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Extensions of Chain Rings

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1. Introduction

It is the purpose of this paper to study the construction of integral domains R in which the lattice of right (left) ideals is linearly ordered. Such rings are called *right* (*left*) *chain rings*. The unique maximal ideal is denoted by $J(R)$ and the group of units by $U(R)$. If R contains exactly n prime ideals, not counting R and (0), we say that R has *rank n*. The question whether every right ideal in a right chain ring of rank one has to be two-sided was left unanswered in [3].

Using ideas of Mathiak and Rohlfing ([8, 9]) we construct a localization R_1 of an Ore extension of a chain ring R_0 with a monomorphism σ and a σ-derivation δ. The ring R_1 is an extension of R_0 with $J(R_1)\cap R_0=J(R_0)$ and is again a chain ring. However R_1 can turn out to be a rank one right chain ring without any two-sided ideals besides R_1, $J(R_1)$ and (0) even if R_0 is commutative or has infinite rank. This follows from an investigation of the relationship between right ideals (Sect. 3) twosided ideals, completely prime ideals (Sect. 4) and prime ideals in R_0 and R_1 (Sect. 5).

The construction discussed here fails to provide an answer to the question whether prime ideals exist in chain rings which are not completely prime. It will be proved (in the case $\delta=0$) that such an ideal can exist in the extension R_1 only if one existed in R_0 already, see Theorem 4.

The value group of a rank one commutative or right invariant right chain ring is isomorphic to a subgroup of the real numbers under addition. Examples of rank one chain rings can be constructed with the above mentioned method whose value groups can be considered as generalizations of Chehatas simple ordered group (Sect. 6).

2. The Construction

Let R_0 be a right chain ring with monomorphism σ and σ-derivation δ. We construct the Ore skew polynomial ring $R=R_0[x,\sigma,\delta]=\{\sum a_i x^i;\ a_i \text{ in } R_0\}$ with multiplication in R defined by $xa=\sigma(a)x+\delta(a)$ [6].

It is assumed that σ and δ are compatible with $J(R_0)$ in the following sense:

(V) i) $\sigma(r)$ is in $J(R_0)$ if and only if r is in $J(R_0)$ for r in R_0.
ii) $\delta(J(R_0))$ is contained in $J(R_0)$.

Let S be the subset of polynomials $\sum a_i x^i$ in R with at least one a_i in $U(R_0)$. To show that S is multiplicatively closed we consider $f(x)=\sum f_k x^k$, $g(x)=\sum g_t x^t$ in S with f_i and g_j the highest coefficient in $U(R_0)$. The coefficient h_{i+j} of x^{i+j} in $h(x)=f(x)g(x)$ has the form $h_{i+j}=\sum_{k+t=i+j} f_k \sigma^k(g_t)+\sum_{k+t>i+j} f_k g'_t$. Here g'_t denotes an element obtained from a coefficient g_t of $g(x)$ by applying a sequence of mappings σ or δ with δ appearing at least once. Since $k+t>i+j$ in the second sum, either f_k or g_t is in J and hence either f_k or g'_t is in $J(R_0)$ using condition (V). Every summand in the first sum is in $J(R_0)$ except the term $f_i\sigma^i(g_j)$ which is a unit. This shows that h_{i+j} is a unit and $h(x)$ is in S. This argument also shows that $f(x)g(x)$ can not be an element of S if one of the factors is not in S.

Since R_0 is a right chain ring, it is a right Ore-ring and the skew field of fractions $Q=Q(R_0)=\{ab^{-1};\ b\neq 0,\ a \text{ in } R_0\}$ exists. We need one further condition in order to make use of Q: given finitely many elements p_i, $i=1,\dots,n$ in Q then there exists $0\neq d$ in R_0 with dp_i in R_0 for all i. However, the following lemma shows that this condition holds if and only if R_0 is also a left chain ring.

Lemma 1. *Let R_0 be a right chain ring and Q its skew field of fractions. If for an arbitrary element q in Q an element $0\neq d$ in R_0 exists with dq in R_0 then R_0 is a left chain ring.*

Proof. Let $a, b\neq 0$ be elements in R_0 and $q=ab^{-1}$ in Q follows. Assume $dab^{-1}=r$ is in R_0 for some $d\neq 0$ in R_0. Then either $r=dr_1$ and $ab^{-1}=d^{-1}r=r_1$ or $d=rd_1$ and $d^{-1}r=d_1^{-1}$ for elements d_1, r_1 in R_0. In the first case $ab^{-1}=r_1$, $a=r_1 b$ and $R_0 b\supseteq R_0 a$ follows, in the second case $ab^{-1}=d_1^{-1}$, $b=d_1 a$ and $R_0 a\supseteq R_0 b$.

From now on it is assumed that R_0 is a right and left chain ring - chain ring for short. This implies that the elements of Q have the form a or a^{-1} where a is in R_0.

Lemma 2. *Let R_0 be a chain ring, σ a monomorphism, δ a σ-derivation of R_0 satisfying condition* (V). *Then there exists a natural extension of σ and δ respectively to Q defined through $\sigma(a^{-1})=\sigma(a)^{-1}$ and $\delta(a^{-1})=-\sigma(a)^{-1}\delta(a)a^{-1}$.*

The *proof* of this lemma can be given by considering several cases and will be omitted.

