which are not invariant, and hence not semi-invariant (Lemma 2.4). Let \( G \) be any abelian linearly ordered group of rank \( n - 2 \) and \( H \) an isomorphic copy of \( G \). We form the group ring \( A \) of the direct sum \( G \oplus H \) over a (commutative) field \( F \). The quotient field \( K \) of \( A \) contains two isomorphic valuation ring \( V_1 \) and \( V_2 \) of rank \( n - 2 \) such that \( V_1 \not\subseteq V_2 \subseteq V_1 \) holds. Let \( \sigma \) be the automorphism of \( K \) obtained by extending the isomorphism between \( V_1 \) and \( V_2 \) to \( K \).

Next we form the skew power series ring \( W = K[[x, \sigma]] \) with elements \( \sum_{i=0}^{\infty} x^i k_i \), \( k_i \in K \), and \( k x = x k^\sigma \) determining the multiplication.

Finally consider the subring
\[
R = \left\{ v + \sum_{i=1}^{\infty} x^i k_i \in W \mid v \in V_1, \ k_i \in K \right\}
\]
of \( W \) consisting of those elements whose constant term is in \( V_1 \). (We assume \( \sigma \) maps \( V_1 \) onto \( V_2 \).) Then it follows that \( R \) is a chain ring without zero divisors, and since \( V_1 \not\subseteq V_2 \subseteq V_1 \) holds we have \( Rx \not\subseteq xR \subseteq Rx \), i.e. \( R \) is not semi-invariant.

In order to determine the prime ideals in \( R \) let \( (0) = Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_{n-2} \subseteq Q_{n-1} = V_1 \) be the chain of prime ideals of \( V_1 \) and define
\[
P_j = \left\{ v + \sum_{i=1}^{\infty} x^i k_i \in R \mid v \in Q_j \right\}
\]
for \( j = 0, 1, \ldots, n-2 \).

Every \( P_j \) is a completely prime ideal \( \neq (0) \) in \( R \).

If any prime ideal \( P \) in \( R \) is given we define
\[
Q = \left\{ r \in V_1 \mid r + \sum_{i=1}^{\infty} x^i k_i \in P \right\}.
\]
If \( Q \neq 0 \) and \( a = r + \cdots \) is an element in \( P \) with \( r \neq 0 \) in \( V_1 \), then \( ax(r^\sigma)^{-1} = x(1 + \cdots) \) is in \( P \). But \( (1 + \cdots) \) is a unit and \( P_0 \subseteq P \) follows. The ideal \( Q = P \cap V_1 \) is a prime ideal in \( V_1 \) and \( P = P_j \) for some \( j \in \{1, \ldots, n-2\} \) follows.

We are left with the case \( Q = (0) \). This implies \( P \subseteq P_0 \). But \( \bigcap P_0 = (0) \) implies \( P = P_0 \) or \( P = (0) \).

Problems. 1. We don’t know if there exists a chain ring without zero divisor with exactly two prime ideals which is non-invariant.

2. We could not decide if non-invariant \( H \)-rings with exactly two prime ideals exist. This kind of existence problem was encountered by Osofsky in [5], where rings are classified whose cyclic modules have cyclic injective hulls, and appears as an unsolved problem in geometry [11].
3. Semi-invariant chain rings with d.c.c. for prime ideals. We will prove the following result: The rings described in the above title are invariant.

The proof of this result will consist of a series of steps which in turn provide us with some better understanding of the structure of the lattice of prime ideals of a semi-invariant chain ring.

Lemma 3.1. Let $R$ be a chain ring in which every non unit is a left zero divisor. Then $(Ra)^l = Ra$ holds for all $a$ in $R$.

Proof. Assume $a \neq 0$. We always have $Ra \subseteq (Ra)^l$. Let $y \in (Ra)^l$. We have either $ha = y$ or $hy = a$ for some element $h$ in $R$. If $hy = a$ holds, $(Rh)^r \cap yR = (0)$ follows and we have $yR = (0)$ or $(Rh)^r = (0)$. Using our assumption we conclude in the second case that $h$ is a unit and $y \in Ra$ follows.

