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GROUP RINGS AND GENERALIZED VALUATIONS

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0. Introduction

If R is a ring and Γ is a group, then it is possible to define various ring-structures on the free left R-module with basis Γ using group homomorphisms $\sigma : \Gamma \to \mathrm{Aut}(R)$ by defining $\gamma r = r^{\sigma(r)}\gamma$ for all $\gamma \in \Gamma$ and $r \in R$ where Aut(R) denotes the automorphism group of R.

A consideration of rings $R^{\sigma}[\Gamma]$ for special rings R, e.g. $R = Z/pZ$, the field with p elements, can often be used to collect information about the group Γ. However, there is another use of group rings. They provide a useful tool for the construction of examples in ring theory.

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In this paper, the latter approach is taken. It was initiated by the construction of nearly-simple chain rings, and in particular by the analysis of an example by Dubrovin [3]. Here, a ring R is called a chain ring if its set of right and left ideals is linearly ordered under inclusion, and a local ring R is called nearly--simple if $\{0\}$, $J(R)$ and R are the only two-sided ideals of R where $J(R)$ denotes the Jacobson radical of R.

While most constructions of chain rings are based on an idea by Neumann [6] on generalized power series, this paper is concerned with the possibility of using group rings and semigroup rings over right ordered groups, and their generalizations.

In a first step groups Γ are considered which are the union of a smooth ascending subnormal series $\{\Gamma_\alpha\}_{\alpha<\kappa}$ such that $\Gamma_{\alpha+1}/\Gamma_\alpha$ is a torsion-free abelian group. In Theorem 2.2, it is shown that these groups are exactly the groups Γ that can be right ordered in such a way by a positive cone Γ^+ such that the set of convex subgroups is well ordered and for all $\alpha,\beta \in \Gamma^+$ there is $0 \neq n < \omega$ such that $(\alpha\beta)^n\alpha^{-1}\beta^{-1} \in \Gamma^+$.
The construction of chain rings via group rings uses localization techniques that require that $R^\sigma[\Gamma]$ is a right Ore ring whenever R is.

In *Section 3*, it will be shown that the groups considered in Section 2 satisfy this requirement. The class of these groups contains the class of locally nilpotent, torsion--free groups. By this method numerous chain rings can be constructed via group resp. semigroup rings. These rings have a crucial decomposition property which gives rise to introduce the concept of generalized valuation firstly used by Radó [7], now lightened from another point of view. The disadvantage of Radó's nonsymmetric definition is overcome in this paper by the consideration of two conjugated valuations instead of one. However, our examples which are derived from semigroup rings admit valuations whose value sets are the underlying sets of a right (left) ordered groups which cannot be assumed in the general case. Nevertheless, this described concept of valuations has a much larger field of applications as it was known till now.

1. Preliminaries

The purpose of this section is to summarize the notations and basic results that will be used throughout this paper.

A linear order $\leq$ on a group Γ is a *right (left) order* if $\alpha \leq \beta$ implies $\alpha\gamma \leq \beta\gamma$ ($\gamma\alpha \leq \gamma\beta$) for all $\alpha,\beta,\gamma \in \Gamma$. If $\leq$ is a right and left order, then Γ is linearly ordered. Associated with every right order $\leq$ on Γ is a subsemigroup Γ^+ of Γ defined by $\Gamma^+ = \{\gamma \in \Gamma | \gamma \geq e\}$ which is the generalized *positive cone* of Γ. Here e denotes the identity element of Γ. Γ^+ satisfies

i) if $\gamma \in \Gamma\backslash\Gamma^+$, then $\gamma^{-1} \in \Gamma^+$,

ii) Γ^+ contains the identity e of Γ, and

iii) if $\gamma, \gamma^{-1} \in \Gamma^+$, then $\gamma = e$.

Conversely, every subsemigroup Π of Γ satisfying these three conditions induces a right (left) order on Γ by defining $\alpha \leq_r \beta$ ($\alpha \leq_\ell \beta$) if and only if $\beta\alpha^{-1} \in \Pi$ ($\alpha^{-1}\beta \in \Pi$). However, $\leq_\ell$ and $\leq_r$ do not agree in general. It is easy to show that this is the case exactly if $\gamma^{-1}\Gamma^+\gamma \subset \Gamma^+$ for all $\gamma \in \Gamma$.

In the remainder of this paper group rings over right ordered groups and their generalizations are investigated. *Rings* R are *not necessarily commutative* with a unit $1 \in R$, however we restrict ourselves to the case that R is *zero-divisor-free*. In particular, the following is of interest. Let R be any ring and Γ a group. Suppose $\sigma : \Gamma \to \mathrm{Aut}(R)$ is a group homomorphism where Aut(R) denotes the automorphism group of R. A ring structure is defined on the free left R-module with basis Γ by

$$\alpha r = r^{\sigma(\alpha)}\alpha.$$

This ring is called a *skew group ring* and denoted by $R^\sigma[\Gamma]$.

Lemma 1.1: Let R be a ring with no zero-divisors, $(\Gamma, \leq_r)$ a right ordered group with positive cone Γ^+ and $\sigma : \Gamma \to \mathrm{Aut}(R)$ a group monomorphism. For every $0 \neq a \in R^\sigma[\Gamma]$ in the skew group ring $R^\sigma[\Gamma]$ there are unique elements u, v with $u, v \in S$ $= \{\Sigma_{\gamma\in\Gamma^+} \gamma r_\gamma | r_e \neq 0$ with e the identity of $\Gamma\}$ and $\alpha, \beta \in \Gamma$ such that

$$a = u\alpha = \beta v$$

Moreover, $\alpha \in \Gamma^+$ if and only if $\beta \in \Gamma^+$.