We will denote the extension of σ and δ to Q by σ, δ again. The skew polynomial ring $Q[x,\sigma,\delta]=\{\sum c^{-1}a_i x^i,\ 0\neq c,\ a_i \text{ in } R_0\}$ can be constructed and is a left principal ideal domain. We used the fact that R_0 is a left chain ring to write the elements in Q in the form $c^{-1}a$ with $0\neq c$, a in R_0. The skew field D of quotients of the ring $Q[x,\sigma,\delta]$ exists and elements in this skewfield have the form

$$(\textstyle\sum c^{-1}a_i x^i)^{-1}(\sum c^{-1}b_j x^j)=(\sum a_i x^i)^{-1}(\sum b_j x^j)$$

in R_1. This will be done in special cases, but not in general. Helpful in the following discussion if the introduction of an extension $\hat{R}_0$ of R_0 which allows an extension $\hat{\sigma}$ of the monomorphism σ of R_0 such that $\hat{\sigma}$ is an automorphism.

Lemma 4. *Assume R_0 is a chain ring with a monomorphism σ. Then there exists a chain ring $\hat{R}_0$ containing R_0 and an extension $\hat{\sigma}$ of σ such that $\hat{\sigma}$ is an automorphism of $\hat{R}_0$ and for every α in $\hat{R}_0$ there exists an n with $\hat{\sigma}^n(\alpha)$ in R_0. Condition* (V) *holds for $\hat{R}_0, \hat{\sigma}$ if it holds for R_0 and σ.*

Proof. We consider $R_2 = R_0[y, \sigma] = \{\sum a_i y^i;\ a^i \text{ in } R_0\}$; $ya = \sigma(\alpha) y$. The set $M = \{y^n, n = 0, 1, 2, \ldots\}$ is an Ore-system in R_2 and the ring $M^{-1} R_2 = \{x^{-n} \sum a_i x^i;$ a_i in $R_0\}$ exists. $R_3 = M^{-1} R_2$ contains the subset $\hat{R}_0 = \bigcup y^{-n} R_0 y^n$, $n = 0, 1, 2, \ldots$. We have $R_0^{(n)} = y^{-n} R_0 y^n \supseteq y^{-n} \sigma(R_0) y^n = y^{-(n-1)} R_0 y^{(n-1)} = R_0^{(n-1)}$ and it follows that $\hat{R}_0$ is a subring and again a chain ring. If one defines $\hat{\sigma}(y^{-n} a y^n) = y^{-n} \sigma(a) y^n$ it follows that this is the mapping induced on $\hat{R}_0$ by the inner automorphism of R_3 that sends u to yuy^{-1}. The same is true for the inverse of these mappings and this shows that $\hat{\sigma}$ is an automorphism of $\hat{R}_0$ and an extension of σ. Further, $J(\hat{R}_0) = \bigcup y^{-n} J(R_0) y^n$. Hence $\hat{R}_0$ and $\hat{\sigma}$ satisfy (V) if R_0 and σ do and this proves the lemma.

Can the σ-derivation δ also be extended from R_0 to the ring $\hat{R}_0$? This is possible if δ and σ commute i.e. if $\sigma\delta = \delta\sigma$. In that case one defines $\hat{\delta}(y^{-n} a y^n) = y^{-n} \delta(a) y^n$ where $y^{-n} a y^n$ is in $\hat{R}_0$ and a in R_0. It must be checked that this is well defined and is a $\hat{\sigma}$-derivation of $\hat{R}_0$. This means that $y^{-n} a y^n = y^{-m} b y^m$ must imply $y^{-n} \delta(a) y^n = y^{-m} \delta(b) y^m$ for elements a, b in R_0. We can assume that $n > m$ and

$$y^{-(n-m)} a y^{n-m} = b \quad \text{or} \quad a = \sigma^{n-m}(b) \quad \text{follows.}$$

Hence $y^{-n} \delta(a) y^n = y^{-n} \delta(\sigma^{n-m}(b)) y^n = y^{-n} \sigma^{n-m}(\delta(b)) y^n = y^{-m} \delta(b) y^m$ and the mapping $\hat{\delta}$ is well defined. It follows very easily that $\hat{\delta}(\alpha + \beta) = \hat{\delta}(\alpha) + \hat{\delta}(\beta)$ and $\hat{\delta}(\alpha\beta) = \hat{\sigma}(\alpha) \hat{\delta}(\beta) + \hat{\delta}(\alpha) \beta$ for elements α, β in $\hat{R}_0$. Finally, if the condition (V) holds for R_0, σ and δ it also holds for $\hat{R}_0$, $\hat{\sigma}$ and $\hat{\delta}$.

The ring $\hat{R}_0$ plays a crucial role in the proof of the following theorem.

Theorem 2. *Let R_0 be a chain ring, σ a monomorphism of R_0 and δ a σ-derivation satisfying condition* (V). *Then $W_r(R_1) = \{x^{-n} a R_1, a \text{ in } R_0\}$ and $W_l(R_1) = \{R_1 a x^n, a \text{ in } R_0\}$ for non-negative integers n, provided σ is an automorphism or δ is equal to zero. Further, $x^{-n} a R_1 = x^{-m} b R_1$ if and only if $\sigma^m(a) R_0 = \sigma^n(b) R_0$ and $x^{-n} a R_1 \subseteq x^{-m} b R_1$ if and only if $\sigma^m(a) R_0 \subseteq \sigma^n(b) R_0$.*

Proof. If σ is an automorphism or $\delta = 0$ it is clear that the ring R_0 constructed above can be considered as a subring of R_1 and in fact that $R_1 = S^{-1} R_0[x, \sigma, \delta] = \hat{S}^{-1} \hat{R}_0[x, \hat{\sigma}, \hat{\delta}] = \hat{R}_1$. To prove the last statement we observe that R_1 contains the subring $\hat{R}_0[x, \hat{\sigma}, \hat{\delta}]$ with the subset $\hat{S} = \{\sum \alpha_i x^i;\ \alpha_i \text{ in } \hat{R}_0$, at least one α_i in $U(\hat{R}_0)\}$. The set $\hat{S}$ is an Ore system. The element x in R_1 remains algebraically independent over $\hat{R}_0$, since $\sum \alpha_i x^i = 0$ implies $\sum x^n \alpha_i x^{-n} x^i = 0$ with $x^n \alpha_i x^{-n}$ in R_0 for all i and suitable n. Hence $x^n \alpha_i x^{-n} = 0$ and $\alpha_i = 0$ for all i.