Notation. We write $ACB$ for elements $A \subseteq B$ in some lattice $L$ if $ALCLB$ implies $C = A$ or $C = B$ for a third element $C \in L$ and say $A$ is a lower neighbor of $B$ or $B$ is an upper neighbor of $A$ in $L$.

We observe that $ACB$ holds for left ideals $A$ and $B$ in a chain ring $R$ if and only if $B$ is of the form $Ra$ and $A$ of the form $J(R)a$ for some $a \neq 0$ in $R$.

Corollary 3.2. Let $I$ be a left ideal in a chain ring $R$ in which every non unit is a left zero divisor. Then $I = I^l$ or $I \subseteq I^l$.

Proof. Let $I \subseteq I^l$. Then there exists $x \in I^l \setminus I$ and $I \subseteq Rx \subseteq I^l$ follows. We obtain $I^l \subseteq (Rx)^l = Rx$ (using 3.1) and further $Rx = I^l$ which means $I \subseteq I^l$.

Corollary 3.3. Let $P = J(R)$ be a completely prime ideal in a chain ring in which every non unit is a left zero divisor. Then $P^l = P$ holds.

Proof. Assume $P$ is lower neighbor of $P^l$. It follows that $P$ has the form $P = Ja$ for some $a$ in $J$. But with $a^2 \in P$, $a \notin P$ we reach a contradiction.

Lemma 3.4. Let $R$ be a chain ring and $P \neq J(R)$ a completely prime ideal $\neq (0)$. Then $P$ is not finitely generated neither as a left nor as a right ideal.

Proof. Assume $P = Ra$. For $x \in J(R) \setminus P$ there exists an element $s$ in $R$ with $xs = a$. This implies $s \in P$ and $s = ra$ for some $r$ in $R$. But $xs = xra = a$ implies $a = 0$ and $P = (0)$.

Lemma 3.5. A prime ideal in a semi-invariant chain ring $R$ is a completely prime ideal.

Proof. Following [10] it suffices to show that $x^2 \in P$ implies $x \in P$. If $x \notin P$ there exists an element $t$ with $txt \notin P$ and we obtain $x^2 t_1$ or $t_2 x^2$ not in $P$ according to $Rx \subseteq xR$ or $xR \subseteq Rx$ for some elements $t_1$, $t_2$ in $R$. This is of course a contradiction.

Theorem 3.6. A semi-invariant chain ring with d.c.c. for prime ideals is invariant.

Let $R$ be a semi-invariant chain ring and assume $mR \subseteq Rm$ for some $m$ in $R$. This is equivalent with the relation $mJ \subseteq Jm$. In order to show $Rm = mR$ it is sufficient.
to check if in the factor ring $R/Jm = T$ the equation $T\varphi(m) = \varphi(m)T$ holds where $\varphi$ is the canonical homomorphism from $R$ onto $T$. We observe that in $T$ every non unit is a left zero divisor.

We restrict ourselves now to semi-invariant chain rings in which the zero ideal has an upper neighbor $Rm$ in the lattice of left ideals. We say in that case:

$R$ has property $(\ast)$.

Before we can prove Theorem 3.6 we need a few more results.

**Lemma 3.7.** Let $R$ be a ring with property $(\ast)$, $P \neq J$ a prime ideal in $R$. Then we have:

a) $Jr = Rm$ and $J = (mR)^l = (Rm)^l$.

b) $P \neq (aR)^l$ for every $a$ in $R$.

**Proof.** $Rm \subseteq Jr$ is obvious. If $Jx = (0)$ holds and $x \notin Rm$, we have $Rx \supseteq Rm \supseteq (0)$ and $Rx \cap Jx \supseteq Rm \neq (0)$, a contradiction. To prove b) let $P = (aR)^l$ and $ta = m$. Then $t \in J$, (3.7a), $t \notin P$, but $t^2 \in P$, a contradiction to 3.5.

