Reprinted from JOURNAL OF ALGEBRA
All Rights Reserved by Academic Press, New York and London

Vol. 87, No. 2, April 1984
Printed in Belgium

Skew Power Series Rings and Derivations

H. H. BRUNGS*

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada

AND

G. TÖRNER

Fachbereich Mathematik, Universität Duisburg, 41 Duisburg, West Germany

Communicated by P. M. Cohn

Received March 10, 1982

0. INTRODUCTION

In 1933 Ore [4] defined a noncommutative multiplication for polynomials in an indeterminate z over a skew field F. Since this multiplication was supposed to respect the degree function, it was determined by the rule $az = za^{\sigma} + a^{\delta}$ for elements a in F with σ a monomorphism and δ a σ-derivation for F.

If one attempts to define an order-preserving noncommutative multiplication for power series rings in one indeterminate z over F one obtains

$$az = za^{\delta_0} + z^2 a^{\delta_1} + z^3 a^{\delta_2} + \cdots + z^{n+1} a^{\delta_n} + \cdots \tag{M}$$

and conditions (i) and (ii$_n$) hold:

(i) The δ_i's are additive mappings from F to F and $a \neq 0$ implies $a^{\delta_0} \neq 0$.

(ii$_n$) $(ab)^{\delta_n} = \sum_{i=0}^{n} a^{\Delta_i^n} b^{\delta_i}$ for $n = 0, 1, 2, \ldots$. Here, Δ_i^n is the coefficient of t^{n+1} in $(\sum_0^\infty t^{k+1}\delta_k)^{i+1}$, where $\sum_0^\infty t^{k+1}\delta_k = f(t)$ is a generating function for the δ_k's with commuting indeterminate t.

* The author acknowledges NSERC support and hospitality experienced during the summer 1981 at the University of Duisburg.

0021-8693/84 $3.00

Condition (i) follows easily from the fact that $(a+b)z = az + bz$ and that for $a \neq 0$ the order of az equals 1. Condition (ii$_n$) follows from the associative law $(ab)z = a(bz)$, see also [2, 5]. This means, in particular, that δ_0 is a monomorphism and that $\delta_1 \delta_0^{-1}$ is a δ_0-derivation of F.

We will say that a sequence $(\delta_0, \delta_1, \dots, \delta_n, \dots)$ of mappings δ_i from F to F is admissible if (i) and (ii$_n$) hold for all n. Such a sequence could be finite.

We want to investigate the set of all admissible sequences for a given field F and we assume from now on that F is commutative. One known admissible sequence is $(\mathrm{id}, \delta, \delta^2, \delta^3, \dots, \delta^n, \dots)$ for an ordinary derivation δ of F. It will be proved that $(\mathrm{id}, 0, 0, \dots, 0, \delta = \delta_k, 0, 0, \dots, 0, ((k+1)/2)\,\delta^2 = \delta_{2k}, 0, \dots, 0, 0, \dots, 0, ((2k+1)(k+1)/(2 \times 3))\,\delta^3 = \delta_{3k}, \dots)$ is admissible for a derivation δ of F. This sequence is just a special case of a general type of sequence discussed in this paper in which the $\delta_i = g_i(\delta)$ for a derivation δ of F and where the $g_i(x)$ are certain polynomials in one variable x with coefficients in $K = \{a \in F; a^\delta = 0\}$. Even though it seems likely that these sequences are admisssible we were not able to prove this in general. Further, it is shown that finite admissible sequences do exist which cannot form the beginning segment of a longer admissible sequence.

Another obvious problem, not dealt with here, is the investigation of equivalence classes of admissible sequences under the equivalence relation given by the isomorphism of the corresponding rings $F[[z, \delta_0, \delta_1, \dots, \delta_n, \dots]]$ with multiplication defined by (M).

1. Structure of Complete Right Chain Rings

Let R be a ring with unit element in which $aR \supseteq bR$ or $bR \supseteq aR$ holds for every pair of elements a, b in R. Such a ring is called a right-chain ring. If we assume further that R is right noetherian then R is a principal right-ideal ring and every right ideal is two sided. Let $J(R) = zR$ be the maximal right ideal of such a ring. We assume that $\bigcap_0^\infty (zR)^n = \bigcap_0^\infty z^n R = (0)$ and that R is complete with respect to the topology defined by using the $z^n R$ as neighborhoods of 0.