Similarly there exists an integer n for a given element $\hat{f}(x)$ in $\hat{S}$ such that $x^n \hat{f}(x) x^{-n}$ is in R_0 and hence in S. This shows that every element in $\hat{S}$ is invertible in R_1 and $\hat{R}_1 = R_1$ follows. The ring $\hat{R}_0[x, \hat{\sigma}, \hat{\delta}]$ is a right and left

Ore ring, since $\hat{\sigma}$ is an automorphism. Further, $\hat{S}^{-1}\hat{R}_0[x,\hat{\sigma},\hat{\delta}]=\hat{R}_0[x,\hat{\sigma},\hat{\delta}]\hat{S}^{-1}=R_1$ since $\hat{S}=\{\sum x^i\alpha_i,\ \alpha_i$ in R_0, at least one α_i in $U(R_0)\}$ using again the fact that $\hat{\sigma}$ is an automorphism.

A principal right ideal in R_1 has the form $f^{-1}(x)g(x)R_1$ which is equal to $\hat{g}_1(x)\hat{f}_1^{-1}(x)R_1$ for elements $\hat{f}_1(x)$ in $\hat{S}$ and $\hat{g}_1(x)$ in $\hat{R}_0[x,\hat{\sigma},\hat{\delta}]$. Therefore $f^{-1}(x)g(x)R_1=x^{-n}ax^nR_1=x^{-n}aR_1$ for a certain element a in R_0 and a certain non-negative integer n.

The principal left ideals in R_1 have the form $R_1f^{-1}(x)g(x)=R_1g(x)$ with $g(x)$ in $R_0[x,\sigma,\delta]$, $g(x)=\sum a_ix^i=\sum x^i\alpha_i$ for suitable elements α_i in $\hat{R}_0$. Therefore, $R_1g(x)=R_1\alpha=R_1bx^n$ for a certain element $\alpha=x^{-n}bx^n$ in $\hat{R}_0$, b in R_0.

To prove the last part of the theorem we consider an element a in $R_0\cap U(R_1)$. This implies $a=f^{-1}(x)g(x)$ with $f(x)=\sum a_ix^i$, $g(x)=\sum b_jx^j$ in S. Hence, $\sum a_i\sigma^i(a)x^i=\sum b_jx^j$ with at least one b_i in $U(R_0)$. This implies that $\sigma^i(a)$ is a unit and because of (V) that a is in $U(R_0)$. Assume $x^{-n}aR_1=x^{-m}bR_1$. We multiply this equation from the left with x^{n+m} and $\sigma^m(a)R_1=\sigma^n(b)R_1$ follows. Either $\sigma^m(a)r=\sigma^n(b)$ or $\sigma^m(a)=\sigma^n(b)r$ for some element r in R_0. We saw that $\sigma^m(a)\varepsilon=\sigma^m(b)$ for a unit ε in R_1 and r in $U(R_0)$ follows, using the above observation. The remaining statement is proved in the same fashion.

We single out the following facts:

Remark. Let R_0, σ, δ and R_1 be as in Theorem 1.

Then
$$U(R_1)\cap R_0=U(R_0);\qquad J(R_1)\cap R_0=J(R_0).$$

Corollary 1. *Assume R_0 is a chain ring, σ a monomorphism of R_0, δ a σ-derivation satisfying condition* (V). *If in addition δ is an inner σ-derivation, i.e. $\delta(a)=za-\sigma(a)z$ for some z in R_0 then every principal right ideal in R_1 has the form $(x-z)^{-n}aR_1$ for some a in R_0, some nonnegative integer n.*

Proof. We have $xa=\sigma(a)x+za-\sigma(a)z$ which implies $(x-z)a=\sigma(a)(x-z)$. The corollary follows from Theorem 2 if x is replaced by $x-z$.

Corollary 2. *Assume R_0,σ,δ are as in Theorem* 1, *with $\sigma\delta=\delta\sigma$ and $\sigma(a)R_0\supseteq\delta(a)R_0$ for all a in R_0 as additional condition. Then $W_r(R_1)=\{x^{-n}aR_1$, a in R_0, n a nonnegative integer$\}$.*