Proof: Let α be the smallest element resp. $\leq_r$ in the support of a. By factoring α on the right side we obtain $u \in S$. Let $\leq_\ell$ be the left order on Γ induced by Γ^+. Write $a = \Sigma_{i=0}^n r_i\gamma_i$ with $r_i \neq 0$ and $\gamma_0 \leq_\ell \ldots \leq_\ell \gamma_n$. Then

$$a = \gamma_0 \Sigma_{i=0}^n r_i^{\sigma(\gamma_0^{-1})} \gamma_0^{-1}\gamma_i$$

and $\gamma_0^{-1}\gamma_i \in \Gamma^+$ since $\gamma_0 \leq_\ell \gamma_i$. Choose $\beta = \gamma_0$ and

$v = \Sigma_{i=0}^n r_i^{\sigma(\gamma_0^{-1})}\gamma_0^{-1}\gamma_i$. The rest of the lemma is now obvious. In the case that $\leq_\ell$ and $\leq_r$ agree, then $\alpha = \beta$ holds in the last lemma. This is for instance the case if Γ is commutative. In the following, $a = u\alpha$ $(= \beta v)$ is called the *canonical right (left) decomposition* of a. As S is a multiplicatively closed subset we conclude.

1.2 Corollary: Skew group rings $R^\sigma[\Gamma]$ with R zero-divisor-free and Γ right orderable have no zero-divisors.

One of the main concerns of this paper is to investigate when the set S in Lemma 1.1 is a right Ore set in $R^{\sigma}[\Gamma]$. Here, a multiplicatively closed subset T not containing 0 of a ring R is a *right Ore set* if for all $t \in T$ and $0 \neq r \in R$ the set $rT \cap tR$ is non-empty. R itself is a *right Ore ring* if $R\setminus\{0\}$ is a right Ore set. The localization of R at T is denoted by R_T.

As mentioned in the introduction we are interested in construction methods for chain rings. Hence, in order to guarantee localization in the skew group ring $R^{\sigma}[\Gamma]$ as well as right orderablity for Γ we will study the following type of groups:

Definition 1.3: Let Γ be a group. Γ has a subnormal series $\{\Gamma_\alpha\}_{\alpha<\kappa}$ of length κ where κ is an ordinal number if

i) $\Gamma_0 = \{e\}$,

ii) Γ_α is a normal subgroup of $\Gamma_{\alpha+1}$ for all $\alpha < \kappa$,

iii) $\Gamma_\alpha = \cup_{\beta<\kappa}\Gamma_\beta$ if α is a limit ordinal, and

iv) $\Gamma = \cup_{\alpha<\kappa}\Gamma_\alpha$.

The groups $\Gamma_{\alpha+1}/\Gamma_\alpha$ are the factors of the subnormal series.

2. A Special Class of Right Ordered Groups

One of the important results on right ordered groups is that every torsion-free abelian group can be right ordered in such a way that the set of its convex subgroups satisfies the minimum condition with respect to inclusion. Here, a subgroup B of a right ordered group Γ is convex if $e \leq_r \gamma \leq_r \beta$ and $\beta \in B$ implies $\gamma \in B$ for all $\gamma \in \Gamma$.

Lemma 2.1: Let Γ be a group with a subnormal series $\{\Gamma_\alpha\}_{\alpha<\kappa}$ of length κ whose factors are torsion-free abelian. Then, Γ can be right ordered in such a way that the convex subgroups of Γ in this right order satisfy the minimum condition with respect to inclusion.

Proof: Since every torsion-free abelian group has a subnormal series whose factors are subgroups of the rational numbers Q^+, one can assume that $\Gamma_{\alpha+1}/\Gamma_\alpha$ is isomorphic to a subgroup of Q^+ for all $\alpha < \kappa$. If this is not the case, then refine the original chain.

Suppose, one has already defined positive cones Π_β of Γ_β such that $\Pi_\beta \subset \Pi_\gamma$ for $\beta < \gamma < \alpha$.
If α is a limit ordinal, then let $\Pi_\alpha = \cup_{\beta<\alpha}\Pi_\beta$. On the other hand, if $\alpha = \beta + 1$, then $\Gamma_{\beta+1}/\Gamma_\beta$ is right ordered as a subgroup of Q^+. By [1, Theorem 3.7], there is a right order on Γ_α induced by the one on $\Gamma_{\beta+1}/\Gamma_\beta$ whose positive cone Π_α contains Π_β. Moreover, Γ_β is convex in $\Gamma_{\beta+1}$. $\Pi = \cup_{\alpha<\kappa}\Pi_\alpha$ is a generalized positive cone in Γ. The lemma is proved if it is shown that the Γ_α's and Γ

are the only convex subgroups of Γ in the right order induced by Π. Suppose that Γ_α is not convex in Γ for some $\alpha<\kappa$. Then, there is $\beta \in \Gamma_\alpha$ and $\gamma \in \Gamma$ such that $e \leq_r \gamma \leq_r \beta$ and $\gamma \notin \Gamma_\alpha$. Choose σ minimal with $\gamma \in \Gamma_\sigma$. Obviously, $\sigma > \alpha$ and $\sigma -1$ exists by condition iii) in the definition of a subnormal series. Since $\Gamma_{\sigma-1}$ is a convex subgroup of Γ_σ containing β, γ is in $\Gamma_{\sigma-1}$ too. However, this contradicts the choice of σ. Conversely, let Δ be a proper convex subgroup of Γ; then choose σ to be minimal in $\{\nu < \kappa : \Gamma_\sigma \cap (\Gamma\setminus\Delta) \neq \emptyset\}$. Because of

iii) in Definition 1.1, $\sigma -1$ exists. Moreover, $\Gamma_{\sigma-1} \subset \Delta \subset \Gamma_\sigma$. Therefore, $\Delta/\Gamma_{\sigma-1}$ is a proper convex subgroup of $\Gamma_\sigma/\Gamma_{\sigma-1}$. Furthermore, $\Gamma_\sigma/_\Delta$ is torsion-free since Δ is convex in Γ_σ. Because $\Gamma_\sigma/\Gamma_{\sigma-1}$ is isomorphic to a subgroup of Q^+, this implies $\Delta = \Gamma_{\sigma-1}$.