One can ask whether the structure theorems of Cohen [2] for noetherian commutative complete local rings can be extended in some way to the above situation. The first question then would be: when does there exist a skew field D of representatives of R/zR in R? This is trivially the case if $R/zR \cong \mathbb{Q}$ or $\mathbb{Q}(t)$, the function field in one variable over $\mathbb{Q}$, the field of rational numbers. If $\mathrm{char}(R) = \mathrm{char}(R/zR) = p$ a prime, then again a field of representatives exists if $R/zR \cong GF(p)$ or $GF(p)(t)$.

Vidal in [7] showed that there exists a right-chain ring R with maximal right ideal zR and $z^2 = 0$, $R/zR \cong GF(p)(t_1, t_2) = K$, $\mathrm{char}(R) = p$, and R does not contain a field of representatives of K. Whereas not much seems to be known in this general case, very detailed information is available in the finite case, where either R is a finite ring or R is a finite dimensional algebra over a field.

Let us now assume that R contains a field F (a skew field) of representatives of R/zR. In this case (M) defines the multiplication of R and the corresponding sequence $(\delta_0, \delta_1, ...)$ is a (possibly finite) admissible sequence of F. We have in this case $R \cong F[[z, \delta_0, \delta_1, ...]]$ the type of ring described in the Introduction). It is possible that $z^{n+1} = 0$ and $(\delta_0, \delta_1, ..., \delta_n)$ is a finite sequence.

2. Admissible Sequences

We will assume that F is a commutative field and that $(\delta_0, \delta_1, ..., \delta_n, ...)$ is an admissible sequence. Before we obtain some results we want to describe the Δ_i^n that appear in the definition of an admissible sequence. It follows that $\Delta_i^n = \sum \bar{\delta}_{j_1} \cdots \bar{\delta}_{j_{i+1}}$ with the sum taken over all ordered $(i+1)$-tuples $j_1, ..., j_{i+1}$ of positive integers j_k with $j_1 + j_2 + \cdots + j_{i+1} = n+1$ and $\bar{\delta}_{s+1} = \delta_s$ for all s. We compute Δ_2^4 as an example and obtain

> $i = 2$, $n = 4$; $4 + 1 = 5$ must be partitioned into $2 + 1 = 3$ summands, taking the order into consideration.

Hence, $5 = 1 + 1 + 3 = 1 + 3 + 1 = 3 + 1 + 1 = 1 + 2 + 2 = 2 + 1 + 2 = 2 + 2 + 1$. This translates into

$$\Delta_2^4 = \delta_0^2 \delta_2 + \delta_0 \delta_2 \delta_0 + \delta_2 \delta_0^2 + \delta_0 \delta_1^2 + \delta_1 \delta_0 \delta_1 + \delta_1^2 \delta_0.$$

Lemma 1. *Let $A = (\delta_0, \delta_1, ..., \delta_n)$, $n \geqslant 1$ be an admissible sequence, and let $B = (\delta_0', \delta_1', ..., \delta_{n-1}', \delta_n')$ be a sequence of additive mappings from F to F with $\delta_i = \delta_i'$ for $i = 0, ..., n-1$. Then B is an admissible sequence if and only if $\delta = \delta_n' - \delta_n$ satisfies*

$$(ab)^{\delta} = a^{\delta} b^{\delta_0} + a^{\delta_0^{n+1}} b^{\delta}.$$

We use the notation $'\Delta_i^k$ for the Δ_i^k's derived from the sequence B and Δ_i^k for those obtained using the δ_i's in the sequence A. We have

$$(ab)^{\delta_n'} = a^{'\Delta_0^n} b^{\delta_0'} + a^{'\Delta_1^n} b^{\delta_1'} + \cdots + a^{'\Delta_n^n} b^{\delta_n'}.$$

It is clear that $'\Delta_i^n = \Delta_i^n$ for $i = 1, ..., n-1$ and that $'\Delta_n^n = \Delta_n^n = \delta_0^{n+1}$. Therefore $(ab)^{\delta_n' - \delta_n} = a^{\delta_n' - \delta_n} b^{\delta_0} + a^{\delta_0^{n+1}} b^{\delta_n' - \delta_0}$. This shows that $\delta = \delta_n' - \delta_n$ satisfies the above condition. Reversing the argument proves the other half of the lemma.