Proof. We consider the ring $\hat{R}_0$. We saw (Sect. 3) that both σ and δ can be extended to $\hat{R}_0$ with $\hat{\sigma}\hat{\delta}=\hat{\delta}\hat{\sigma}$, since $\sigma\delta=\delta\sigma$. The principal right ideals of $\hat{R}_1=\hat{S}^{-1}\hat{R}_0[x,\hat{\sigma},\hat{\delta}]$ are of the form $\alpha\hat{R}_1$ with α in $\hat{R}_0$, since $\hat{\sigma}$ is an automorphism (Theorem 2). We will show that a principal ideal $\alpha\hat{R}_1$ of $\hat{R}_1$ can be written in the form $x^{-n}b\hat{R}_1$ for some b in R_0. This is done by induction on n with $\hat{\sigma}^n(\alpha)$ in R_0; the statement being true for $n=0$. Assuming α in $\hat{R}_0$ with $\hat{\sigma}^n(\alpha)$ in R_0, we have $x\alpha=\hat{\sigma}(\alpha)x+\hat{\delta}(a)$. However, $\alpha=\hat{\sigma}^{-n}(a)$ for some a in R_0 and $\sigma(a)r=\delta(a)$ for a certain r in R_0 by assumption. Hence $\hat{\delta}(\alpha)=\hat{\delta}\hat{\sigma}^{-n}(a)=\hat{\sigma}^{-n}(\hat{\delta}(a))=\hat{\sigma}^{-n}(\sigma(a)r)=\hat{\sigma}^{-(n-1)}(a)\hat{\sigma}^{-n}(r)$. We obtain: $x\alpha=\hat{\sigma}(\alpha)x+\hat{\sigma}^{-(n-1)}(a)\hat{\sigma}^{-n}(r)=\hat{\sigma}(\alpha)(x+\hat{\sigma}^{-n}(r))$ and $\alpha=x^{-1}\hat{\sigma}(\alpha)(x+\sigma^{-n}(r))$ follows. The element $(x+\hat{\sigma}^{-n}(r))$ is in $\hat{S}$ and $\hat{\sigma}(\alpha)\hat{R}_1=x^{-m}b\hat{R}_1$, b in R_0 by induction. Therefore $\alpha\hat{R}_1=x^{-(m+1)}b\hat{R}_1$ and the above statement is proved.

We complete the proof by showing that $\alpha\hat{R}_1 = v\hat{R}_1 = w\hat{R}_1$ for v, w in R_1 is true only if $vR_1 = wR_1$. This will follow if we show that an element $f^{-1}(x)g(x)$ in R_1 is a unit in $\hat{R}_1$ only if it is already a unit in R_1. We can assume that $f(x) = 1$ and write $g(x) = \sum c_t x^t = (\sum \hat{a}_i x^i)^{-1}(\sum \hat{b}_j x^j)$ with c_t in R_0, $\hat{a}_i, \hat{b}_j$ in $\hat{R}_0$ for all t, i, j. This implies $\sum \hat{a}_i x^i \sum c_t x^t = \sum \hat{b}_j x^j$ and at least one of the $\hat{b}_j$'s is a unit in $\hat{R}_0$. Using condition (V) we see that at least one of the c_t's has to be in $U(\hat{R}_0)$ and hence in $U(R_0)$, (Remark after Theorem 2).

4. Two-sided Ideals

Let R_0, σ, δ and R_1 be as in Theorem 1. The two-sided ideals of R_1 are described in terms of certain two-sided ideals of R_0. To prepare the definition of these ideals in R_0 we consider the commuting rule for elements a in R_0 and a power x^n of x. We have $x^n a = \Delta_n^n(a)x^n + \Delta_{n-1}^n(a)x^{n-1} + \ldots + \Delta_i^n(a)x^i + \ldots + \Delta_0^n(a)$ where Δ_i^n is defined through $(\sigma t + \sigma)^n = \sum_{i=0}^{n} \Delta_i^n t^i$ if t is a commuting indeterminate. Hence $\Delta_2^3 = \sigma^2\delta + \sigma\delta\sigma + \delta\sigma^2$, $\Delta_n^n = \sigma^n$ and $\Delta_0^n = \delta^n$ for all n.

Definition. A two-sided ideal I in R_0 is called (σ, δ)-*compatible* if conditions i) and ii) are satisfied:

i) $\sigma(I) \subseteq I$; $\delta(I) \subseteq I$.

ii) The element a in R_0 is contained in I if there exist $f_0, \ldots, f_n$ in R_0, not all in $J(R_0)$, with $\sum_{j \geqq i}^{n} f_j \Delta_i^j(a)$ in I for all i, $0 \leqq i \leqq n$.

Remark 1. The condition (V) is equivalent with the following condition: $J(R_0)$ is (σ, δ)-compatible.

It must be proved that ii) follows from (V) for the ideal $J(R_0)$. Assume an element a exists in R_0, not in $J(R_0)$, but $\sum_{j \geqq i}^{n} f_j \Delta_i^j(a)$ in J for $i = 0, \ldots, n$ for certain elements $f_0, \ldots, f_n$ in R_0, not all in $J(R_0)$. For $m = \max\{j, f_j \text{ a unit}\}$ we have $\sum_{j \geqq m}^{n} f_j \Delta_m^j(a)$ not in J, since $f_m \Delta_m^m(a)$ is not in J, but $f_j \Delta_m^j(a)$ is in J for $j > m$ — a contradiction.

Remark 2. An ideal I in R_0 is $(\sigma, 0)$-compatible if and only if $\sigma(I) \subseteq I$ and $\sigma(a)$ in I implies a in I for any element a in R_0.

We assume first that I is $(\sigma, 0)$-compatible: If a is in I then $\sigma(a)$ is in I by definition. If $\sigma(a)$ is in I we choose $n = 1$, $f_0 = 0$, $f_1 = 1$ and $f_0 \Delta_0^0(a) + f_1 \Delta_0^1(a) = f_0 a + f_1 \delta(a) = 0$ is in I. Since $f_1 \Delta_1^1(a) = \sigma(a)$ is also in I we can conclude that a is in I. Conversely, let I be an ideal with $\sigma(a)$ in I if and only if a is I. We must show that condition ii) holds. We observe that $\Delta_i^n = 0$ for $i < n$, since $\delta = 0$. We therefore assume $f_j \Delta_j^j(a) = f_j \sigma^j(a)$ in I, at least one f_j a unit and a in I follows.