**Lemma 3.8.** Let $P \neq J$ be a prime ideal in a ring with property $(\ast)$. Then

$$Pr = \{a \in R \mid \exists t \in R \setminus P : ta = 0 \}.$$ 

**Proof.** Let $a \in Pr$. If $ta \neq 0$ for all $t \in R \setminus P = Pc$, we have $a \notin (Rt)r$ for all $t$ and $\bigcup (Rt)r \subseteq a$ follows where the union is taken over all $t$ in $Pc$. Therefore $(aR)^l \subseteq \bigcap (Rt)r = \bigcap Rt$. Since $P \neq J$ cannot be a lower neighbor we obtain $(aR)^l \subseteq \bigcap Rt = P$. Since $aR \subseteq Pr$ is given, $Pr = P \subseteq (aR)^l \subseteq P$ follows, and we have a contradiction to Lemma 3.7 b). This shows that $Pr$ is contained in

$$\{a \in R \mid \exists t \in Pc : ta = 0 \}.$$ 

The reverse inclusion is obvious.

We recall a definition from [12]: Let $I$ be a two-sided ideal in $R$. We define

$$S_t(I) = \{s \in R \mid st \in I \text{ implies } t \in I \text{ for any } t \in R\},$$

$$S_r(I) = \{s \in R \mid ts \in I \text{ implies } t \in I \text{ for any } t \in R\}.$$ 

The following result holds ([12]):

**Lemma 3.9.** Let $R$ be a chain ring and $I$ a two-sided ideal in $I$. Then $R \setminus S_t(I)$ and $R \setminus S_r(I)$ are completely prime ideals in $R$.

**Lemma 3.10.** Let $R$ be a ring with property $(\ast)$, $P$ a prime ideal in $R$. Then:

a) $Pc \subseteq S_t(Pr)$.

b) $Pr = S_r(Pr)c$.

**Proof.** We have $Pr \neq (0)$ since $Jr = Rm \neq (0)$. To prove a) let $P = J$ first. Then $Jr = Rm$ and $U(R) = Jc \subseteq S_t(Rm)$. For $P \neq J$, $s \in Pc$, $st \in Pr$ we conclude with Lemma 3.8 that an element $r$ exists in $Pc$ with $rst = 0$. But $rs$ is in $Pc$ as well and the same lemma shows that $t$ is in $Pr$. To prove b) let $0 \neq x \in S_r(Pr)$. Since $Pr \neq (0)$,
there exists an element \( 0 \neq z \in P^r \) and an element \( s \in R \) with \( sx = z \). It follows that \( s \) is in \( P^r \) and \( x \) is not in \( P^{rr} \). We therefore obtain the relation \( P^{rr} \subseteq S_r(Pr)^c \). If on the other hand \( c \) is an element in \( S_r(Pr)^c \) then an element \( t \notin P^r \) exists with \( tx \in P^r \). Let \( x \in P^r \). We finish the proof by showing that \( zx = 0 \) holds. Since \( t \notin P^r \), there exists an element \( u \) in \( R \) with \( ut = z \). If \( u \) is an element in \( P^c \) we have \( t \in P^r \) by part a) and this is a contradiction. We obtain \( u \in P \) and \( zx = u(tx) = 0 \).

**Corollary 3.11.** Let \( R \) be a ring with property \((*)\), \( P \) a prime ideal in \( R \). Then \( P^{rr} \) is a completely prime ideal in \( R \).

We are now in a position to prove Theorem 3.6.

**Proof.** Let \( R \) be ring with property \((*)\) and \( \text{d.c.c.} \) for prime ideals, i.e. a semi-invariant chain ring with \( \text{d.c.c.} \) for prime ideals. We assume as before that \((0) \) has an upper neighbor \( Rm \) in the lattice of left ideals and \( mR \subseteq Rm \) holds. We have \( J^1 \subseteq mR \), since \( x \in J^1 \), \( x \notin mR \) implies \( m = xj \) and \( m = 0 \), a contradiction. Therefore \( J^1 \subseteq mR \subseteq Rm \) and \( J^r \) and in the case \( J^1 = J^r \) the desired equation \( mR = Rm \) follows.