In Lemma 2 we need the assumption that the field F is commutative of characteristic 0. Further, we use the fact that for $\delta_0 = 1$, the identity, and δ an ordinary derivation of F the sequence $(1, \delta, \delta^2, \delta^3, \ldots, \delta^n, \ldots)$ is admissible. This follows from [2, p. 38] or from results proved later in this paper.

LEMMA 2. *Assume there exist two derivations δ_1, δ_2 for F with $\delta_i \neq c\delta_j$ for every c in F and $i, j = 1, 2$; i.e., $\{\delta_1, \delta_2\}$ is linearly independent over F. Then $A = (1, \delta, \delta_1^2, \delta_1^3, \ldots, \delta_1^{n-1} + \delta_2)$, $n - 1 \geqslant 2$, is an admissible sequence which cannot be the initial segment of a longer admissible sequence.*

Proof. Clearly A is admissible, by Lemma 1. We assume that such a sequence,

$$B = (1, \tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n),$$

exists with $\tau_i = \delta_1^i$ for $i = 1, \ldots, n-2$ and $\tau_{n-1} = \delta_1^{n-1} + \delta_2$. Then $(ab)^{\tau_n} = a^{\tau_n}b + a^{\Delta_1^n}b^{\tau_1} + \cdots + a^{\Delta_{n-1}^n}b^{\tau_{n-1}} + ab^{\tau_n}$. We have

$$\begin{aligned} \Delta_1^n &= \tau_0\tau_{n-1} + \tau_{n-1}\tau_0 + \tau_1\tau_{n-2} + \tau_{n-2}\tau_1 + \cdots + \tau_i\tau_{n-(i+1)} + \cdots \\ &= 2\tau_{n-1} + (n-2)\,\delta_1^{n-1} = 2\delta_2 + n\delta_1^{n-1} \end{aligned}$$

and

$$\Delta_i^n = \binom{n}{i}\delta_1^{n-i} \qquad \text{for} \quad i = 2, \ldots, n-1.$$

Since the $\Delta_i^n = \binom{n}{i}\delta_1^{n-i}$ for $i = 1, \ldots, n$ for the admissible sequence $(1, \delta_1, \delta_1^2, \ldots, \delta_1^n)$, and hence,

$$(ab)^{\delta_1^n} = a^{\delta_1^n}b + na^{\delta_1^{n-1}}b^{\delta_1} + \cdots + \binom{n}{i}a^{\delta_1^{n-i}}b^{\delta_1^i} + \cdots + ab^{\delta_1^n},$$

we obtain

$$(ab)^{\tau_n} = (ab)^{\delta_1^n} + 2a^{\delta_2}b^{\delta_1} + na^{\delta_1}b^{\delta_2} - (a^{\delta_1^n - \tau_n}b + ab^{\delta_1^n - \tau_n}).$$

Since F is commutative, we have $(ab)^{\tau_n} = (ba)^{\tau_n}$ and this equation leads to

$$2a^{\delta_2}b^{\delta_1} + na^{\delta_1}b^{\delta_2} = 2b^{\delta_2}a^{\delta_1} + nb^{\delta_1}a^{\delta_2}$$

or $(n-2)\,a^{\delta_1}b^{\delta_2} = (n-2)\,a^{\delta_2}b^{\delta_1}$, and since $n \geqslant 3$,

$$a^{\delta_1}b^{\delta_2} = a^{\delta_2}b^{\delta_1} \qquad \text{for all } a, b \text{ in } F.$$

If we choose a fixed a_0 with $a_0^{\delta_1} \neq 0$ in F we obtain $b^{\delta_2} = cb^{\delta_1}$ for a constant $c = a_0^{\delta_2}(a_0^{\delta_1})^{-1}$ and all b in F—a contradiction.

Remark. The condition $n \geqslant 3$ is necessary. Any admissible sequence

$(1, \delta_1)$ can be continued to a longer admissible sequence; δ_1 is just an ordinary derivation of F.