Theorem 3. *Let R_0, R_1 be as in Theorem* 1. *There exists a one-to-one correspondence between ideals in R_1 and (σ, δ)-compatible ideals in R_0.*

Proof. Let I be an ideal in R_1. The intersection $I_0 = I \cap R_0$ is an ideal in R_0. We prove that it is (σ, δ)-compatible. Let a be in I_0 and $xa = \sigma(a)x + \delta(a)$ follows. Either $\sigma(a)r = \delta(a)$ or $\sigma(a) = \delta(a)r$ for some r in R_0. In the first case we have $\sigma(a)x + \delta(a) = \sigma(a)(x+r)$ in I and hence $\sigma(a)$ in I and I_0, since $x+r$ is in S. Similarly, we conclude that $\delta(a)$ is in I_0 in the second case and $\sigma(a)$, $\delta(a)$ in I_0 follows for every a in I_0. Let $f_0, \ldots, f_n$ be elements in R_0, not all in $J(R_0)$. The element $\sum_0^n f_j x^i$ is in S. Assume a is an element in R_0 such that $c_i = \sum_{j \geqq i}^n f_j \Delta_i^j(a)$ is in I_0 for $i = 0, \ldots, n$. It follows that $c_{i_0} R_0 \supseteq c_i R_0$ is true for a certain index i_0 and all $i = 0, \ldots, n$. That means that

$$f(x)a = \left(\sum_{i=0}^n f_i x^i\right) \cdot a = \sum_{i=0}^n \left(\sum_{j \geqq i} f_j \Delta_i^j(a)\right) x^i = c_{i_0} \tilde{f}(x)$$

is in I, $\tilde{f}(x)$ is in S and a in I_0 follows. This proves the (σ, δ)-compatibility of I_0.

We prove next that $R_1 I_0 R_1 = I$, and $R_1 I_0 R_1 \subseteq I$ is obvious. Any element in R_1 has the form $f_1^{-1}(x) g(x) = f_1^{-1}(x) a g_1(x)$ with a in R_0, $f_1(x)$, $g_1(x)$ in S. If such an element is also in I, then a is certainly in I_0. Hence $R_1 I_0 R_1 \supseteq I$ and $I = R_1(I \cap R_0) R_1$ is proved.

To complete the proof of the theorem we must show that $R_1 I_0 R_1 \cap R_0 = I_0$ for any (σ, δ)-compatible ideal I_0 in R_0. Let a be in I_0, $f^{-1}g$, $h^{-1}k$ in R_1, then it must be proved that b is in I_0 if $b = f^{-1} g a h^{-1} k$ is in R_0. Using the (σ, δ)-compatibility of I_0 we see that $ga = a' g_1$ with a' in I_0 and g_1 in S. Using the Ore-property of S we obtain $f^{-1} g a h^{-1} k = f^{-1} a' g_1 h^{-1} k = f^{-1} a' h_1^{-1} g_2 k$ for certain elements h_1, g_2 in S. By the same property there exist elements u, w in S, c in R_0 with $ua' = cwh_1$ and as before $ua' = a'' u_1$ for u_1 in S, a'' in I_0. Since $a'' u_1 = cwh_1$ and a'' is in I_0, u_1, w, h_1 are in S, the element c is in I_0 and $a' h_1^{-1} = u^{-1} cw$ and finally $b = f^{-1} u^{-1} c w g_2 k$ is in R_0 with c in I_0. If we write $t = uf$ and $s = wg_2 k$, we obtain $t(x) b = c s(x)$ with $t = t(x)$ and $s = s(x)$ in S. Comparing coefficients we get: $\sum_{j \geqq i}^n t_j \Delta_i^j(b)$ is an element in I_0 for all i if $t(x) = \sum_{i=0}^n t_i x^i$. It follows that b is in I_0 and the theorem is proved.

Using the last theorem it is easy to describe the completely prime ideals in R_1.

Lemma 5. *An ideal I in R_1 is completely prime if and only if $I_0 = I \cap R_0$ is a completely prime ideal in R_0.*

Proof. One direction is trivial and we assume that I_0 is completely prime in R_0. It follows from the above theorem that I_0 is (σ, δ)-compatible. It is sufficient to show, [4], that $f^{-1} g f^{-1} g$ in I implies $f^{-1} g$ in I for any element $f^{-1} g$ in R_1. We write $f^{-1} g = f^{-1} a g_1$ for a in R_0, not in I_0, f^{-1}, g_1 in S and will show that $f^{-1} a g_1 f^{-1} a g_1$ is not an element in I. Using the (σ, δ)-

compatibility of I_0 we know that for $s(x)$ in S there are elements a', a'' not in I_0 with

$$s(x)\,a = a' s_1(x) \quad \text{and} \quad a s^{-1}(x) = s_2^{-1}(x)\, a''\, t(x)$$

for certain elements $s_1(x)$, $s_2(x)$ and $t(x)$ in S.

Hence:

$$\begin{aligned} f^{-1} g f^{-1} g &= f^{-1} a g_1 f^{-1} a g_1 = f^{-1} a f_1^{-1} g_2 a g_1 \\ &= f^{-1} f_2^{-1} a'' w g_2 a g_1 = f_1^{-1} f_2^{-1} a'' a' w_1 g_3 g_1 \end{aligned}$$

for $f, f_1, f_2, g_1, g_2, g_3, w, w_1$ in S, a, a', a'' in R_0, not in I_0. It follows from the above theorem that $f^{-1} g f^{-1} g$ is not in I, since $a'' a'$ is not in I_0.

