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PRIME IDEALS IN RIGHT CHAIN RINGS

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Our investigation of prime ideals in right chain rings has two different roots. During a classification of Hjelmslev planes according to their ideal type [9] the third author came across the following at that time hypothetical case: a Hjelmslev ring with just two proper ideals, namely the Jacobson radical J and (O), and $J^2 = J$ in contrast to the case of the well known uniform planes. Also the question about the existence of a special Hjelmslev ring with few two-sided ideals was left open in a module classification by Osofsky [7].

Essential in both situations was the question whether in chain rings there exist prime ideals which are not completely prime. For the internal understanding of chain rings this question is crucial as the structure of the lattice of prime ideals is closely related to the invariance properties of the ring.

Recently Dubrovin constructed examples of the type above in two papers [4,5] which became the starting point for [1] and this article.

The first section contains some useful preliminary results and demonstrates typical arguments for chain rings. In the following section we obtain criteria to decide whether certain ideals are completely prime. Moreover, it is shown that in chain rings finitely generated non-zero prime ideals are maximal, either as completely prime ideals or as not completely prime prime ideals. Section 3 is devoted to the analysis of the segment between two prime ideals. It turns out that a prime ideal which is not completely prime is always closely connected to a completely prime ideal.

Also, the existence of a not completely prime prime ideal implies special properties of the lattice of right ideals above it. In section 4 we sketch the arithmetic of not completely prime prime ideals, and in section 5 we describe the segment between a not completely prime prime ideal and the next completely prime ideal (or (O)) below it for chain rings. The paper ends with a section in which we construct an example of a not finitely generated not completely prime prime ideal, using a result of Dubrovin [5].

1. Preliminaries

All rings R are not necessarily commutative and have a unit $1 \in R$. We denote by $J(R) = J$ the Jacobson radical; $U(R) = U$ stands for the group of units.

DEFINITION 1.1: *A local ring* R *is called nearly simple if* J *and* (O) *are the only twosided ideals of* R.

DEFINITION 1.2: *A right chain ring* R *is a ring with* $aR \subseteq bR$ *or* $bR \subset aR$ *for any two elements* a,b *in* R. *If also* $Ra \subseteq Rb$ *or* $Rb \subset Ra$ *holds for all* $a,b \in R$, R *is called a chain ring. A right chain ring* R *is a right Hjelmslev ring (sometimes affine Hjelmslev ring) if each element in* J *is a twosided zero divisor. Analogously, we speak of a Hjelmslev ring (sometimes projective Hjelmslev ring).*

The following results will be used repeatedly

<u>LEMMA 1.3:</u> *Let* R *be a right chain ring,* $a \in R$, $r \in J$. *Then there exist elements* $r_1 \in U$, $r_2 \in J$ *with* $r = r_1 r_2 = r_2 r_1$ *and* $r_2 a = a r_2'$ *for some* $r_2' \in R$.

<u>PROOF:</u> If $ra = ar_2'$ we are done. Otherwise $ras = a$ for some s in J and $ra(1+s) = (1+r)a$. But r,s in J implies that $(1+r)^{-1}$ and $(1+s)^{-1}$ exist and $(1+r)^{-1}ra = a(1+s)^{-1}$. The lemma follows with $r_1 = 1+r$, $r_2 = (1+r)^{-1}r$.

<u>COROLLARY 1.4:</u> *Let* R *be a right chain ring. For* x,y *with* $Ux \subseteq yR$ *we have* $Rx \subseteq yR$.

<u>LEMMA 1.5:</u> *Let* I *be a right ideal in a right chain ring* R, *Then* $I_1 = \cup uI$ *is the minimal twosided ideal containing* I *and* $I_2 = \cap uI$ *is the maximal twosided ideal contained in* I *where* u *runs through* U.

<u>PROOF:</u> By Lemma 1.3 we have $I_1 = RI$. This is clearly the minimal twosided ideal over I. Any twosided ideal contained in I is contained in uI for every $u \in U$. That I_2 is twosided follows from Lemma 1.3.

2. Prime ideals in right chain rings

DEFINITION 2.1: *Let* R *be a ring and* P *a right ideal.*

(*i*) P *is called completely prime* (*c.p. for short*) *if and only if* $xy \in P$ *implies* $x \in P$ *or* $y \in P$.

(*ii*) P *is called prime if and only if* $xRy \subseteq P$ *implies* $x \in P$ *or* $y \in P$.

In [8] it was proved that $x^2 \in P$ implies $x \in P$ is necessary and sufficient for a prime ideal P to be completely prime. For right chain rings we even have the following:

LEMMA 2.2: *Let* R *be a right chain ring and* P *a twosided ideal. Then we have:*
P *is a completely prime ideal if and only if* $x^2 \in P$ *implies* $x \in P$.