With this the following characterization of the groups described in Lemma 2.1 can be given.

Theorem 2.2: For a group Γ, the following are equivalent:

a) Γ has a subnormal series with torsion-free abelian factors.

b) There is a generalized positive cone Π on Γ inducing a right order $\leq_r$ such that the convex subgroups with respect to this order satisfy the minimum condition, and for α, $\beta \in \Pi$ there is $0 \neq n < \omega$ such that $\beta\alpha \leq_r (\alpha\beta)^n$.

Proof: a) $\Rightarrow$ b): Let Π be the positive cone constructed in Lemma 2.1 using the subnormal series $\{\Gamma_\alpha\}_{\alpha<\kappa}$ with factors isomorphic to a subgroup of Q^+. It is left to show only the last condition in b). But by Lemma 2.1, if Δ_1 is normal in Δ_2, and Δ_2/Δ_1 is isomorphic to a subgroup of $\mathbb{R}$. By [1, Theorem 4.1], the last condition of b) is satisfied too.

b) $\Rightarrow$ a): The chain of convex subgroups of Γ is well--ordered. By [1, Theorem 4.1], C is normal in D if D is the successor of C in this order, an D/C is isomorphic to a subgroup of $\mathbb{R}$. This proves a).

It shall be remarked, that if one considers canonical right and left decompositions of elements of $R^\sigma[\Gamma]$, then α is in a convex subgroup Δ of Γ if and only if β is.

3. Group Rings as Ore Rings

In the last section, right ordered groups Γ which have a subnormal series with torsion-free abelian factors have been investigates. Now it will be shown that $R^\sigma[\Gamma]$ is a right Ore ring if R is one, and $\sigma:\Gamma \to \mathrm{Aut}(R)$ is a group homomorphism.

Lemma 3.1: Let R be a ring, Γ be a group, and be $\sigma:\Gamma \to$

Aut(R) a group homomorphism. If Γ is the semi-direct product of a normal subgroup N by a subgroup B, then a ring-structure is defined on the free left R [N]-module with basis B by $\beta(r\gamma) = r^{\sigma(\beta)}(\beta\gamma\beta^{-1})\beta$. The ring $(R^{\sigma}[N])^{\sigma}[B]$ obtained this way is isomorphic to $R^{\sigma}[\Gamma]$.

Proof: Consider the map $\varphi: (R^{\sigma}[N])^{\sigma}[B] \to R^{\sigma}[\Gamma]$ defined by $\varphi(\sum_{\beta\in B}(\sum_{\nu\in N} r_{\nu,\beta}\nu)\beta) = \sum_{\beta\in B}\sum_{\nu\in B} r_{\nu,\beta}\nu\beta$. Since Γ is the

semi-direct product of N and B, and $(R^{\sigma}[N])^{\sigma}$ [B] is a free left R[N]-module, this is an isomorphism of abelian groups.

Moreover, if $\sum_{\alpha\in B}(\sum_{\nu\in N} r_{\nu,\beta}\nu)\alpha$ and $b = \sum_{\beta\in B}(\sum_{\mu\in N} s_{\mu,\beta}\mu)\beta$ are elements of $(R^{\sigma}[N])^{\sigma}$ [B]

then

$$ab = \sum_{\alpha\in B}\sum_{\beta\in B}\sum_{\nu\in N}\sum_{\mu\in N}(r_{\nu,\alpha}(s_{\mu,\beta}{}^{\sigma(\alpha)})^{\sigma(\nu)}\nu)\alpha\mu\beta$$

and

$$\varphi(ab) = \sum_{\alpha\in B}\sum_{\beta\in B}\sum_{\nu\in N}\sum_{\mu\in N} r_{\nu,\alpha}(s_{\mu,\beta})^{\sigma(\nu\alpha)}\nu\alpha\mu\alpha^{-1}(\alpha\beta)$$

On the other hand,

$$\varphi(a)\varphi(b) = (\sum_{\alpha\in B}\sum_{\nu\in N} r_{\nu,\alpha}\nu\alpha)(\sum_{\beta\in B}\sum_{\mu\in N} s_{\mu,\beta}\mu\beta)$$

$$= \sum_{\alpha\in B}\sum_{\nu\in N}\sum_{\beta\in B}\sum_{\mu\in N} r_{\nu,\alpha}\, s_{\mu,\beta}{}^{\sigma(\nu\alpha)}\nu\alpha\mu\beta$$

Therefore, φ is an isomorphism of rings.

Lemma 3.2: Let R be a right Ore domain, and Γ be a torsion-free group that has a normal subgroup N such that Γ/N is torsion-free abelian and $R^{\sigma}[N]$ is a right Ore ring where $\sigma:\Gamma \to \mathrm{Aut}(R)$ is a group homomorphism. Then, $R^{\sigma}[\Gamma]$ is a right Ore ring.

Proof: Let $0 \neq a, b \in R[\Gamma]$, say $a = \sum_{i=1}^{n} r_i\gamma_i$, $b = \sum_{i=1}^{n} s_i\gamma_i$. Then, there is a subgroup U of Γ containing N and the set $\{\gamma_1, \ldots, \gamma_n\}$ such that U/N is finitely generated. Thus, there is $m < \omega$ such that $U/N \cong \oplus_{i=\ell}^{m} Z$. Obviously, it is enough to show that $R^{\sigma}[U]$ is a right Ore ring.

If $m = 1$, then there is a subgroup $B = \langle x\rangle$ of U such that U is the semi-direct product of N by B. By Lemma 3.1, it is enough to show that $R^{\sigma}[N]^{\sigma}[B]$ is a right Ore ring.

