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Locally Archimedian Right-Ordered Groups and Locally Invariant Valuation Rings

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In the last years there has been a rapid development in noncommutative valuation theory, in particular pursued by Mathiak in [11, 12, 13]. His approach overcomes the difficulties that the (generalized) value groups are no longer linearly ordered as well as the associated valuation rings have "only a few" two-sided ideals. From the point of view of ring theory valuation rings (in the sense of Mathiak) are chain rings (without zero divisors), namely rings whose lattices of right and left ideals are linearly ordered. There are numerous articles answering ring-theoretical questions on that class of noncommutative local rings.

Although Mathiak's approach is (in some sense) as general as possible mainly those "strongly" noncommutative valuations are of interest which can still be treated; e.g., a modified approximation theorem for these valuations is valid. It was Gräter [9] who introduced a special class of noncommutative valuations, namely the so called "locally invariant valuations."

In Section 1 of this article we investigate a class of right chain rings which contains the valuation rings of Gräter's locally invariant valuations. However, our definition is more general including right chain rings which are not necessarily chain rings and admitting zero-divisors. It turns out, that these rings can be characterized by the existence of not too many, but "sufficiently many" two-sided ideals, to be more precise: between any two prime ideals there is a further two-sided ideal (see Proposition 1.3). As the multiplicative semigroups of these rings are closely related to certain right ordered groups, Section 2 is devoted to a new characterization of a class of right-ordered groups (Theorem 2.8), called locally archimedian (or sometimes poly-ordered) groups, which has earlier been studied by Conrad [5]. A positive answer to a question of Ault [2] is given.

In Section 3 we prove that there is a close connection between the locally invariant chain domains introduced by Gräter and these right-ordered

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groups (Theorem 3.1). The starting point for this investigation was the formal similarity of a theorem of Gräter on valuations and a theorem of Conrad on right-ordered groups.

The analysis leads to a construction method for certain locally invariant chain rings as localization of skew semigroup rings over the positive cone of a locally archimedian right-ordered group (Theorem 4.3). The resulting chain ring is naturally associated to the semigroup of positive elements (Dubrovin [7]).

All rings are not necessarily commutative and have an identity. $J = J(R)$ denotes the Jacobsonradical of the ring R whereas $U = U(R)$ stands for the group of units.

1. Locally Archimedian Right Chain Rings

In this section rings may have zero divisors. A *right chain ring* is a ring in which the right ideals are linearly ordered by inclusion. Similarly, a left chain ring is defined. A *chain ring* is a ring which is both a right and left chain ring.

As we need the existence of some nontrivial two-sided ideals for further calculations, we will consider rings which satisfy the following condition, which is not very strong:

Definition 1.1. *A right chain ring R is called locally archimedian if for any $a \in R$ the right ideal $\bigcap_n a^n R$ is two-sided.*

Note that this is in fact only a condition on the non-units which are not nilpotent. The property above is related to a condition for groups resp. semigroups as defined in Conrad [5, 4.1], so that one might obtain results analogous to those in case of semigroups. Conversely semigroup rings over such semigroups lead to the mentioned kind of rings as we will see later.

Lemma 1.2. *Each prime ideal in a locally archimedian right chain ring is completely prime.*

Proof. If there exists a not completely prime prime ideal Q, then we would have a completely prime ideal P with P/Q simple [3, Theorem 3.5]. In this case we have elements $x \in P \setminus Q$ with $Q \subsetneqq \bigcap x^n R$ [3, Lemma 3.6] which contradicts the property of R being locally archimedian.

The following result gives other characterizations of locally archimedian right chain rings:

PROPOSITION 1.3. *Let R be a right chain ring. The following properties are equivalent*:

(i) *R is locally archimedian.*

(ii) *For any $a \in R$ we have $Ra^2 \subseteq aR$.*

(iii) *For any $a \in R$ there exists $n = n(a) \in \mathbb{N}$ such that $Ra^n \subseteq aR$.*

(iv) *Between any two prime ideals there is another two-sided ideal.*

Proof. (i) $\Rightarrow$ (ii) Let $a \in R$, $a \neq 0$. If $Ra^2 \nsubseteq aR$ then $a = ua^2s$ for some $u \in U$, $s \in J$. Thus $a = (ua)^n as^n$ for all $n \in \mathbb{N}$, hence $a \in \bigcap_n (ua)^n R$. By assumption, this is a two-sided ideal, so $ua \in \bigcap_n (ua)^n R$ which implies $ua = 0$ - contradiction.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv) Let $Q \subsetneqq P$ be prime and suppose P/Q is simple. Then by [3, Lemma 3.4] we can find $x \in P \backslash Q$ such that $Q \subsetneqq x^nR \subseteq xR \subsetneqq P$ for all n. Hence by (iii) we get $Q \subsetneqq Rx^nR \subsetneqq P$ for some $n \in \mathbb{N}$. Hence P/Q is never simple.