PROOF: If x,y are not in P and xy is in P we have either $x = ys$ or $y = xs$ for some s in R. In the first case $x^2 = xys$ is in P and hence x in P. In the second case we have $y^2 = yxs$ in P if yx is also in P and y in P follows. This last assummption is always satisfied since $(yx)^2$ is in P.

With the following result we have a method at hand to get completely prime ideals in right chain rings.

THEOREM 2.3: *Let R be a right chain ring.*

(*i*) *Nonzero idempotent ideals are completely prime.*

(*ii*) *If I is an ideal which is not nilpotent then* $\bigcap_n I^n$ *is a completely prime ideal.*

(*iii*) *If* $t \in R$ *is not nilpotent, then* $P = \bigcap_n t^n R$ *is a prime right ideal. Moreover, if P is a two-sided ideal, then P is completely prime.*

PROOF: (i) Let $(0) \neq I = I^2$. Suppose $a \notin I$ but $a^2 \in I$. Then $I \subseteq aR$ and thus $I = I^2 \subseteq aI \subseteq a^2R \subseteq I$. Hence $I = a^2R$. But nonzero idempotent ideals are never finitely generated as right ideals. This contradiction shows that I is c.p.

(ii) Set $P = \bigcap_n I^n$. If $t \notin P$, then there exists $n \in \mathbb{N}$ with $I^n \subseteq tR$. Suppose $t^2 \in P$ then we obtain $I^{2n} \subseteq tI^n \subseteq t^2R \subseteq P \subseteq I^{4n} \subseteq I^{2n}$. Hence $P = I^{2n} = I^{4n}$ is idempotent and thus by (i) c.p. as I is not nilpotent.

(iii) For the first assertion it suffices to show that $xRx \subseteq P$ implies $x \in P$. Suppose not, then $t^nR \subseteq xR$ for some n, hence $t^{2n}R \subseteq t^nxR \subseteq xRxR \subseteq P$. But $t^{2n} \notin P$ as t is not nilpotent, contradiction. Now assume that P is a twosided ideal. Let $x \in R$ with $x^2 \in P$. If $x \notin P$ then $t^n = xa$ for some n, where $a \notin xR$ because $t^n \notin P$. Hence $x = ar$ for some $r \in J \setminus P$. Now for some m we have $t^m = rs$ with $s \in R$ and thus $t^{n+m} = xars = x^2s \in P$ - contradiction.

REMARK 2.4: *It is easy to prove the following:*

(i) *If I is an ideal of R, then there is no prime ideal P with* $\bigcap_n I^n \subsetneqq P \subsetneqq I$.

(ii) *If* $t \in R$ *then there is no completely prime ideal P with* $\bigcap_n t^n R \subsetneqq P \subsetneqq tR$.

REMARK 2.5: *The example in [4] (see also section 6) shows that* $\bigcap_n t^n R$ *is in general not a c.p. right ideal for* $t \in R$ *not nilpotent.*

3. The ideal lattice between two neighbour prime ideals

Let R be a right chain ring and $P \supset Q$ neighbour prime ideals, that means there are no further prime ideals different from P, Q and lying between P, Q. If there exists a twosided ideal I with $P \supset I \supset Q$ we get a chain of twosided ideals

$P \supset I \supset I^2 \supset \ldots \supset I^n \supset \ldots \supset \bigcap I^n = Q$, hence Q is completely prime!

Even more is true:

PROPOSITION 3.1: *Let* R *be a right chain ring,* $P = P^2$ *a completely prime ideal. Then*

(i) *For any ideal* $I \subsetneq P$ *which is not prime,* P/I *is not simple.*

(ii) *If* P/Q *is not simple, then we have: for any* $x \in P \setminus Q$ *there exists an ideal* I *with* $xR \subseteq I \subsetneq P$.

PROOF: (i) Suppose $I \subsetneq P$ is a twosided ideal with P/I simple. Now let X, Y be ideals of R with $XY \subseteq I$. If $X \not\subseteq I$ and $Y \not\subseteq I$, then we must have $P \subseteq X$ and $P \subseteq Y$ as P/I is simple. Hence $P = P^2 \subseteq XY \subseteq I$. Contradiction. Thus $X \subseteq I$ or $Y \subseteq I$. But this shows that I is a prime ideal.

(ii) Suppose not, that is, for any ideal X with $Q \subsetneq X \subsetneq P$ we have $X \subsetneq xR$. Set $I = \bigcup_{X \triangleleft R,\, X \subseteq xR} X \subseteq xR \subsetneq P$
As R is a right chain ring, I is an ideal of R and $Q \subsetneq I$ as P/Q is not simple. Now by (i), P/I is not simple, hence there exists an ideal I' with $I \subsetneq I' \subsetneq P$. Moreover, by our assumption $I' \subsetneq xR$. But then, by definition of I, $I' \subseteq I$. Contradiction. Thus there must be an ideal I with $xR \subseteq I \subsetneq P$.

COROLLARY 3.2: *Let* R *be a right chain ring,* P, Q *neighbour prime ideals and* P/Q *not simple. Then for any* $x \in P \setminus Q$ *we have* $\bigcap_n x^n R = Q$.