As before let F be a commutative field of characteristic 0, δ a derivation of F, $\delta \neq 0$, and $K = \{a \in F; a^\delta = 0\}$ the subfield of constants under δ. It follows from Lemma 1 that the terms $\delta_1, \delta_2, ..., \delta_{n-1}$ determine δ_n in an admissible sequence $(1, \delta_1, \delta_2, \delta_3, ..., \delta_n, ...)$ up to a derivation of F. We will write $\delta_n = f_n(\delta_1, \delta_2, ..., \delta_{n-1}) + \tau_n$ with τ_n an ordinary derivation of F. Lemma 2 shows that τ_n cannot be arbitrary in the case of an infinite admissible sequence. We want to investigate the case in which τ_n equals $a_n \delta$ for a_n in K and for the fixed derivation δ. This means $\delta_1 = a_1 \delta$ and $\delta_n = f_n(\delta_1, ..., \delta_{n-1}) + a_n \delta$ for $n \geqslant 2$.

We point out that our assumed condition is more restrictive than appears necessary in the light of Lemma 2. However, the case a_n in F, not only in K, seems to pose additional difficulties. It is now the problem to compute these functions f_n. Using the definition of an admissible sequence one can compute f_n directly for small n. The f_n's turn out to be polynomials $g_n(\delta) - a_n \delta$ in δ, where the coefficients are determined by the a_i's in a particular way. Here are the first of these polynomials:

$$\delta_1 = g_1(\delta) = a_1 \delta;$$

$$\delta_2 = g_2(\delta) = a_1^2 \delta^2 + a_2 \delta;$$

$$\delta_3 = g_3(\delta) = a_1^3 \delta^3 + \tfrac{5}{2} a_1 a_2 \delta^2 + a_3 \delta;$$

$$\delta_4 = g_4(\delta) = a_1^4 \delta^4 + \tfrac{13}{3} a_1^2 a_2 \delta^3 + (3a_1 a_3 + \tfrac{3}{2} a_2^2)\, \delta^2 + a_4 \delta;$$

$$\delta_5 = g_5(\delta) = a_1^5 \delta^5 + \tfrac{77}{12} a_2 a_1^3 \delta^4 + (6a_1^2 a_3 + \tfrac{35}{6} a_1 a_2^2)\, \delta^3 + \tfrac{7}{2}(a_1 a_4 + a_2 a_3)\, \delta^2 + a_5 \delta.$$

To compute the above listed terms directly is a rather lengthy and tedious exercise. An induction step is not obvious if at all possible; the appearance of partitions in the buildup of the Δ_i^n from the δ_i's is one of the stumbling blocks.

In any case, one can check directly that the sequence $(1, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$ is an admissible sequence for any choice of the a_i's in K and a given δ and F.

3. A Conjecture and Some Special Cases

The notation in this section is as in Section 2. We will discuss a relationship between the polynomials $g_n(x)$ which was obtained by studying examples. We are not able to prove, in general, that the sequence of $\delta_n = g_n(\delta)$ obtained from those $g_n(x)$ is, in fact, an admissible sequence. The functions $g_n(x)$ are considered as polynomials over the commutative field K of characteristic 0 and we write $g_n'(x)$ for the usual first derivative of $g_n(x)$.

We consider the following relationship between these polynomials:

$$g_n'(x) = na_1 g_{n-1}(x) + (n-1) a_2 g_{n-2}(x) + (n-2) a_3 g_{n-3}(x) + \cdots + 2a_{n-1} g_1(x) + a_n g_0(x), \tag{I}$$

with $g_0(x) = 1$ and $g_n(0) = 0$ for $n > 0$. The derivative $h'(x)$ of a polynomial $h(x) = \sum_{i=0}^{m} c_i x^i$ in $K[x]$ is the formal derivative $h'(x) = \sum_{i=1}^{m} i \cdot c_i x^{i-1}$. If we form the generating function,

$$H = H(x, y) = \sum_{n=0}^{\infty} g_n(x) y^{n+1},$$

then (I) means that H is a solution of the partial differential equation,

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} (a_1 y^2 + a_2 y^3 + \cdots + a_{n-1} y^n + \cdots), \tag{II}$$

with initial condition $H(0, y) = y$. Here again the formal derivatives with respect to x or y are used, (see [3]). The previous statement can be checked easily by comparing the coefficients of y^{n+1} in (II). We have $g_n'(x)$ on the left-hand side and $na_1 g_{n-1}(x) + (n-1) a_2 g_{n-2}(x) + \cdots + a_n g_0(x)$ with $g_0(x) = 1$ on the right-hand side.