Let $0 \neq a, b \in R^{\sigma}[N]^{\sigma}[B]$. To show that $ab \neq 0$, write $a = \sum_{i\geq q} r_i x^i$ and $b = \sum_{j\geq k} s_j x^j$ where $r_i, s_j \in R^{\sigma}[N]$ and $r_q \neq 0 \neq s_k$. Then, $ab = r_q s_k^{\sigma(x^q)} x^{q+k}$ + terms with higher exponent in x. But $r_q s_k^{\sigma(x^q)} \neq 0$ implies $ab \neq 0$.

To show that there are $c, d \in R^{\sigma}[N]^{\sigma}[B]$ with $0 \neq ac = bd$, write $a = \sum_{i=-n}^{n} r_i x^i$ and $b = \sum_{i=-n}^{n} s_j x^j$. It is enough to find $d_{-n}, \ldots, d_n$ and $c_{-n}, \ldots, c_n$ such that

$c = \sum_{i=-n}^{n} c_i x^i$ and $d = \sum_{j=-n}^{n} s_j x^j$ satisfy ac = bd and not all the c_i 's and d_j 's are equal to zero. But the condition ac = bd gives rise to an homogenous linear equation system for the c_i's and d_i's having 4n + 1 equations for 4n + 2 variables. Since $R^\sigma[N]$ is a right Ore domain, this system has a non-zero solution over $R^\sigma[N]$. This proves the existence of c and d with $0 \neq ac = bd$.

If $U/N \cong \oplus_{i=1}^{m} Z$, then there is a normal subgroup N_1 of U such that $U/N \cong Z$ and $N_1/N \cong \oplus_{i=1}^{m-1} Z$. By the case m = 1 and the induction hypothesis, $R^\sigma[U]$ is a right Ore ring.

In order to prove main theorem of this section, one more lemma is needed.

Lemma 3.3: Let Γ be a group that is the union of an ascending chain $\{\Gamma_i\}_{i \in I}$, and let R be a right Ore ring. If $\sigma : \Gamma \to \mathrm{Aut}(R)$ is a group homomorphism and $R^\sigma[\Gamma_i]$ is a right Ore ring, then $R^\sigma[\Gamma]$ is a right Ore ring.

Proof: Since every finitely generated subgroup of Γ is contained in Γ_i for some $i \in I$, every two elements of $R^\sigma[\Gamma]$ are contained in $R^\sigma[\Gamma_i]$ for some $i \in I$.

Theorem 3.4: Suppose that Γ is a group with a subnormal series with torsion-free, abelian factors. If R is a right (left) Ore ring and $\sigma:\Gamma \to \mathrm{Aut}(R)$ is a group homomorphism, then $R^{\sigma}[\Gamma]$ is a right (left) Ore ring.

Proof: Suppose, $\{\Gamma_{\alpha}\}_{\alpha<\kappa}$ is the subnormal series in Γ with torsion-free abelian factors. Assume that it has been shown that $R^{\sigma}[\Gamma_{\beta}]$ is a right Ore ring for $\beta < \alpha$. If α is a limit ordinal, then apply Lemma 3.3 to show that $R^{\sigma}[\Gamma_{\alpha}]$ is right Ore. On the other hand, if $\alpha = \beta + 1$, then apply Lemma 3.2. Another application of Lemma 3.3 proves that $R^{\sigma}[\Gamma]$ is a right Ore ring. The case that R is a left Ore ring is treated similarly.

Obviously, if $R^{\sigma}[\Delta]$ is a right Ore ring for every finitely generated subgroup Δ of Γ, then $R^{\sigma}[\Gamma]$ is a right Ore ring. Since the factors of the ascending central series of a torsion-free nilpotent group are torsion-free abelian, this last statement and Theorem 3.4 suffice to prove.

Corollary 3.5: Let Γ be a locally nilpotent group. If R is a right (left) Ore ring and $\sigma:\Gamma \to \mathrm{Aut}(R)$ is a group homomorphism, then $R^{\sigma}[\Gamma]$ is a right (left) Ore ring.

It is well-known that the conditions in Theorem 3.4 cannot be ommited in general; as Neumann [6, p. 213/214] pointed out there exist group rings over a field and linearly ordered group which are not Ore.

This section concludes with an illustration, how the results of Theorem 3.4 can be applied to the construction of chain rings. The reader shall be reminded of a notation introduced in Section 1. If Γ is a right ordered group with generalized positive cone Γ^+, R a right Ore ring, and $\sigma:\Gamma \to \mathrm{Aut}(R)$ a homomorphism, then denote by S the subset $\{\sum_{\gamma\in\Gamma^+} r_\gamma \gamma \mid \gamma_e \neq 0\}$ of $R^\sigma[\Gamma]$.

Theorem 3.6: Let Γ be a group with generalized positive cone Γ^+. If R is a right Ore ring and $\sigma:\Gamma \to \mathrm{Aut}(R)$ a group homomorphism such that $R^\sigma[\Gamma]$ is a right Ore ring then S is a right Ore set of both $R^\sigma[\Gamma^+]$ and $R^\sigma[\Gamma]$. Moreover, the rings $(R^\sigma[\Gamma^+]_S$ and $(R^\sigma[\Gamma])_S$ are right chain rings.

Proof: Let $0 \neq x \in R^\sigma[\Gamma]$ and $s \in S$. Since $R^\sigma[\Gamma]$ is a right Ore ring, there are $u,v \in R^\sigma[\Gamma]$ with $0 \neq xu = sv$. By Lemma 1.1, one has $\alpha \in \Gamma$ and $a \in S$ with $u = a\alpha$. Consequently, $0 \neq xa = s(v\alpha^{-1})$ which shows that S is an Ore set in $R^\sigma[\Gamma]$. Moreover, if $x \in R^\sigma[\Gamma^+]$, then choose $b \in S$ and $\beta \in \Gamma$ with $v = b\beta$. Consequently, $0 \neq xa = sb(\beta\alpha^{-1})$. A comparison of coefficients shows that $\beta\alpha^{-1} \in \Gamma^+$. Therefore, S is an Ore set in $R^\sigma[\Gamma]$.
Finally, if $0 \neq y_1, y_2 \in (R^\sigma[\Gamma])_S$, then use Lemma 1.1 to write $y_i = s^{-1}\gamma_i c_i$ with $\gamma_i \in \Gamma$ and $s, c_i \in S$ for $i = 1,2$.