5. Prime Ideals

The relationship between prime ideals in R_1 and R_0 is studied in this section.

Lemma 6. *If I_0 is a (σ, δ)-compatible prime ideal in R_0 then $I = R_1 I_0 R_1$ is a prime ideal in R_1.*

Proof. Let $f_1^{-1}(x) a g_1(x) = u$ and $f_2^{-1}(x) b g_2(x) = v$ be elements in R_1 not in I with $f_i(x)$, $g_i(x)$ in S, a, b in R_0. This implies that a, b are not in I_0 and an element r exists in R_0 with arb not in I_0. Hence, $u g_1^{-1} r f_2 v$ is an element in R_1, not in I, and I is a prime ideal in R_1.

Given a prime ideal I in R_1, we cannot make a definite statement about $I \cap R_0$. However, a related problem is discussed. Since it is not known whether prime ideals in chain rings must be completely prime ideals, it is of interest to decide whether a prime ideal which is not completely prime can appear in R_1 without such a prime ideal existing in R_0. The following theorem answers this question in the case $\delta = 0$.

Theorem 4. *Let $R_0, R_1 = S^{-1} R_0[x, \sigma]$ be as in Theorem 1 with $\delta = 0$. If there exist prime ideals in R_1 which are not completely prime then such prime ideals do also exist in R_0.*

We recall the following result from [4] (Lemma 2): Given elements a, r in a right chain ring then $ra = uar'$ for a unit u and an element r' in the ring. Using this we first prove the following technical result.

Lemma 7. *Let R_0, σ and R_1 be as in Theorem 4, $f(x)$ an element in S and a an element in R_0. Then there exist integers $i, j \geqq 0$ and elements r_1, r_2 in R_0, u in $U(R_0)$ with $a f^{-1}(x) = s(x)^{-1} r_1 \sigma^i(a) t(x)$ and $f(x) a = u \sigma^j(a) r_2 f_1(x)$ with $s(x)$, $t(x)$, $f_1(x)$ in S.*

Proof of Lemma 7. We have $a f^{-1}(x) = s(x)^{-1} b t(x)$ for elements $s(x), t(x)$ in S, b in R_0. Hence, $s(x) a = b t(x) f(x)$ with $s(x) = \sum c_i x^i$. There exists an i with $c_i \sigma^i(a) R_0 \supseteq c_j \sigma^j(a) R_0$ for all j and b can be chosen equal to $c_i \sigma^i(a)$ since $t(x) f(x)$ is in S, proving the first part of the lemma with $r_1 = c_i$. To prove the second part let $f(x)=\sum d_i x^i$ and $f(x)a=\sum d_i \sigma^i(a)x^i$ follows. As before, $d_j\sigma^j(a)R_0 \supseteq d_i\sigma^i(a)R_0$ for all i and for a certain index j. Using the above cited result we have $d_j\sigma^j(a)=u\sigma^j(a)r_2$ for a unit u in R_0, r_2 in R_0 and $f(x)a=d_j\sigma^j(a)f_1(x)=u\sigma^j(a)r_2 f_1(x)$ follows with $f_1(x)$ in S.

Proof of Theorem 4. Let $Q \neq (0)$ be a prime ideal in R_1, not completely prime. Let P be the smallest, completely prime ideal in R_1 containing Q. We know ([4], Theorem 3) that there are no two-sided ideals properly between P and Q. The intersections $P_0=P\cap R_0$ and $Q_0=Q\cap R_0$ are σ-compatible ideals in R_0 with P_0 completely prime (Lemma 5), and Q_0 not completely prime. If Q_0 is prime we are done and we assume from now on that Q_0 is not prime. The proof of the theorem will be completed by exhibiting a prime ideal in R_0 which is not completely prime.

By assumption there exists an element a in R_0 with a not in Q_0, but aR_0a in Q_0. Let P' be the smallest completely prime ideal in R_0 containing a, and $P' \subseteq P_0$ follows.

We write $I^{(0)}=\bigcup_{v\in U(R_0)} vaR_0$; $I^{(n)}=\bigcup_{v\in U(R_0)} v\sigma^n(a)R_0$ and $I^{(n)}$ is the smallest two-sided ideal in R_0 containing $\sigma^n(a)$ ([4], Lemma 2) for all nonnegative integers n.

The ideal $(I^{(0)})^2$ and the element $(\sigma^n(a))^2$ are contained in Q_0.

Case 1. $I^{(1)} \subseteq I^{(0)}$. This implies $\sigma(a)=var$ for v in $U(R_0)$, r in R_0. Therefore, $\sigma^n(a)=\sigma^{n-1}(v)\sigma^{n-1}(a)\sigma^{n-1}(r)$ is contained in $I^{(n-1)}$ and $I^{(n)} \subseteq I^{(n-1)}$ follows for all $n \geqq 1$. The product $I^{(i)}I^{(j)}$ is contained in $(I^{(0)})^2$ and hence in Q_0, which implies that $\sigma^i(a)v\sigma^j(a)$ is an element of Q_0 for every v in $U(R_0)$ and non-negative integers i,j. We know from Theorem 3, iv in [4] that there exists a unit u in $U(R_1)$ with $(au)^n$ not in Q for all positive n. With $u=f^{-1}(x)g(x)$ $-f(x)$, $g(x)$ in S – we obtain

$$(au)^2=af^{-1}(x)g(x)af^{-1}(x)g(x)=f_1^{-1}(x)r_1\sigma^i(a)u_0\sigma^j(a)r_2g_1(x)f^{-1}(x)g(x)$$

for elements r_1, r_2 in R_0, u_0 in $U(R_0)$, $f_1(x)$, $g_1(x)$ in S using Lemma 6. This proves that $(au)^2$ is in Q for every u in $U(R_1)$ – a contradiction.