PROOF: If $P \neq P^2$, then for any $x \in P \setminus Q$ we have $\bigcap_n x^n R \subseteq \bigcap_n P^n = Q$ where the last equation follows from the fact that $\bigcap_n P^n$ is the minimal prime ideal below P. Moreover, $Q = \bigcap_n P^n$ is a completely prime ideal by Theorem 2.3. Therefore $Q \subseteq \bigcap_n x^n R$, hence $Q = \bigcap_n x^n R$. Thus we can now assume $P = P^2$. Let $x \in P \setminus Q$. By Proposition 3.1 there exists an ideal I with $xR \subseteq I \subsetneqq P$. Hence $\bigcap_n x^n R \subseteq \bigcap_n I^n$ and $\bigcap_n I^n = Q$, as $\bigcap_n I^n$ is the minimal prime ideal below I. Again, Q is a completely prime ideal and thus we also have $Q \subseteq \bigcap_n x^n R$.

Now we turn our attention to the case where P/Q is simple. Note that there are chain rings R constructed by Mathiak [6] and Dubrovin [4] with P = J and Q = (O) completely prime, hence nearly simple.

PROPOSITION 3.3: *Let* R *be a right chain ring,* $A \subseteq B$ *ideals of* R *with* B/A *simple. Let* $x \in B \setminus A$ *with* $x^2 \notin A$ *and* $xR \neq B$. *Then there exists a unit* u *with* $A \subsetneqq xR \subseteq \bigcap_n (ux)^n R$, *in particular also* $A \subsetneqq \bigcap_n (xu)^n R$.

PROOF: Let $x \in B \setminus A$, $u \in U$. If $ux \in xR$, then $ux^2 \in xR$. If $x = uxw$ for some $w \in R$, we consider the following two cases:

(i) $x = wq$. Then $ux^2 = uxwq = xq \in xR$.

(ii) $w = xq$. Then $x = uxw = ux^2q = ux(ux^2q)q = (ux)^2xq^2 = (ux)^n xq^n$ for all $n \in \mathbb{N}$. Thus $x \in \bigcap_n (ux)^n R$.

Hence in this case $A \subsetneqq xR \subseteq \bigcap_n (ux)^n R$, and as A is a twosided ideal also $A \subsetneqq \bigcap_n (xu)^n R$. But if this case does not occur we get $ux^2 \in xR$ for any $u \in U$ and thus $Rx^2R \subseteq xR$. By assumption $x^2 \notin A$ and $xR \neq B$, hence $A \subsetneqq Rx^2R \subseteq xR \subsetneqq B$ contradicting the fact that B/A is simple.

COROLLARY 3.4: *Let R be a right chain ring, P,Q neighbour prime ideals with P/Q simple. Then for any $x \in P \setminus Q$ there exist units $u,v \in U$ with $Q \subsetneqq xR \subseteq \bigcap_n (uxv)^n R$ and $Q \subsetneqq \bigcap_n (xvu)^n R$. If Q is completely prime we can choose $v = 1$.*

PROOF: First of all, note that $P^2 = P$, since otherwise P/Q is not simple. This implies that $xR \subsetneqq P$ for any $x \in P$. If Q is c.p., then clearly $x^2 \notin Q$ for any $x \in P \setminus Q$. If Q is prime but not c.p., then for any $x \in P \setminus Q$ there exists a unit v with $(xv)^2 \notin Q$, since otherwise $xRx \subseteq Q$, contradicting the fact that Q is prime. Now apply Proposition 3.3 to x or xv, respectively.

See Example 6.5 where this "effect" is "visualized".

The case P/Q simple is the normal situation if Q is prime, but not completely prime.

<u>THEOREM 3.5:</u> *Let* R *be a right chain ring,* Q *a prime ideal which is not completely prime and* P *the intersection of all completely prime ideals containing* Q. *Then* $P = P^2$ *and there are no twosided ideals between* P *and* Q *different from those two ideals. Moreover,* $Q \neq (0)$ *implies* $Q^2 \neq Q$ *and* Q *is nilpotent or* $\cap Q^n$ *is completely prime.*

<u>PROOF:</u> Let I be a twosided ideal with $Q \subsetneq I \subseteq P$. As Q is prime, $Q \subsetneq I^n$ for all n, hence $Q \subseteq \bigcap_n I^n$. But $\bigcap_n I^n$ is a c.p. ideal by Theorem 2.3, hence $P = \bigcap_n I^n \subseteq I^2 \subseteq I \subseteq P$. This implies $I = P$ and $P = P^2$. The last assertion follows from Theorem 2.3.

We notice that non-completely prime prime ideals are always pairing with a completely prime ideal.

The first example of a not completely prime prime ideal in a chain ring was given by Dubrovin [5] (see section 6).