Since Γ^+ induces a linear left order on Γ, one has either $\gamma^{-1}{}_1\,\gamma_2 \in \Gamma^+$ or $\gamma_2^{-1}\,\gamma_1 \in \Gamma^+$. Assume the first holds, say $\gamma_2 = \gamma_1\gamma$ for some $\gamma \in \Gamma^+$.

Then,
$$\begin{aligned} y_2(R^\sigma[\Gamma])_S &= s^{-1}\,\gamma_2\,c_2\,(R^\sigma[\Gamma])_S \\ &= s^{-1}\,\gamma_2\,\gamma\,(R^\sigma[\Gamma])_S \\ &\subseteq s^{-1}\,\gamma_1\,(R^\sigma[\Gamma])_S \\ &= y_1\,(R^\sigma[\Gamma])_S \end{aligned}$$

A similar argument is used to show that $(R^\sigma[\Gamma^+])_S$ is a right Ore ring.

Clearly, the arguments used in this section are right-left symmetric. This and a standard technical calculation are enough to prove.

Corollary 3.7: Let Γ be a group with a subnormal series whose factors are torsion-free abelian. If K is a field and $\sigma:\Gamma \to \mathrm{Aut}(K)$ a group-hommorphism, then both rings $(K[\Gamma^+])_S$ and $(K[\Gamma])_S$ are left and right chain rings for each generalized positive cone Γ^+ in Γ. Furthermore, $(K[\Gamma^+])_S$ is nearly simple if and only if for all $x \in \Gamma^+\backslash\{e\}$ one has $\Gamma^+\backslash\{e\} = \Gamma^+ x\Gamma^+$.

4. Generalized Valuations

The skew-group and skew-semigroup rings constructed in Section 3 will be considered from a more general point of view which allows to give a larger class of examples for a concept of valuation which has been introduced by Radó [7].

The idea using generalized valuation originates from Lemma 1.1. The importance of this lemma is the existence of a decomposition property for a class of group rings resp. semigroup rings. This property by itself, not the fact that the rings are group rings, will be fundamental for the following.

An analysis of Lemma 1.1 and the special situation where a ring R has the form $R^{\sigma}[\Gamma]$ shows

(i) R contains multiplicative semigroups
S (namely the S as defined in 1.1) and
H (namely Γ^+) with
$S \cap H = \{1\}$

(ii) If $\alpha, \beta \in H$, then $\alpha\beta^{-1} \in H$ or $\beta\alpha^{-1} \in H$

(iii) There is a group Γ with a generalized positive cone Γ^+
and a homomorphism $|\ |: H \to \Gamma$ of semigroup such that
(a) $|H| \supseteq \Gamma^+$
(b) $|\alpha| = e \in \Gamma$ implies that α is a unit in V.
(In Lemma 1.1 we have to choose $|\ |$ as the identity.)

(iv) Every element $0 \neq r \in R$ has a unique decomposition $r = u\alpha = \beta\varphi$ with $u, v \in S$ and $\alpha, \beta \in H$ satisfying

(a) $|\alpha| \in \Gamma^+$ exactly if $|\beta| \in \Gamma^+$,

(b) $|\alpha| = e$ if $|\beta| = e$, and

(c) if $x_1 = u_1\alpha_1 = \beta_1 v_1$ and $x_2 = u_2\alpha_2 = \beta_2 v_2$ are elements of R with $|\alpha_1|, |\alpha_2| \in \Gamma^+$, then $0 \neq x_1 + x_2 = u\alpha = \beta v$ implies $|\alpha|, |\beta| \in \Gamma^+$.

Definition 4.1: A ring R has the weak decomposition property (WDP) if and only if conditions (i) through (iv) are satisfied.

The group rings resp. semigroup rings constructed in Section 3 as well as the rings of generalized powerseries [6] have this property.

In the next step, rings with (WDP) are considered in view of the concept of valuations by Radó [7] mentioned above. Besides giving an easier way to understand the arithmetic of ideals of the ring, one obtains a class of examples for these valuations far away from the invariant case.

Lemma 4.2: Suppose R is a ring with (WDP) and Γ the associated group with the generalized positive cone Γ^+. Then there is a pair $(|\ |_\ell, |\ |_r)$ of maps

$|\ |_\ell : R \setminus \{0\} \to (\Gamma, \leq_\ell)$, respectively

$|\ |_r : R \setminus \{0\} \to (\Gamma, \leq_r)$

satisfying:

(i) $|x|_\ell \leq_\ell |y|_\ell$ implies $|zx|_\ell \leq_\ell |zy|_\ell$,
respectively
$|x|_r \leq_r |y|_r$ implies $|xz|_r \leq_r |yz|_r$,

(ii) If $x \neq y$, then $\min\{|x|_\ell, |y|_\ell\} \leq_\ell |x - y|_\ell$ and $\min\{|x|_r, |y|_r\} \leq_r |x - y|_r$,

(iii) $|x|_\ell = |1|_\ell$ iff $|x|_r = |1|_r$, and

(iv) $|x|_\ell \geq_\ell |1|_\ell$ iff $|x|_r \geq_r |1|_r$.

<u>Proof:</u> Because of the decomposition property, $x = u\alpha = \beta v$ for each $x \neq 0 \in R$ with $u, v \in S$ and $\alpha, \beta \in H$. Define $|x|_r = |\alpha|$ resp. $|x|_\ell = |\beta|$.