(iv) $\Rightarrow$ (i) Let $a \in R$. Clearly, we can assume that $a \in J$ and a is not nilpotent. Thus there exist minimal prime ideals Q and P with $Q \subsetneqq aR \subseteq P$. Using [3, Corollary 3.2] we get $\bigcap_n a^nR = Q$ - done.

COROLLARY 1.4. *Let R be a locally archimedian right chain ring. Then we have*

(i) *For $a \in R$ with $a^2 \neq 0$ and P the minimal prime ideal above aR we have $Pa \subseteq aP$. Moreover, $P = P^{\Pi}$ where P^{Π} is the minimal prime ideal with $Pa \subseteq aP^{\Pi}$.*

(ii) *For $a \in R$ with $a^2 \neq 0$ and P the minimal prime ideal above aR we have $sa = ar$ with $s \notin P$ implies $r \notin P$.*

Proof. (i) Of course, we can assume $a \in J$. Let $x \in P$. We consider the cases $xay = a$ and $xa = ay$. $xay = a$ implies $x^nay^n = a$ for all $n \in \mathbb{N}$. However there exists $k \in \mathbb{N}$ with $x^k = ar$, $r \in R$ which leads to $aray^k = a$. Contradiction. If in the case $xa = ay$ we have $y \notin P$, then we would obtain $x^na = ay^n$ with $ay^n \notin a^2R$ for all $n \in \mathbb{N}$. On the other hand we know $x^n = ar$ for a suitable $n \in \mathbb{N}$. Hence $ara = ay^n$, $a(ra - y^n) = 0$ which implies $ay^n = 0$. As P is minimal above $a \in R$, all prime ideals are completely prime and $Pa \subseteq aP$ is valid, $P^{\Pi} = P$ trivially holds.

(ii) We suppose $s \notin P$ and $r \in P$, hence for a suitable $n \in \mathbb{N}$ we have $r^n = a^3w$, $w \in R$. As $s^na = ar^n = a^4w$ and $s^nz = a$ for some $z \in R$, we obtain $s^n(a - za^3w) = 0$ and by Proposition 1.3 (ii) $s^na = 0$; together with (i) we conclude $a^2 = 0$. Contradiction.

We will now describe for a right aR the minimal two-sided ideal $\overline{aR}$ containing aR as well as the maximal twosided ideal $\underline{aR}$ contained in aR.

PROPOSITION 1.5. *Let R be a locally archimedian right chain ring, $a \in R$. Then $\underline{aR} = aP$, where P is a prime ideal, whereas $\overline{aR} = \bigcup xR$ with x running over $xs = a$ for arbitrary $s \in S = R \setminus P$.*

Proof. Obviously, we have $\underline{aR} = \bigcap_{u \in U} uaR$, so $ax \in \underline{aR}$ if and only if $uax \in aR$ for all $u \in U$. But this is equivalent to $Rax \subseteq aR$, hence $\underline{aR} = aP$ with $P = \{x \in R \mid Rax \subseteq aR\}$. Clearly, P is a right ideal. To show that P is prime let X be an ideal of R with $X \not\subseteq P$. Then $aR \subsetneq RaX$, which implies $RaX \subseteq RaX^2$, so $aR \subsetneq RaX^2$ that is $X^2 \not\subseteq P$. Now let $P_1 = \{x \in R \mid RaRx \subseteq aR\}$, hence $P_1 \subseteq P$. However, P_1 is a two-sided prime ideal which is maximal as a two-sided ideal in P; hence P/P_1 is simple. If P_1' denotes the minimal prime ideal above P_1, by Proposition 1.3(iv) we get a two-sided ideal $I \neq P_1$ with $\bigcap I^n = P_1$ using the fact that P_1 is completely prime. However, this is a contradiction to P/P_1 simple, hence $P = P_1$. Now let $y \in \overline{aR} \setminus aR$. Then $y \in uaR$ for some $u \in U$ with $uas = a$. We conclude $s \in S$ because aP is two-sided. Hence $\overline{aR} = \bigcup_{xs = a, s \in S} xR$.