<u>LEMMA 3.6:</u> *Let* R *be a right chain ring,* $P \supset Q$ *neighbour prime ideals and* Q *not completely prime. Further let* $x \in P \setminus Q$ *be not* Q*-nilpotent. Then* $Q \subsetneq \bigcap_n x^n R$.

PROOF: As x is not Q-nilpotent, $Q \subseteq \bigcap_n x^nR$. If $Q = \bigcap_n x^nR$, then Theorem 2.3 (iii) would imply Q completely prime, contradicting our assumption on Q.

It is natural to ask the following question:

PROBLEM: *Let P/Q be simple and Q completely prime. Does there always exist an element $x \in P\setminus Q$ with $\bigcap_n x^nR = Q$?*

By Corollary 3.4 elements which are not Q-nilpotent are "everywhere" between P and Q; the same is true for Q-nilpotent elements.

PROPOSITION 3.7: *Let R be a right chain ring, $P \supset Q$ neighbour prime ideals and Q not completely prime.*

(i) For each $x \in P\setminus Q$ exists at least one unit v with $(xv)^2 \in Q$.

(ii) Let $x \in P\setminus Q$ be Q-nilpotent. If Q is nilpotent, then $\bigcap_n x^nR = \{0\}$. If Q is not nilpotent, then $\bigcap_n x^nR = \bigcap_n Q^n$ is a completely prime ideal.

PROOF: (i) Let x be in $P\setminus Q$ and $Q = \bigcap uxR$, where u runs through U, follows from Lemma 1.5. If xUx is contained in $P\setminus Q$ we can conclude that Q is completely prime: Let y be not in Q. Then there exists $s \in J$, $u \in U$ with $ys = ux$.

Let $s_1 \in U$ with $s_1sy = sy'$ (Lemma 1.3). Using $s_1 \in U$, $ux \in P \setminus Q$ and our assumption we have $uxs_1ux = yss_1ys = ys_1sys = y^2s's$ in $P \setminus Q$ and y^2 is not in Q. Lemma 2.2 shows that Q is completely prime. The contradiction shows that our assumption xUx in $P \setminus Q$ for $x \in P \setminus Q$ is wrong and proves statement (i).

(ii) If x is Q-nilpotent then $\bigcap_n x^nR \subseteq \bigcap_n Q^n$. For nilpotent Q the assertion is clear. If Q is not nilpotent then $\bigcap_n Q^n$ is a completely prime ideal. Now $\bigcap_n x^nR \subsetneqq \bigcap_n Q^n$ would imply $x^mR \subseteq \bigcap_n Q^n$ for some m, hence - as $\bigcap_n Q^n$ is completely prime - $x \in \bigcap_n Q^n \subseteq Q$. Contradiction. Thus $\bigcap_n x^nR = \bigcap_n Q^n$.

4. Arithmetic of not completely prime prime ideals

In the following, Q will always denote a prime ideal which is not c.p., and P the minimal c.p. ideal containing Q.

PROPOSITION 4.1: *Let* R *be a right chain ring. Then we have:*

(i) *Let* $s \notin P$. *Then* $ts \in Q$ *implies* $t \in Q$.

(ii) $PQ \subseteq QP$.

PROOF: (i) As $s \notin P$, $P \subseteq sR$. Thus $tP \subseteq tsR \subseteq Q$. Now Q is prime and $P \nsubseteq Q$, hence $t \in Q$.

(ii) Set $I = \{x \in P \mid xQ \subseteq QP\}$. As Q is a twosided ideal, so is I.

Obviously, $Q \subseteq I \subseteq P$. As $P \setminus Q$ is simple (by Theorem 3.5) we must have $I = Q$ or $I = P$. Using Proposition 3.7 we get $x \in P \setminus Q$ with $x^2 \in Q$. Now let $z \in Q$. As R is a right chain ring, $z = xa$ for some $a \in R$. By (i) $a \in P$, hence $xz = x^2 a \in QP$. Thus $I = P$.

COROLLARY 4.2: *Let* R *be a chain ring. Then we have:*
(i) *Let* $s \notin P$ *and* $st \in Q$, *then* $t \in Q$.
(ii) $PQ = QP$.

PROBLEM: *If* R *is a chain ring, do we always have* $PQ \neq Q$?

It seems that only under further conditions a stronger result (see Proposition 4.5) can be obtained for right chain rings.

First a general lemma:

LEMMA 4.3: *Let* Q *be a nonzero ideal of a local ring* R *with* $Q = aR = Rb$. *Then we also have* $Q = Ra = bR$.

PROOF: There exist $r, s \in R$ with $a = rb$, $b = as$. Hence $a = ras = ar's$ for some $r' \in R$, as $ra \in Q = aR$. If $r's \in J(R)$, then we must have $a = 0$, contradiction. Hence $r's \in U$ and so $s \in U$. Similarly, $r \in U$. So $aR = Rb = Rrb = Ra$ and $bR = asR = aR = Rb$.