Case 2. $I^{(0)} \subsetneqq I^{(1)}$. We assume first that $I^{(n)} \subsetneqq I^{(n+1)}$ for $n=0,1,2,\ldots$ and want to show that $(I^{(n)})^2 \subseteq Q_0$. The element $u\sigma^n(a)v\sigma^n(a)=\sigma^{n+1}(a)rv\sigma^n(a)$ for some r in $J(R_0)$ where u, v are in $U(R_0)$. As mentioned earlier, we can factor $r=r_1r_2$ with r_1 in $U(R_0)$, r_2 in $J(R_0)$ such that $r_2v\sigma^n(a)=v\sigma^n(a)r_2'$ for some r_2' in R_0. We get $\sigma^{n+1}(a)r_1v\sigma^n(a)r_2'=(\sigma^{n+1}(a))^2sr_2$ for some s in $J(R_0)$. This element is in Q_0, since a^2 is in Q_0. The ideal $I=\bigcup I^{(n)}$ satisfies $I^2 \subseteq Q_0$ and a contradiction follows as in case 1.

We consider the last possibility: $I^{(0)} \subsetneqq I^{(1)} \subsetneqq \ldots \subsetneqq I^{(n)}=I^{(n+1)}$. This implies $I^{(n)}=I^{(n+i)}$ for all nonnegative integers i. We obtain, as in the above argument, $(I^{(k)})^2 \subseteq Q_0$ for $k=0,1,\ldots,n-1$. Let $Q_0'=\bigcap_{u\in U(R_0)} u\sigma^n(a)R_0$. This is a two-sided ideal and there are no two-sided ideals properly between Q_0' and $I^{(n)}=I$ ([4], Lemma 2).

Let z, y be in Q_0' and $z=\sigma^n(a)r$, $ry=\sigma^n(a)s$ for r, s in R_0 follows. This implies that $zy=\sigma^n(a)ry=(\sigma^n(a))^2s$ is contained in Q_0 and $(Q_0')^2 \subseteq Q_0$.

The ideal $I^{(n)}$ is idempotent, otherwise $I^2 \subsetneqq I = I^{(n)}$ and $I^2 \subseteq Q_0'$ follows. This in turn implies $I^4 \subseteq Q_0$ and $(au)^4$ in Q_0 for every unit u in R_1 - a contradiction as in the previous cases. Hence, $I^2 = I$ is a complete prime ideal ([4], Theorem 1 (ii)).

The prime ideal I can not be finitely generated ([4], Theorem 2) as a right ideal and $\sigma^n(a)R_0 \subsetneqq I$. Therefore a unit u exists in R_0 with $\sigma^n(\mathrm{a})R_0 \subsetneqq u\sigma^n(a)R_0$ and $u^{-1}\sigma^n(a)R_0 \subsetneqq \sigma^n(a)R_0$. This implies $Q_0' \subsetneqq \sigma^n(a)R_0$ and $\sigma^n(a)^2$ contained in Q_0' shows that Q_0' is not completely prime. All conditions of Theorem 4 in [4] are satisfied for the pair I and Q_0' which proves that Q_0' is a prime ideal in R_0 which is not completely prime.

6. Further Remarks and Examples

We consider σ-compatible (i.e. $(\sigma, 0)$-compatible) ideals and begin with the following definition. Let I be a right ideal in R_0. The largest σ-compatible ideal $\bar{I}$ contained in I is called the *σ-kernel* of I.

Lemma 7. *Let R be a chain ring, σ an automorphism of R.*

i) *Every ideal is σ-compatible in R if σ has finite order n.*

ii) *If P is a completely prime ideal in R and not σ-compatible then there exist non-finite strictly ascending and strictly descending chains of completely prime ideals.*

iii) *The σ-kernel $\bar{I}$ of an ideal I of R is completely prime if $\bar{I} \subseteq \bigcap I^n$.*

Proof. i) If $\sigma(a) = ar \neq 0$ we obtain $\sigma^n(a) = ar\sigma(r)\ldots\sigma^{n-1}(r) = a$, and r is a unit. A similar argument shows that s is a unit if $\sigma(a)s = a$.

ii) We assume $\sigma(P) \supsetneqq P$, replacing σ by σ^{-1} if necessary. Then $\sigma^{n+1}(P) \supsetneqq \sigma^n(P)$ and $\sigma^{-n}(P) \supsetneqq \sigma^{-(n+1)}(P)$ and all these ideals are completely prime.

iii) Let I be any ideal in R with $\sigma(I) \subseteq I$. We show that $D = \bigcap \sigma^n(I) = \bar{I}$. Obviously we have $\sigma(D) \subseteq D$. Assume that $\sigma(a)$ is in $\sigma^n(I)$ for all n, then $a = \sigma^{n-1}(b_n)$ for some b_n in I and all positive n. Hence, $\sigma(a)$ in D implies a in D and D is σ-compatible. Let L be any σ-compatible ideal between D and I. Then $L \subseteq \sigma^n(I)$ for all n and $L \subseteq D$, which shows $L = D = \bar{I}$. The same type of argument shows that $\bar{I} = \bigcap \sigma^{-n}(I)$ in case $\sigma(I) \supseteq I$.