We try to solve (II) and begin by rewriting this equation,

$$\frac{\partial H}{\partial x} - \frac{\partial H}{\partial y} (a_1 y^2 + a_2 y^3 + \cdots) = 0.$$

In order to use the theory of linear partial differential equations we now assume that the coefficients a_i are contained in the prime field $\mathbb{Q}$ of K. Using [6, p. 304] we obtain

$$\frac{dx}{du} = 1,$$

$$\frac{dy}{du} = -(a_1 y^2 + a_2 y^3 + \cdots),$$

$$\frac{dH}{du} = 0.$$

This leads to $\int dx + \int dy/(a_1 y^2 + a_2 y^3 + \cdots) = C_1 = C_1(x, y)$ a constant and $\int dH = H = C_2$ a constant.

If one now considers any function $\phi(w, v)$ in two variables and solves the equation $\phi(C_2, C_1) = 0$ for $C_2 = H = H(x, y)$ one obtains a solution of (II).

In order to find the function ϕ_1 that leads to the solution with the right initial condition one has to substitute the condition $C_2 = H(0, y) = y$ into the equation for $C_1 = C_1(0, y) = C_1(0, C_2)$ which will lead to a relationship $\phi_1(C_1, C_2) = 0$. This then is the function ϕ to be used to find H.

We consider a special case,

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y}\, y^{k+1} \qquad \text{with} \qquad H(0, y) = y; \quad k = 1, 2, \ldots . \tag{II'}$$

As described above we must solve

$$dx = \frac{dy}{-y^{k+1}} \qquad \text{and obtain} \qquad x - \frac{1}{ky^k} = C_1,$$

and $H = H(x, y) = C_2$. Any function $\phi(x - (1/ky^k), H) = 0$ solved for H would lead to a solution of (II′). In order to satisfy the initial condition we consider $C_1(0, y) = -(1/ky^k)$ and substitute $C_2 = H(0, y) = y$ for y in this equation. We obtain $C_1 = -(1/kC_2^k)$, and $\phi_1(C_1, C_2) = C_1 + (1/kC_2^k)$ is the correct function ϕ. Hence $x - (1/ky^k) + (1/kH^k) = 0$ and $H = (y/\sqrt[k]{1 - kxy^k})$is the solution of (II′).

This means that for $H = (y/\sqrt[k]{1 - kxy^k}) = \sum_0^\infty g_n(x)\, y^{n+1}$ we obtain a sequence $(1, \delta_1, \delta_2, \ldots, \delta_n, \ldots)$ with $\delta_n = g_n(\delta)$, where δ is an ordinary derivation of the field F. We prove next that this sequence is in fact an admissible sequence for F.

Using the expression just obtained, $H = y(1 - kxy^k)^{-1/k}$ for the generating function, we want to compute the corresponding δ_n's, the Δ_i^n's, and finally prove that the relationship that defines an admissible sequence holds for $(1, \delta_1, \delta_2, \ldots, \delta_n, \ldots)$. From $1/(1 - z)^{\alpha+1} = \sum_{i=0}^\infty \binom{i+\alpha}{i} z^i$ for α any real number and $z < 1$, we obtain

$$H = \frac{y}{(1 - kxy^k)^{1/k}} = \frac{y}{(1 - kxy^k)^{(1-k)/k+1}} = y \sum_{i=0}^{\infty} \binom{i + (1-k)/k}{i} (kxy^k)^i$$

$$= \sum_{i=0}^{\infty} \binom{k(i-1)+1}{i}^{*} x^i y^{ik+1}$$

$$\text{with} \qquad \binom{k(i-1)+1}{i}^{*} = \frac{\prod_{j=1}^{i} (k(i-j)+1)}{i!}.$$

We conclude that in this case $g_n(x) = 0$ for $k \nmid n$ and

$$g_{ik} = \binom{k(i-1)+1}{i}^{*} x^i$$

$$= \frac{k(i-1)+1}{i}\, x \cdot g_{(i-1)k}(x) \qquad \text{for} \quad i \geqslant 1; \;\; g_0(x) = 1.$$

To this sequence corresponds the sequence of mappings,

$$(1, 0,\ldots, 0, \delta_k, 0,\ldots, 0, \delta_{2k},\ldots, \delta_{ik},\ldots), \tag{S}$$

with $\delta_{ik} = \binom{k(i-1)+1}{i}^* \delta^i$, where δ is an ordinary derivation of F.