PROPOSITION 4.4: *Let R be a right chain ring which is a domain and Q finitely generated as left and right ideal. Then* $st \in Q$, $s \notin P$ *implies* $t \in Q$.

PROOF: By Lemma 4.4 there exists $q \in R$ with $Q = qR = Rq$. As P is completely prime, $st \in Q \subseteq P$ implies $t \in P$. By 3.7 we can assume that $t^2 \in Q$. There exists $r \in R$ such that $st = rq$, because $st \in Q = Rq$. Now suppose $t \notin Q$. Then $q = ta$ for some $a \in P$ and $qa = bq$ for some $b \in R$. Now $sq = sta = rqa = rbq$ and as R is a domain this implies $s = rb$, so $b \notin P$. Hence, $st = rbt = rq$, so $q = bt$. Thus $q^2 = btta = bcqa$ for some $c \in R$, as $t^2 \in Q$, and we get $q^2 = bcbq$, hence $q = bcb$. Since $b \notin P$, by 4.1 (i) we obtain $bc \in Q$. So $bc = qd$ for some $d \in R$. Therefore $q = qdb$ and thus $1 = db$. But now $t = dbt = dq \in Q$. Contradiction.

In chain rings prime ideals are "seldom" finitely generated:

THEOREM 4.5: *Let R be a chain ring, P a prime ideal of R. If P is finitely generated as right ideal, we have one of the following situations:*

(i) $P = (0)$

(ii) $P = J$ *and* $P = Ra = aR$ *for some* $a \in R$

(iii) *P is the maximal prime ideal below J, P is not completely prime and* $P = Ra = aR$ *for some* $a \in R$.

PROOF: Assume $0 \neq P \neq J$ and $P = aR$. Let $x \in J \setminus P$. Then there exists $r \in J$ with $a = rx$. If P is c.p. then $r = as$ for some s, hence $a = asx$ and $sx \in J$ implies $a = 0$, contradiction. Thus P is not completely prime. Now let Q be the minimal prime ideal above P; this is c.p. by Theorem 3.5. Suppose $Q \neq J$. Let $x \in J \setminus Q$. Then $a = rx$ for some $r \in R$. By Proposition 4.1(i), $r \in P$, say $r = as$. But now $a = asx$ and $sx \in J$ implies $a = 0$ - contradiction. Thus $Q = J$. It remains to show $P = Ra$ if $P \neq 0$. Clearly, $Ra \subseteq aR$. If $ar \notin Ra$ for some $r \in R$, then $a = sar$ for some $s \in J$. For $P = J$ $s = at$ for some $t \in R$ and hence $a = atar$. As $tar \in J$ this implies $a = 0$. Contradiction. If $P \neq J$, the above and Proposition 4.1(ii) gives $sa \in aJ$, say $sa = at$ with $t \in J$. Now $a = atr$ with $tr \in J$ again implies $a = 0$. Contradiction.

REMARKS 4.6: *(a) This theorem is not true for right chain rings* [1].

(b) In Section 6 we shall discuss an example for situation (iii) given by Dubrovin [5].

(c) The proof above shows: if $J = aR$ *in a left chain ring, then also* $J = Ra$.

5. Investigation of $Q/\bigcap_n Q^n$

It is plausible that the fact that there are no ideals between a not completely prime prime ideal Q and the minimal completely prime ideal P containing Q has consequences for other parts of the ideal lattice, in particular for the following segment. As the situation for chain rings is much clearer we restrict ourselves to this class of rings. First an observation:

LEMMA 5.1: *Let* R *be a right chain ring,* Q *prime but not completely prime,* P *the minimal completely prime ideal over* Q.

(i) *For any* $a \in Q$ *there are* $x,y \in P \setminus Q$ *with* $aR \subseteq xyR \subseteq Q$.

(ii) *If* x, y *are in* $P \setminus Q$ *with* $xy \in Q$, *then* $xy \in Q \setminus Q^2$ *if* $Q^2 \neq (0)$.

PROOF: (i) By Proposition 3.7 there exists $x \in P \setminus Q$ with $x^2 \in Q$. Now if $a \in Q$ then $a = xy$ for some $y \in P$ by 4.1. If $y \notin Q$ we are done. So we can assume $y \in Q$. But then $y = xs$ for some $s \in J$ and we obtain $aR = xyR = x^2sR \subseteq x^2R$ with $x \in P \setminus Q$.

(ii) Suppose $xy = ab$ with $a,b \in Q$. Then $a = xr$ for some $r \in R$, and $xy = xrb$. Now $y - rb \notin Q$ as $y \notin Q$ but $b \in Q$. Hence $Q^2 \subseteq xQ \subseteq x(y-rb)Q = (0)$. Contradiction.