We have the assumption $\bar{I} \subseteq \bigcap I^n$, and $\bar{I}$ is a completely prime ideal if $\bar{I} = \bigcap I^n = P$ using [4], Theorem 1 (iii). The containment $\sigma(I) \subseteq I$ implies $\sigma(P) \subseteq P$ and $\bar{I} \subseteq \bar{P} = \bigcap \sigma^n(P) \subseteq \bigcap \sigma^n(I) = \bar{I}$ implies $\bar{I} = \bar{P}$ is a completely prime ideal as the intersection of the $\sigma^n(P)$. Similarly, $\bar{I} = \bar{P} = \bigcap \sigma^{-n}(P)$ in case $I \subseteq \sigma(I)$ which proves the lemma.

The group of values of a commutative rank one valuation ring is isomorphic to a subgroup of the additive group of real numbers [7]. The construction discussed in the previous sections provides us with examples of rank one chain rings. The following generalization of the value group of a valuation ring was defined in [2] for an integral domain R. Let $W = \{aR;\, a \neq 0 \text{ in } R\}$ and $\tilde{H}(R) = \{\tilde{r};\, r \neq 0 \text{ in } R\}$ where $\tilde{r}$ is the mapping from W to W defined by $\tilde{r}(aR) = raR$.

With $\tilde{r}_1 \cdot \tilde{r}_2 = \widetilde{r_1 r_2}$ and $\tilde{r}_1 \geqq \tilde{r}_2$ if $r_1 aR \subseteq r_2 aR$ for all a in R we consider $\tilde{H}(R)$ as a partially ordered semigroup. We are not able to determine $\tilde{H}(R)$ for all rank one chain rings, but will compute this semigroup in a typical example.

Example 1. Consider $R_0 = K[[t]]$ the commutative power series ring in one variable over a commutative field K. Let σ be the monomorphism of R_0 defined by $\sigma(t) = t^2$, $\sigma(a) = a$ for a in K. This monomorphism is $J(R_0)$-compatible. One can therefore construct $R_1 = S^{-1} R_0[x, \sigma]$ and it follows that the principal right ideals of R_1 are of the form $x^{-n} t^k R_1 = x^{-n} t^k x^n R_1$. However, $x^{-1} t x$ is an element mapped onto t by the inner automorphism of R_1 induced by x. This inner automorphism is an extension of σ. We therefore write $x^{-n} t^k R_1 = t^{k/2^n} R_1$ and this provides us with a description of all principal right ideals of R_1, i.e. $W(R_1) = \{t^{k/2^n} R_1, k, n \text{ nonnegative integers}\}$. This set corresponds to the set W' of nonnegative rational numbers $\frac{k}{2^n}$, however $t^{q_1} R_1 \geqq t^{q_2} R_1$ iff $q_1 \leqq q_2$ in W'. Using this observation one can represent the elements in $\tilde{H}(R_1)$ by functions from W' to W' whose graphs consist of finitely many linear pieces. Let for example $f(x) = t x^2 + (t^3 + t^8) x + t^{10}$ in R_1.

Then

$$\tilde{f}(t^z R_1) = \begin{cases} t^{1+4z} R_1 & \text{for } 0 \leqq z \leqq 1 \\ t^{3+2z} R_1 & \text{for } 1 \leqq z \leqq 7 \\ t^{10+z} R_1 & \text{for } 7 \leqq z, \quad z \text{ in } W'. \end{cases}$$

The function $\tilde{f}$ can be represented by a function $\hat{f}$ defined in W' with $\hat{f}(z) = \operatorname{Min}(1+4z,\ 3+2z,\ 10+z)$ for z in W'. In this example R_0, tR_0 and (0) are the only σ-compatible ideals of R_0 which implies that R_1 is a rank one right chain ring with no other two-sided ideals beside R_1, $J(R_1)$ and (0). However, in this case $\operatorname{rank} R_0 = \operatorname{rank} R_1$ and the following (archimedian) property still holds: given u in $J(R_1)$, $0 \neq v$ in R_1 then $u^n R_1 \leqq v R_1$ for some n.

Rohlfing in [9] has computed further examples of groups related to $\tilde{H}(R_1)$ which can be considered as generalizations of the simple ordered groups constructed by Chehata in [5].

Example 2. We conclude with an example in which $\operatorname{rank}(R_0) = \infty$, $\operatorname{rank}(R_1) = 1$ even though $W(R_0)$ may be considered as a subset of $W(R_1)$ as it is always the case in the above construction. Let $G = \bigoplus Z_i$, $i = 1, 2, 3, \ldots$ where $Z_i = (Z, +)$ is the additive group of the integers for every i. Then G is an ordered group under the lexicographical ordering and contains the subsemigroup $H = \{g \text{ in } G, g \geqq \text{identity} = e\}$. Let K be a commutative field. The semigroup ring $K[H] = \{\sum a_h h;\ a_h \text{ in } K, h \text{ in } H \text{ almost all } a_h = 0\}$ contains the multiplicatively closed subset M consisting of elements $\sum a_h h$ in $K[H]$ with $a_e \neq 0$. We set $R_0 = M^{-1} K[H]$ and $\tilde{H}(R_0) \cong H$.

The mapping σ of H with $\sigma(z_1, z_2, \ldots) = (c_1, c_2, \ldots)$ in H with $c_1 = 0$, $c_{i+1} = z_i$ for $i = 1, 2, \ldots$ induces a monomorphism of R_0 which is again called σ. The only σ-compatible ideals of R_0 are R_0, $J(R_0)$ and (0). Hence, R_1 can be