4.1 (iv) implies (iii) and (iv). To show (i), let $x = u_1\alpha_1, y = u_2\alpha_2$ and $z = u_3\alpha_3$, i.e. $|x|_r = |\alpha_1|$, $|y|_r = |\alpha_2|$ and $|z|_r = |\alpha_3|$; further suppose $|\alpha_1| \leq_r |\alpha_2|$.

Because of conditions 4.1 (ii) and (iii) there is $\rho \in H$ with $\alpha_2 = \rho\alpha_1$, $|\rho| \in \Gamma^+$. Moreover, $\alpha_1 u_3 = v_1\alpha_1'$ and $\alpha_2 u_3 = v_2\alpha_2'$.
Then $|xz|_r = |u_1 v_1 \alpha_1' \alpha_3|_r = |\alpha_1'\alpha_3|$ and $|yz|_r = |u_2 v_2 \alpha_2' \alpha_3|_r = |\alpha_2'\alpha_3|$. Furthermore, $\rho v_1 = v_1'\rho'$, implies $\alpha_2 u_3 = \rho\alpha_1 u_3 = \rho v_1 \alpha' = v_1'\rho'\alpha_1'$. On the other hand, $\alpha_2 u_3 = v_2\alpha_2'$. Consequently $v_2\alpha_2' = v_1'\rho'\alpha_1'$ and hence $\alpha_2' = \rho'\alpha_1'$ with $|\rho'| \in \Gamma^+$. In this case, $|\rho'||\alpha_1'\alpha_3| = |\rho'||xz|_r = |\alpha_2'\alpha_3| = |yz|_r$ implies

$|xz|_r \le_r |yz|$. That $|\ |_\ell$ satisfies (i) is proved in the same way.

It is left to show (ii). By symmetry, it is enough to consider $|\ |_r$ only. Write again $x = u_1\alpha_1$, $y = u_2\alpha_2$. Without loss of generality, $\rho\alpha_1 = \alpha_2$ for some $\rho \in H$ and $|\rho| \in \Gamma^+$. Then, $x = u_1\alpha_1$ and $y = u_2\rho\alpha_2$. In this case, $x - y = (u_1 - u_2\rho)\alpha_1$. Write $u_1 - u_2\rho = v\pi$ with $|\Pi| \in \Gamma^+$ (4.1 (iv) c). Therefore, $x - y = v\pi\alpha$,
i.e.
$|x-y|_r = |\pi\alpha_1| \ge_r |\alpha_1| = \min\{|x|_r, |y|_r\}$.

Theorem 4.3: Suppose R is a right and left Ore ring with (WDP). Then, the localization R_S exists. R_S is a chain ring and the maps $|\ |_r$ resp. $|\ |_\ell$ given by Lemma 4.2 can be extended to R_S such that (i) through (iv) are still satisfied.

Proof: By definition, S is a semigroup. It is left to show that for all $0 \neq r \in R$, $s \in S$, there are $r' \in R$, $s' \in S$ such that $rs' = sr'$. Since R is a right Ore ring, there are elements $s', r' \in R$ with $rs' = sr'$. Write $s' = u\alpha$, $r' = v\beta$ with $u, v \in S$, $\alpha, \beta \in H$. Then, $ru\alpha = sv\beta$, and the result follows if $\beta\alpha^{-1} \in H$. Suppose $\beta\alpha^{-1} \notin H$, hence $\alpha\beta^{-1} \in H$. Let be $r = w\gamma$. Consequently, $sv = ru\alpha\beta^{-1} = w\gamma u\alpha\beta^{-1} = wu\gamma'\alpha\beta^{-1}$ implies $\gamma'\alpha\beta^{-1} = 1 \in H$ which contradicts $\beta\alpha^{-1} \notin H$. The proof that S is left

Ore is analogous, hence $R_S = {}_S R$.

Let be $as^{-1}, bs^{-1} \in R_S$ with $a = \alpha u$, $b = \beta v$, $\alpha\ \beta \in H$, $u, v \in S$ and $\alpha\beta^{-1} \in H$. Then, $as^{-1} = su^{-1}\alpha^{-1}\beta vs^{-1} = bs^{-1}$ which shows that R_S is a right chain ring.

By similar arguments R_S is a left chain ring too.

If $as^{-1} \in R_S$ is given then define $|as^{-1}|_\ell = |a|_\ell$. Since $as^{-1} = bt^{-1}$ iff there are $p, q \in R$ with $ap = bq$, and $sp = tq$. Then $|a|_\ell = |b|_\ell$ because one can assume that $p, q \in S$, i.e.
$|p|_\ell = |q|_\ell = |1|_\ell = e \in \Gamma$, without loss of generality.

To prove condition 4.2(i), it suffices to prove:
$|a|_\ell \leq_\ell |b|_\ell$ implies $|s^{-1}a|_\ell \leq_\ell |s^{-1}b|_\ell$ for $a, b \in R$ and $s \in S$. For $a = \alpha u$ and $b = \beta v$, one has $|\alpha| \leq_\ell |\beta|$. Hence $\alpha\rho = \beta$ with $\rho \in H$. $s^{-1}\beta = s^{-1}\alpha\rho = \alpha' s'^{-1}\rho = \alpha' \rho' s''^{-1}$ with $|\rho'| \in \Gamma^+$ shows $|s^{-1}a|_\ell \leq_\ell |s^{-1}b|_\ell$.

Similarly, the proofs of (ii), (iii) and (iv) are straightforward applications of the Ore condition and the properties 4.2(i) through (iv) of the ring R. By symmetry the same holds for $|\ |_r$.

Valuations with the functional properties 4.2(i) and (ii) have been considered for the first time by Radó [7]. Here in, it was of essential importance that the condition $|xy| = |x|\ |y|$ was replaced be (i), because, in general, the value set has no algebraic structure at all. Radó assumed only that the range $|R\setminus\{0\}|$ is a linearly ordered set. This lack of structure influenced e.g. Mathiak [5], to introduce an equivalent concept restricted on division rings, and to investigate it from a different point view.