<u>LEMMA 5.2:</u> *Let R be a right chain ring, I a non-zero ideal. Then the following are equivalent:*

(i) I *is not finitely generated as right ideal.*

(ii) I = IJ

<u>PROOF:</u> If I is finitely generated as right ideal, say I = qR, then obviously $q \notin qJ = IJ$. So (ii) implies (i). If I is not finitely generated as right ideal, take $a \in I$. Then there exists $b \in I$ with $aR \subsetneqq bR$, hence $a = bs$ with $s \in J$. Thus $a \in IJ$.

<u>PROPOSITION 5.3:</u> *Let* R *be a chain ring,* Q *a not completely prime prime ideal with* J *the minimal completely prime ideal containing* Q . *If* I *is a twosided ideal with* $Q^n \subset I \subset Q^{n-1}$ *or* $(0) \subset I \subset Q^{n-1}$ *with maximal n, then one of the following holds:*

(i) $I = Q^{n-1}J$

(ii) $I = sR$ *with* $Js \subseteq sJ = Q^n$ *or* $Js = sJ = (0)$.

<u>PROOF:</u> We can assume that $Q^n = (0)$. Suppose $I \neq Q^{n-1}J$. Then there exists $a \in Q^{n-1}J \setminus I$ with $aJ \not\subseteq I$, since if $aJ \subseteq I$ for all $a \in Q^{n-1}J \setminus I$ we would get $I \supseteq (Q^{n-1}J)J = Q^{n-1}J$ - contradiction. Now the right annihilator I^r of I is a twosided ideal with $Q \subseteq I^r \subseteq J$. By Theorem 3.5 we get $I^r = Q$ or $I^r = J$.

CASE 1: $I^r = Q$. Let $b \in Q^{n-1}J \setminus I$. As $I \subset Rb \subset Q^{n-1}$ we have $Q \subseteq (Q^{n-1})^r \subseteq (Rb)^r \subseteq I^r = Q$, hence $Q = (Rb)^r = I^r$. Now let $x \in J \setminus Q$. By 3.7 there exists $u \in U$ with $(xu)^2 \in Q$. If $ax \notin I$, then also $axu \notin I$ and by the above $Q = (Raxu)^r$. Thus $axuxu = 0$ since $(xu)^2 \in Q = (Ra)^r$, but $xu \notin Q$. Contradiction. Hence $ax \in I$ for any $x \in J \setminus Q$ and so $aJ \subseteq I \subsetneq aR$ which implies $I = aJ$, contradicting the choice of a.

CASE 2: $I^r = J$. Suppose I is not finitely generated as right ideal. Let $0 \neq r \in I$, then there exists $s \in I$ with $rR \subsetneq sR$, hence $r = st$ with $t \in J$ and thus $r = 0$, contradiction. Therefore I is finitely generated as right ideal, say $I = sR$, and, of course, $sJ = (0)$. In particular, this implies that for $u \in U$ we have $us = sv$ with a unit v. Let $x \in J \setminus Q$. Then there exists a unit $w \in U$ with $(xw)^2 \in Q$. If $xs = sy$ with $y \in U$, then $xwxws \in sU$, contradicting the fact that $Qs = (0)$. Thus $Js \subseteq sJ$.

REMARK: *We have used the fact that* R *is a left chain ring only to get* $I \subset Ra \subset Q^{n-1}$. *It is possible to rewrite the proof so as to use this fact at another point but we could not get rid of this assumption.*

COROLLARY 5.4: *If Q is finitely generated as right ideal, then any ideal* $\bigcap_n Q^n \subsetneqq I \subseteq Q$ *is of the form* Q^i *or* Q^iJ.

PROOF: If Q is finitely generated as right ideal, then so is every Q^i and hence $Q^i \neq Q^iJ$. But by Proposition 5.3 the existence of an ideal which is not of the form Q^n or Q^nJ implies $Q^k = sJ = sJ^2 = Q^kJ$ for some k, contradiction.

If we assume that Q is finitely generated as left ideal we can get a more general result even for a right chain ring:

PROPOSITION 5.5: *Let R be a right chain ring, Q a prime ideal which is not completely prime and P the minimal completely prime ideal containing Q. If Q is finitely generated as left ideal then there is no twosided ideal I with* $Q^n \subsetneqq I \subsetneqq Q^{n-1}P$ *for any* $n \in \mathbb{N}$.

PROOF: Suppose I is an ideal with $Q^n \subsetneqq I \subsetneqq Q^{n-1}P$. Let $L = \{x \in P \mid Q^{n-1}x \subseteq I\}$. Then L is an ideal of R and $Q \subseteq L \subseteq P$. By Theorem 3.5 $L = Q$ or $L = P$. As $I \subsetneqq Q^{n-1}P$ we obtain $L = Q$. As Q is finitely generated as left ideal, $Q = Rq$ for some $q \in R$. Now $q^{n-1}P \supsetneqq I$ since otherwise $Q^{n-1}P = Rq^{n-1}P \subseteq I$. Let $z \in I \setminus Q^n$. Then $z = q^{n-1}r$ with $r \in P \setminus Q$. Hence $Q^{n-1}r = Rq^{n-1}r = Rz \subseteq I$, and by definition of L, $r \in L = Q$. Contradiction.