We are therefore left with the case \( J^1 \subseteq J^r \) and assuming this we will arrive at a contradiction.

We have \((0) \neq Rm = J^r \subseteq J \) and \((0) \neq Rm = J^r \subseteq J^{rr} \subseteq J \). But we have \( J^{rr} \subseteq J \), otherwise \( J^{rr} = J \) and \( J^1 = J^{rr} = (Rm)^r = Rm = J^r \) follows, a contradiction to our assumption.

It follows that the set \( \mathcal{E} = \{ P \in R | P \text{ prime and } P^{rr} \subseteq P \} \) of prime ideals is not empty and contains a minimal element \( Q \). We know (Cor. 3.11) that \( Q \neq Q^{rr} \) is a prime ideal in \( R \). We show that \( Q^{rr} \) itself is a member of \( \mathcal{E} \) and this contradiction proves the theorem.

If \( Q^{rrrr} \subseteq Q^{rr} \) does not hold we have \( Q^{rrrr} \supseteq Q^{rr} \) and then \( Q^{rr} \supseteq Q^{rrrr} \supseteq Q^{rrr} \). If we show that \( Q^{rr} = Q^{rr} \) is true then \( Q^r \supseteq Q^{rr} \) follows and \( Q \subseteq Q^r \subseteq Q^{rrr} \). But \( Q^{rr} \neq J \) implies \( (Q^{rr})^r = Q^{rr} \) (Cor. 3.3) and \( Q \subseteq (Q^{rr})^r = Q^{rr} \subseteq Q \) follows. The only thing left to prove is the relation \( Q^{rr} = Q^r \). We know from Cor. 3.2 that either \( Q^r = Q^{rr} \) or \( Q^r \subseteq Q^{rr} \) is true. In the second case \( Q^r \) has the form \( Q^r = Ja \). We observe further that in this case \( Q \neq J \) holds, since \( J^r = Rm \) implies \( (J^r)^r = J^r = Rm \) (using 3.7 a) and 3.1).

Using Lemma 3.4 we know that \( Q \) is infinitely generated and therefore \( QJ = Q \) holds. We obtain \( Qa = QJa = (0) \) and \( a \in Q^r = Ja \) leads to \( a = 0 \), a contradiction to \( Q^r \neq (0) \). Thus \( Q^{rr} = Q^r \) and the theorem is proved.

**Corollary 3.12.** Let \( R \) be a semi-invariant chain ring with \( \text{d.c.c.} \) for prime ideals. Assume further that the ideal \((0) \) has an upper neighbor in the lattice of left ideals of \( R \). Then \( R \) is an \( H \)-ring.

Finally we note a sufficient condition for a semi-invariant chain ring without zero divisors to be valuation ring which depends on the structure of the maximal ideal.

**Theorem 3.13.** Let \( R \) be a semi-invariant chain ring without zero divisors. If \( J(R) = J \) is the only idempotent or the only non-idempotent prime ideal in \( R \) then \( R \) is a valuation ring.
Proof. Assume \( aR \subseteq Ra \) holds. This implies \( Ja \subseteq aJ \). The set \( P = \{ x | ax \in Ja \} \) is a prime ideal and \( aP = Ja \). If \( P = J \) we are done, since \( aJ = Ja \) implies \( aR = Ra \).

If \( P \neq J \) we distinguish two cases: First let us assume \( J^2 = J \) and \( P^2 = P \). Then \( J^2 = J \) and \( aP = aP^2 = aP = Ja \) is a contradiction. In the remaining case \( J^2 = J \), \( P^2 = P \) we obtain \( Ja = aP = aP^2 = J^2a \subseteq Ja \); a contradiction.

References


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