However, in the author opinion, Radó's approach has not been descussed sufficiently enough. Firstly, Radó does not distinguish, as it is shown here between right and left valuations since he considers only division rings, and in this case, "left properties of elements x,y" can be viewed as "right properties of their inverses x^{-1},y^{-1}" and vice-versa. With respect to this, the approach here is a more ring-theoretic concept and the analysis of the relation between left and right valuations allows to investigate the left-right symmetry of a ring, an idea which will not be considered this paper however.

Secondly, the examples in this paper give a large class of rings in which it is not allowed to calculate multiplicatively with the valuation, but where the range still has a sufficient algebraic structure, namely the value set is the underlying structure of a right (left) ordered semigroup. Generalizing the idea of Lemma 4.2 and having the approach of Radó [7] in mind, one defines:

Definition4.4: If R is a ring, and $(\Omega_\ell, \le_\ell)$, $(\Omega_r, \le_r)$ are linearly ordered sets, then a pair $(|\ |_r, |\ |_\ell)$ of mappings $|\ |_r : R \setminus \{0\} \to \Omega_r$ and $|\ |_\ell : R \setminus \{0\} \to \Omega_\ell$ is a *pair of generalized, conjugated valuations* (Shortly: a generalized valuation of R) if

(i) $|x|_\ell \le_\ell |y_\ell|$ implies $|zx|_\ell \le_\ell |zy|_\ell$,
respectively
$|x|_r \le_r |y_r|$ implies $|xz|_r \le_r |yz|_r$,

(ii) $\min \{|x|_\ell, |y|_\ell\} \le_\ell |x-y|_\ell$ and
$\min \{|x|_r, |y|_r\} \le_r |x-y|_r$ for $x \neq y$,

(iii) $|R \setminus \{0\}|_\ell = \Omega_\ell$ and $|R \setminus \{0\}|_r = \Omega_r$,

(iv) $|x|_\ell = |1|_\ell$ exactly if $|x|_r = |1|_r$, and

(v) $|x|_\ell \ge_\ell |1|_\ell$ iff $|x|_r \ge_r |1|_r$.

$|\ |_\ell$ resp. $|\ |_r$ will often be called left resp. right valuation.

Obviously

$$R_{|\,|} = \{x \mid |x|_\ell \ge_\ell |1|_\ell\} \cup \{0\}$$

is a ring, the *valuation* ring of $|\ |_\ell$. By 4.4(v) the definition is symmetrical.

Definition 4.5: A valuation $(|\ |_\ell, |\ |_r)$ of a ring R is regular if
$|x|_\ell = |1|_\ell$ implies x is a unit of R.

Remarks 4.6: 1. Let D be a division ring with v a valuation (in the sense of Schilling) and Γ its value group. Set $\Omega_\ell = \Gamma = \Omega_r$ and $|\ |_\ell = |\ |_r = v$. Then we have a generalized valuation. The restriction of v to the valuation ring (in the classical sense) induces a regular valuation.

2. Every chain ring R has at least one pair of generalized valuations. Define $\Omega_\ell = \{xR \mid 0 \neq x \in R\}$, $\Omega_r = \{Rx \mid 0 \neq x \in R\}$, and $|x|_\ell = xR$ resp. $|x|_r = Rx$. Then R is the associated (regular) valuation ring of this *canonical* valuation on the chain ring R.

3. Rings with the weak decomposition property satisfy 4.4(i) through 4.4(v) because of Lemma 4.2. Furthermore, one has the following.

Theorem 4.7: (i) Let R be a ring and $(|\ |_r, |\ |_\ell)$ a pair of regular generalized valuations on R. Further, let R satisfy the weak decomposition property with
$|u\alpha|_r = |\alpha|$ resp. $|\beta v|_\ell = |\beta|$
for all $\alpha,\beta \in H$, $u,v \in S$.
Then R is a chain ring.

Proof: Obviously $|u|_r = |1|_r$ implies u an unit. Because of $\alpha\beta^{-1} \in H$ or $\beta\alpha^{-1} \in H$ the right (left) ideals are linearly ordered by inclusion.

For chain rings as described in 4.7 there is a correspondence between the upper segments and one-sided(!) ideals. Thereby, a subset $\pi \neq \emptyset$ of a linearly ordered set $(\Omega,\leq)$ is called an *upper segment* if $\alpha \in \pi$, $\alpha \leq \beta$ implies $\beta \in \pi$.

Lemma 4.8: Let R be a chain ring and $(|\ |_r, |\ |_\ell)$ its canonical valuation ring, $I \neq \emptyset$, $I \subseteq R$ and $|I\setminus\{0\}|_\ell = \pi$. Then the following properties are equivalent:

a) I is a right ideal

b) π is an upper segment with respect to $\leq_\ell$.

The proof is straightforward and therefore omitted. However, not much is known about the admissable order structures of Ω_ℓ resp. Ω_r for chain rings, because there are no general construction methods. Since the examples in Section 5 are rings with the weak decomposition property, i.e. Ω_ℓ resp. Ω_r are derived from right (left) ordered groups, the discussion is restricted to the case completely prime and "two-sided" for this class only.

Lemma 4.9: If R is a chain ring with (WDP), $\emptyset \neq I \subseteq R$ and $|I\setminus\{0\}|_\ell = \pi_\ell$, $|I\setminus\{0\}|_r = \pi_r$ where $\pi_\ell, \pi_r \subseteq \Gamma^+$, then the following are equivalent:

a) I is a twosided ideal

b) (i) $\pi_\ell = \pi_r$

(ii) π_ℓ resp. π_r are upper segments in respect to $\leq_\ell$ resp. $\leq_r$

Furthermore, the following equivalence holds:

a) I is a completely prime ideal.

b) (i) $\pi_\ell = \pi_r = \pi$

(ii) π_ℓ resp. π_r are upper segments in respect to $\leq_\ell$ resp. $\leq_r$.