We will use the generating function H to compute the Δ_i^n for this sequence and show that (S) is an admissible sequence. Therefore we consider (see Section 0)

$$\begin{aligned} H^{i+1} &= \left(\frac{y}{(1-kxy^k)^{1/k}}\right)^{i+1} = y^{i+1}\frac{1}{(1-kxy^k)^{(i+(1-k))/k+1}} \\ &= y^{i+1}\sum_{s=0}^{\infty}\binom{s+(i+1-k)/k}{s}(kxy^k)^s \\ &= \sum_{s=0}^{\infty}\prod_{j=1}^{s}\left(\frac{k(j-1)+i+1}{j}\right)\cdot x^s\cdot y^{ks+i+1}. \end{aligned}$$

That means

$$\Delta_i^{sk+i} = \prod_{j=1}^{s}\left(\frac{k(j-1)+i+1}{j}\right)\cdot\delta^s \tag{T}$$

and

$$\Delta_i^n = 0 \qquad \text{for} \quad n-i \not\equiv 0 \bmod k.$$

Knowing the δ_n and the Δ_i^n explicitly it is now possible to check that $(ab)^{\delta_n} = \sum_{i=0}^n a^{\Delta_i^n} b^{\delta_i}$ which proves that (S) is an admissible sequence.

If $n \not\equiv 0 \bmod k$ we have $\delta_n = 0$ and $\delta_i \neq 0$ possible only for $i \equiv 0 \bmod k$. In this case $n - i \not\equiv 0 \bmod k$ and $\Delta_i^n = 0$. This means the above equation is correct for $n \not\equiv 0 \bmod k$.

We now prove the equation

$$(ab)^{\delta_{nk}} = a^{\Delta_0^{nk}} b^{\delta_0} + a^{\Delta_k^{nk}} b^{\delta_k} + a^{\Delta_{2k}^{nk}} b^{\delta_{2k}} + \cdots + a^{\Delta_{nk}^{nk}} b^{\delta_{nk}}.$$

We write δ for δ_k and using (S) and (T) we obtain as the coefficient of $a^{\delta^{n-s}}b^{\delta^s}$ in the expansion of $(ab)^{\delta_{nk}} = (ab)^{\delta^n}([k+1]\cdots[(n-1)k+1]/n!)$, the term $\binom{n}{s}([k+1][2k+1]\cdots[(n-1)k+1]/n!)$, and on the right-hand side we must compute $a^{\Delta_{sk}^{nk}}b^{\delta_{sk}} = a^{\Delta_{sk}^{(n-s)k+sk}}b^{\delta_{sk}}$ and obtain $([sk+1][(s+1)k+1]\cdots[(n-1)k+1]/(n-s)!)$ as the coefficient of $a^{\delta^{n-s}}b^{\delta_{sk}} = ([k+1][2k+1]\cdots[(s-1)k+1]/s!)a^{\delta^{n-s}}b^{\delta^s}$. The equality

$$\binom{n}{s}\frac{[k+1]\cdots[(n-1)k+1]}{n!}$$
$$=\frac{[sk+1][(s+1)k+1]\cdots[(n-1)k+1]}{(n-s)!}$$
$$\cdot\frac{[k+1][2k+1]\cdots[(s-1)k+1]}{s!},$$

proves that $(ab)^{\delta_{nk}} = \sum_0^{nk} a^{\Delta_i^{nk}} b^{\delta_i}$ and the admissibility of the sequence (S). We formulate this result as Theorem 1.

THEOREM 1. *Let F be a commutative field of characteristic 0, δ a derivation of F, and k an integer $\geqslant 1$. Then (δ_i) is an admissible sequence for F, where $\delta_0 =$ identity, $\delta_j = 0$ for $j \not\equiv 0 \bmod k$, and $\delta_{nk} = (1/n!) \prod_{j=1}^{n} [k(n-j)+1]\,\delta^k$.*

Remark 1. For $k = 1$ we obtain the known admissible sequence $(1, \delta, \delta^2, \delta^3, \ldots)$.