6. Examples

In [1] the concept of rings with weak decomposition property was introduced which was in a modified form already in [5]. Before we will sketch it for the special case of chain rings we state some definitions. Let Γ be a group. A multiplicative semigroup Γ^+ is called a generalized positive cone in Γ, if $\alpha \in \Gamma \setminus \Gamma^+$ implies $\alpha^{-1} \in \Gamma^+$, and $\Gamma^+ \cap (\Gamma^+)^{-1} = \{1\}$. Then a left order is defined by setting $\alpha \leq_\ell \beta$ ($\alpha \leq_r \beta$, respectively) if $\alpha^{-1}\beta \in \Gamma^+$ ($\beta\alpha^{-1} \in \Gamma^+$, respectively). With this definition we get:

$\alpha \leq_\ell \beta$ implies $\gamma\alpha \leq_\ell \gamma\beta$ for all $\gamma \in \Gamma$

$\alpha \leq_r \beta$ implies $\alpha\gamma \leq_r \beta\gamma$ for all $\gamma \in \Gamma$.

Then Γ is also said to be a left - (respectively, right -) ordered group (see Conrad [3]).

The following definitions are recalled from [5].

DEFINITION 6.1: *Let Γ be a group with generalized positive cone Γ^+. A ring R is said to be associated with (Γ, Γ^+), if there is a monomorphism μ of Γ^+ into the multiplicative monoid of the ring R, such that for any $r \in R \setminus \{0\}$ there exist $\alpha_1, \alpha_2 \in \Gamma^+$ with $rR = \mu(\alpha_1)R$ and $Rr = R\mu(\alpha_2)$ and such that $\mu(\Gamma^+ \setminus \{1\}) \subseteq J(R)$.*

This is Dubrovins's definition (note that in [5] the last condition is inadvertently omitted) whereas the weak decomposition property defined in [1] is more general.

Before we shall describe the ideal structure of R by means of semigroups we require some notations.

DEFINITION 6.2: *Let* Γ *be a group with generalized positive cone* Γ^+. *A subset* $\Omega \subseteq \Gamma^+$ *is a right (twosided) ideal of* Γ *if for all* $\alpha \in \Omega$, $\beta \in \Gamma^+$: $\alpha\beta \in \Omega$ $(\alpha\beta, \beta\alpha \in \Omega)$.
An ideal Ω *of* Γ^+ *is called prime if for any ideals* Φ, Ψ of Γ^+, $\Phi\Psi \subseteq \Omega$ *implies* $\Phi \subseteq \Omega$ or $\Psi \subseteq \Omega$.
An ideal Ω *of* Γ^+ *is completely prime (c.p.) if for any* $\alpha, \beta \in \Gamma^+$ $\alpha\beta \in \Omega$ *implies* $\alpha \in \Omega$ *or* $\beta \in \Omega$.

PROPOSITION 6.3: [5] *Let* Γ, Γ^+, R, μ *be as in Definition 6.1. For any right ideal* Ω *in* Γ^+ *define*
$\bar{\mu}(\Omega) = \{r \in R \mid r = \mu(g)s$ *for some* $g \in \Omega$, $s \in R\} = \mu(\Omega)R$.
Then $\bar{\mu}$ *is a bijection from the set of right ideals of* Γ^+ *into the set of non-zero right ideals of* R, *which preserves inclusion. Moreover, if* $\bar{\mu}(\Omega)$ *is an ideal of* R, *then* Ω *is an ideal of* Γ^+.

PROPOSITION 6.4: *Let the notation be as in Proposition 6.3.*

(i) *If Ω is a prime ideal of Γ^+, then $\bar{\mu}(\Omega)$ is a prime ideal of* R.

(ii) *If Ω is a c.p. ideal of Γ^+ and $\bar{\mu}(\Omega)$ is a two--sided ideal of* R, *then $\bar{\mu}(\Omega)$ is a c.p. ideal of* R. *Conversely, if $\bar{\mu}(\Omega)$ is a c.p. ideal of* R, *then Ω is a c.p. ideal of Γ^+.*

(iii) *Ω is a finitely generated ideal of Γ^+ if and only if $\bar{\mu}(\Omega)$ is a finitely generated ideal of* R.

PROOF: Straightforward.