(iii) $\Gamma^+ \setminus \pi$ is a subsemigroup.

Proof: Straightforward.

Problem: An ideal P is *completely prime* if for elements $x,y \in R$, $xy \in P$ implies $x \in P$ or $y \in P$. An ideal P is called prime if for left (right) ideals X and Y

$$XY \subset P \text{ implies } X \subset P \text{ or } Y \subset P.$$

It remains an open question to characterize the non-completely prime ideals of a ring with the decomposition property using the value semigroup. Also, it is left unanswered if chain rings can be obtained with that type of prime ideals by the construction of Section 3. However, a related problem of Skornyakov [9, page 142] for a semigroup Γ^+ is apparently still unsolved.

5. Examples

Example 1: (This has already been settled by the construction of Rohlfing [8] in his dissertation:

Let R be left Ore ring and Ω a linearly ordered commutative group (semigroup) of operators. Then the skew group (semigroup) ring $R[\Omega]$ is left Ore too). Let Γ be an archimedean right ordered group. Conrad [1] showed that Γ is order-isomorphic to a subgroup of the real numbers. In particular, Γ is commutative, and therefore, if one considers a decomposition as in Lemma 1.1, $\alpha = \beta$. Consequently, if R is a right Ore ring then $| \; |_\ell = | \; |_r$ in $R[\Gamma^+]_S$. Since every $\leq_\ell$-upper segment is a $\leq_r$-upper segment, $R[\Gamma^+]_S$ is a two-sided valuation ring of rank 1 in the sense of Schilling.

Example 2: Ou Z x Z define an addition by

$(a_1,a_2) + (b_1,b_2) = (a_1+b_1,a_2 \; (-1)^{b_1}+b_2)$.

According to [1, Example 1] an element $(0,0) \neq (a_1,a_2)$ is positive if $a_1 > 0$ or $a_1 = 0$ and $a_2 > 0$. Then

$$\Gamma^+ = \{a_1,a_2) \in Z \times Z \mid (a_1,a_2) \geq (0,0)\}$$

is a generalized positive cone. The following example illustrates the left resp. right order defined by Γ^+ an Γ.

Right order:

$(0,0) < (0,1) <\ldots< (0,n) <\ldots< (1,10) < (1,0) < (1,-10)<\ldots$

$< (2,-10) < (2,0) < (2,10) < \ldots < (3,10) < (3,0) < (3,-10)<\ldots$

The *left order* is just the usual lexicographic order.

The chain ring R $[\Gamma^+]_S$ is obtained by Theorem 3.4

and 3.6 be localizing the Ore ring $R[\Gamma]$. $R[\Gamma^+]_S$ has the following prime ideals:

J, {O} and

$P_1 = \{r \in R[\Gamma^+]_S \mid \text{There is } a \in Z \text{ with } |r|_\ell \geq (1,1)\}$
$\cup \{O\}$

In the same way,

$P_i = \{r \in R[\Gamma^+]_S \mid \text{There is } a \in Z \text{ with } |r|_\ell \geq (i,a)\} \cup \{O\}$

are twosided ideals exactly the sections $[P_i, P_{i+1}]$ with i even (set $P_O = J$) are twosided. Observe that is not possible to define a twosided order on Γ.

Example 3: Suppose, $(K,\leq)$ is an ordered field and

$$\Gamma = \{x \to ax + b \mid a, b \in K, a > O\},$$

the group of affine-linear functions. Γ is ordered by the usual lexicographic order. It is well-known that it is possible to obtain valuation rings of rank 2 by Neumann's construction of generalized power-series. Since Γ is the semi-direct product of $(K,+)$ with $(K^+ \setminus \{O\},\cdot)$, the construction of Section 3 also leads to semigroup rings of rank 2.

Example 4: Dubrovin [3] has defined a generalized positive cone on this group in the case K = Q. For reasons of a better understanding, his definition is discussed geometrically here.

$$\Gamma^+ = \{f \in \Gamma \mid f\left(\frac{1}{\varepsilon}\right) \geq \frac{1}{\varepsilon}\}$$

where ε is a fixed, positive, irrational number. The graphes of the elements in Γ are the straight lines which intersect the line $x = \frac{1}{\varepsilon}$ in exactly one point. On the other hand, through every point of $x = \frac{1}{\varepsilon}$, there is at most one line in Γ.

If $f_1 \circ f_2$ is defined by $f_1(f_2)$, then the points of intersection define the right order on Γ by using the natural order on the line $x = \frac{1}{\varepsilon}$. Simiarly, the left order is found in the same way on $y = \frac{1}{\varepsilon}$. With this, it is clear that $R[\Gamma^+]_S$ is a nearly-simple chain ring.

Observe that example 3 (with $K = Q$) and example 4 describe subrings of the same quotient ring. Even more chain rings can be found in the same division ring:

Example 5: Γ as described in Example 3 can be considered as the group of order-preserving permutations of Q. There is a standard procedure to define right resp. left orders on Γ.

Fix a wellorder

$$s_1 \dashv s_2 \dashv \ldots\ldots \qquad \text{for } s_1, s_2 \in Q$$

of Q. If $f \in \Gamma$, then $L(f)$ denotes the first element in this wellorder for which $f(L(f)) \neq L(f)$. f is positive, if $f(L(f)) > L(f)$. In our case, however, it suffices to distinguish two points $s_1, s_2 \in Q$.

Remark: The fact that $R[\Gamma^+]_S$ is a chain ring if R is Ore does not depend on the chosen generalized positive cone Γ^+ but is answered by the algebraic structure of Γ. However, the ideal structure of that chain ring, i.e. under which condition the ring is nearly simple heavily depends on Γ^+, for instance an illustration can be found in Corollary 3.7 of this paper.

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