Remark 2. If we keep F and δ fixed and construct the power series rings R_k over F with multiplication defined with the help of the admissible sequence as in Theorem 1, then $R_k \cong R_{k'}$ if and only if $k = k'$.

If we try the above method to compute other admissible sequences from a somewhat more complex case of (II) we encounter greater difficulties. Let us discuss the case,

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y}\,(y^2 + y^3). \qquad \text{(II}'')$$

This time we obtain

$$C_1 = x - \ln y - (1/y) + \ln(y+1),$$
$$C_2 = H = z.$$

The general solution of (II″) will then be obtained if one solves $\phi(C_2, C_1) = \phi(z, x - \ln y - (1/y) + \ln(y+1))$ for $z = H = H(x, y)$, where ϕ is an arbitrary function in two variables.

To find the solution with the correct initial condition $H(0, y) = y$ we have $C_2 = H = z = y$ and $C_1 = -\ln C_2 - (1/C_2) + \ln(C_2 + 1)$ for $x = 0$. This relationship between C_1 and C_2 is the one that leads to the solution,

$$x - \ln y - (1/y) + \ln(y+1) = -\ln z - (1/z) + \ln(z+1).$$

We obtain $e^{-1/z}((z+1)/z) = e^x(e^{-1/y}\cdot(y+1)/y)$ as an implicit equation for the function $z(x, y) = H$. Computing the corresponding sequence of the

$g_n(x)$, the $\delta_n = g_n(\delta)$ and the Δ_i^n from the powers H^{i+1} of H and finally show the admissibility does not seem easy even in this special case.

4

In Section 3 we introduced the following conditions:

$$\text{(I)} \quad g_u'(x) = \sum_{i=1}^{u} (u-i+1)\, a_i\, g_{u-i}(x), \; u = 1, 2, \ldots;$$
$$\text{with } g_0(x) = 1 \text{ and } g_j(x) = 0 \text{ for } j = 1, 2, \ldots .$$

$$\text{(II)} \quad \frac{\partial H}{\partial x} = \frac{\partial H}{\partial y}(a_1 y^2 + a_2 y^3 + \cdots), \; H(0, y) = 0;$$
$$H(x, y) = \sum_{0}^{\infty} g_n(x)\, y^{n+1}.$$

These are equivalent descriptions of the function $g_n(x)$ with $g_n(\delta) = \delta_n$. We conjectured further that such a sequence $(1, \delta_1, \delta_2, \ldots)$ defined in this way would lead to an admissible sequence. We will now discuss two additional conditions which are also equivalent to (I) and (II).

The following notation is introduced: a_{ij} (or sometimes $a_{i,j}$) is the coefficient of x^i in $g_{i+j-1}(x)$, $i, j \geqslant 1$. This means $g_n(x) = \sum_{i+j-1=n} a_{ij} x^i$, $i, j \geqslant 1$ and $a_{1,j} = a_j$, where the a_j has the meaning as in (I) and (II).

Condition (I) is equivalent to condition (III):

$$\text{(III)} \quad a_{n+1,m} = \frac{1}{n+1} \sum_{i+j=m+1} (i+n)\, a_{1,j} a_{n,i}$$
$$= \frac{1}{n+1} [a_{11}(m+n)\, a_{n,m} + a_{1,2}(m+n-1)\, a_{n,m-1}$$
$$+ \cdots + a_{1,m}(1+n)\, a_{n,1}].$$

To prove that (I) and (III) are equivalent one compares the coefficients of x^k in (I) and obtains on the left hand the term $(k+1)\, a_{k+1,u-k}$ and on the right-hand side the sum $u a_{11} a_{k,u-k} + (u-1)\, a_{12} a_{k,u-k-1} + (u-2)\, a_{13} a_{k,u-k-2} + \cdots + (k+1)\, a_{1,u-k} a_{k1}$. Therefore, assuming (I), one obtains

$$a_{k+1,u-k} = \frac{1}{k+1} \sum_{i+j=u-k+1} (a_{1,j} a_{k,i})(u-j+1).$$

With $n = k$ and $m = u-k$ we get $u-k+1 = m+1$ and $u-j+1 = n+i$, and (III) follows from the last equation. Reversing the above steps shows that conversely (III) implies (I).