EXAMPLE 6.5: *In [4] Dubrovin constructs a ring* R *associated with a right-ordered group such that* R *is nearly-simple with no zero-divisors. Now we want to describe the "height" of the intersection right ideals* $\bigcap_n x^n R$, *for the sake of simplicity we restrict ourselves to elements* $x = \mu(\alpha)$. *However, α can be interpreted as an affine linear function on $\mathbb{Q}$ with* $t \rightarrow at + b$ *(see [1]). To simplify our notation we identify* x *with α. Then $\alpha \in \Gamma^+$ if $\alpha(\varepsilon) \geq \varepsilon$ for a chosen irrational number ε. A short computation shows*

$\alpha^n = (t \rightarrow a^n t + (a^{n-1} + a^{n-2} + \ldots + 1)b)$

CASE 1: $0 < a < 1$ *and* $a\varepsilon + b > \varepsilon$. *Then* $\lim_{n\rightarrow\infty} (t \rightarrow at + b)^n = (t \rightarrow \frac{1}{1-a} b)$

Thus for every proper right ideal yR *we can find suitable* a,b *such that* $yR \subseteq \bigcap_n x^n R$, *where* $x = \mu(t \to at + b)$.

CASE 2: $1 < a$ *and* $\alpha(\varepsilon) = a\varepsilon + b > \varepsilon$. *Set* $\delta = \alpha(\varepsilon) - \varepsilon$. *By induction,* $\alpha^n(\varepsilon) - \varepsilon = (a^{n-1} + a^{n-2} + \ldots + 1)\delta$, *hence* $\lim_{n\to\infty} \alpha^n(\varepsilon) = \infty$. *This means that for* $x = \mu(\alpha)$ *we have* $\bigcap_n x^n R = (0)$.

It is obvious that elements of the above types are on every "level" in the lattice of ideals.

In the following we will start from a construction given by Dubrovin [5], and using this we will obtain a chain ring which has a prime ideal which is not completely prime and not finitely generated. Let $L = \langle y; x_i, i \in \mathbb{N} \mid x_i y^2 x_i = y, x_{i+1}{}^2 y x_{i+1} = x_i \text{ for all } i \in \mathbb{N} \rangle$. L is a right-ordered group with positive cone Q, which is the monoid generated by y and all x_i, $i \in \mathbb{N}$ (see [5]). Set $G = L \times \mathbb{Z}$. Then $P = \{(x,z) \mid x \in Q \text{ or } x = 1 \text{ and } z \in \mathbb{N}_0\}$ is a positive cone for G, which induces the lexicographical order on G. For an ideal S of Q, define $S^{(1)} = \{(g,z) \in P \mid g \in S, z \in \mathbb{Z}\}$, this is an ideal of P.

LEMMA 6.6: *Let* S *be an ideal of* Q.

(*i*) S *is a prime ideal of* Q *if and only if* $S^{(1)}$ *is a prime ideal of* P.

(ii) S *is a c.p. ideal of* Q *if and only if* $S^{(1)}$ *is a c.p. ideal of* P.

PROOF: Straightforward

PROPOSITION 6.7: *Let* K *be a field. Then* KG *is embeddable into a division ring* D, *and there exists a chain ring* R *in* D *which contains* KP *and is associated with* G.

PROOF: KG is embeddable into a division ring, since KL is embeddable into a division ring and $G/L \approx \mathbb{Z}$. The second assertion follows from [5, Theorem 1].

PROPOSITION 6.8: *There exists a prime ideal in* R *which is not c.p. and which is not finitely generated as right ideal.*

PROOF: Set $S = y^3Q$. This is a prime ideal which is not c.p. [5]. Hence $S^{(1)}$ is a prime ideal which is not c.p.. As $y^3 \in Z(L)$, $(y^3,0) \in Z(G)$, hence $(y^3,0) \in Z(R)$. Thus $\bar{\mu}(S^{(1)}) = \mu(S^{(1)})R$ is a prime ideal of R which is not c.p., where μ is the monomorphism of P into R according to Proposition 6.7. Moreover, $S^{(1)}$ is not finitely generated as right ideal. To see this, suppose $S^{(1)} = (g,n)P$ for some $(g,n) \in P$, $g \neq 1$. Now $(y^3,n-1) \in S^{(1)}$, hence there exists $(h,m) \in P$ with $(y^3,n-1) = (gh, n+m)$.

This implies $m < 0$ and $y^3 = gh$ with $h \neq 1$. Thus $g \notin y^3Q$, so $(g,n) \in (g,n)P \setminus S^{(1)}$. Therefore, $\bar{\mu}(S^{(1)})$ is not finitely generated as right ideal by Proposition 6.4.

<u>REMARK 6.9:</u> *The minimal completely prime ideal above* $\bar{\mu}(S^{(1)})$ *is not* J. *To see this, note that we have an inclusion* $S^{(1)} \subsetneqq (1,2)P \subsetneqq P\setminus\{1\}$ *of twosided ideals of* P *which induces an inclusion* $\bar{\mu}(S^{(1)}) \subsetneqq \bar{\mu}((1,2)P) \subsetneqq \bar{\mu}(P\setminus\{1\}) = J$ *of twosided ideals of* R *since* $(1,2) \in Z(R)$. *Hence by Theorem* 3.5 J *is not the minimal c.p. ideal over* $\bar{\mu}(S^{(1)})$.

Note, that this fact also implies that $\bar{\mu}(S^{(1)})$ *is not finitely generated as right ideal.*

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