The recursion formula (III) can be expressed using matrix notation. Given the constants $a_1 = a_{1,1}$, $a_2 = a_{1,2}, \ldots, a_n = a_{1,n}, \ldots$, we form the strictly lower triangular matrix

$$A = (c_{ij}) \qquad \text{with} \qquad c_{ij} = 0 \text{ for } i \leqslant j \text{ and } c_{ij} = (j+1)\, a_{i-j} \text{ for } i > j.$$

Then

$$\text{(IV)} \quad [g_1(x), g_2(x), \ldots, g_n(x), \ldots]^{\mathrm{T}} = \left(xE + \frac{x^2 A}{2!} + \frac{x^3 A^2}{3!} + \cdots + \frac{x^n A^{n-1}}{n!} + \cdots\right) \times [a_1, a_2, \ldots, a_n, \ldots]^{\mathrm{T}}$$

gives the $g_n(x)$ as defined in (I) as components of the vector to the left.

We want to show that (III) and (IV) are equivalent, using the definition of the $g_n(x)$ as defined for (I). We compute the individual summands on the right-hand side of (IV), and obtain $(xE) \cdot [a_1, \ldots, a_n, \ldots]^{\mathrm{T}} = [xa_1, \ldots, xa_n, \ldots]^{\mathrm{T}}$, the vector consisting of the terms of degree 1 for the $g_n(x)$. We use induction for the other summands. We assume that

$$A^{n-1}[a_1, a_2, \ldots, a_n, \ldots]^{\mathrm{T}} = n!\,[0; \ldots; 0; a_{n,1}; a_{n,2}; \ldots]^{\mathrm{T}},$$

with the first $n-1$ components equal to 0. Then

$$A^n [a_1, a_2, \ldots, a_n, \ldots]^{\mathrm{T}}$$

$$= n! \begin{pmatrix} 0 & & & \bigcirc \\ 2a_1 & 0 & & \\ 2a_2 & 3a_1 & 0 & \\ 2a_3 & 3a_2 & 4a_1 & 0 \\ \vdots & & & \end{pmatrix} [0; \ldots; 0; a_{n,1}; a_{n,2}; \ldots]^{\mathrm{T}}$$

$$= n! \begin{pmatrix} 0 & \cdots & & 0 & \cdots \\ a_1 & 0 & \cdots & 0 & \cdots \\ a_2 & a_1 & 0 & \cdots & \\ a_3 & a_2 & a_1 & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$

$$\times [\underbrace{0; \ldots; 0}_{\substack{n-1 \\ \text{components}}}; (n+1)\, a_{n,1}; (n+2)\, a_{n,2}; \ldots; (n+i)\, a_{n,i}; \ldots]^{\mathrm{T}}$$

$$= [0; 0; \ldots; 0; d_{n+1,1}; d_{n+1,2}; \ldots; d_{n+1,m}; \ldots]^{\mathrm{T}},$$

where the first n components equal zero and the $(n+m)$th component,

$$d_{n+1,m} = n! \sum_{i+j=m+1} a_j(a_{n,i})(n+i).$$

This means $d_{n+1,m} = (n+1)!\, a_{n+1,m}$ and

$$A^n[a_1, a_2, \ldots, a_n, \ldots]^{\mathrm{T}} = (n+1)!\, [0; 0; \ldots; 0; a_{n+1,1}; a_{n+1,2}; \ldots]^{\mathrm{T}},$$

with the first n components equal to zero and $a_{n+1,m}$ as the $(n+m)$th component.

This proves that (III) implies (IV) and reversing the arguments shows that these conditions are equivalent.

References

1. I. S. Cohen, On the structure and ideal theory of complete local rings, *Trans. Amer. Math. Soc.* **59** (1946), 54–106.
2. P. M. Cohn, "Free Rings and Their Relations," Academic Press, London/New York, 1971.
3. I. Niven, Formal power series, *Amer. Math. Monthly* (1969), 870–889.
4. O. Ore, Theory of non-commutative polynomials, *Ann. of Math.* **34** (1933), 480–508.
5. T. H. M. Smits, Nilpotent S-derivations, *Indag. Math.* **30** (1968), 72–86.
6. R. L. Street, "Analysis and Solutions of Partial Differential Equations," Wadsworth, Belmont, Calif. 1973.
7. R. Vidal, Anneaux de Cohen, non commutatifs, sur un corps local, *C. R. Acad. Sci. Paris Sér. A* **288** (1979), 181–184.