Now we close the gap to Gräter's approach [9] and his definition of locally invariant chain rings.

PROPOSITION 1.6. *Let R be a chain ring. Then the following conditions are equivalent:*

(i) *R is locally archimedian.*

(ii) *For any $a \in R$ with $a^2 \neq 0$ the minimal prime ideal P above a satisfies $Pa = aP$.*

Proof. (i) $\Rightarrow$ (ii) By Corollary 1.4 we have $Pa \subseteq aP$. On the other hand by Proposition 1.3(iv) R is a (left) locally archimedian left chain ring, so the left version of Corollary 1.3 implies $aP \subseteq Pa$.

(ii) $\Rightarrow$ (i) If $Q \subsetneq P$ are prime ideals, let $a \in P \setminus Q$. Then $Pa = aP$ is a two-sided ideal between P and Q. Now the assertion follows from 1.3.

DEFINITION 1.7. *Let R be a chain ring with no zero-divisors. If R satisfies one of the equivalent conditions of Proposition* 1.6 *the ring R is called locally invariant.*

Locally invariant chain rings (= valuation rings) were first considered by Gräter [9]. His terminology "locally invariant" is justified by the fact that localizations at an arbitrary prime ideal produce a (partially) invariant valuation ring.

2. Locally Archimedian Right-Ordered Groups

One of the starting points of this investigation was the comparison of the next two results and their formal similarity.

Proposition 2.1. (Gräter [9]). *For a valuation v of a field K the following assertions are equivalent*:

(1) *v is locally invariant.*

(2) *If there are no further ϕ-convex subgroups between two ϕ-convex subgroups $U_1 \subset U_2$ of G_v, then U_1 is normal in U_2.*

Theorem 2.2 (Conrad [5]). *The following properties of a right-ordered group G are equivalent*:

(1) *For each pair $a, b \in P$ ($=$positive cone of G) there exists a positive integer n such that $(ab)^n > ba$.*

(2) *If C and C' are convex subgroups of G and C' covers C, then C is normal in C' and C'/C is orderisomorphic to a subgroup of $\mathbb{R}$.*

For further details concerning the terminology in Gräter's result we refer the reader to Section 3 as well as to the respective articles [9, 11, 12]. In this Section, however, we will sum up a few definitions concerning right-ordered groups and study the above-mentioned class of groups.

A *group G* with identity 1 is called *right-ordered* (ro) if there is a linear order $\leqslant$ on G which satisfies: $a \leqslant b \Rightarrow ac \leqslant bc$ for all $c \in G$. This is equivalent to the existence of a semigroup P (the *positive cone*) with $1 \notin P$ and $G = P \cup P^{-1} \cup \{1\}$ and $P \cap P^{-1} = \emptyset$. Given P, we get the right order by defining $a < b$ iff $ba^{-1} \in P$. Similarly, one can define a *left order* on G by $a <' b$ iff $a^{-1}b \in P$. Sometimes we will denote these orders by $<_r$ and $<_l$, respectively.

A subgroup C of the ro-group G is *convex* if $x \in G$ and $1 < x < c \in C$ imply $x \in C$, and the *covering* in 2.2 means that there are no convex subgroups between C and C'.

Before we start to study the connection between 2.1 and 2.2, we quote some lemmata on ro-groups from Conrad [5].

Lemma 2.3. *Let G be a* ro*-group, P its semigroup of positive elements and $a, b \in P$. Then the following three properties of G are equivalent*:

(i) *there exists a positive integer n such that $(ab)^n > ba$;*

(ii) *if* $a<b$, *then there exists a positive integer such that* $ab^na^{-1}>b$;

(iii) *there exists a positive integer* n *such that* $a^nb>a$.

A further result that we need is contained in

LEMMA 2.4. *Suppose that* G *satisfies the conditions in Lemma* 2.3. *If* C, C' *are convex subgroups of* G *such that* C' *covers* C, *and if* $a, b \in (C' \backslash C) \cap P$, *then there exists a positive integer* n *such that* $a^n > b$.

This lemma is the motivation for

DEFINITION 2.5. *Let* G *be an* ro-*group which satisfies the* (*equivalent*) *conditions in Lemma* 2.3. *Then* G *is called locally archimedean.*

Remark. In [1] a ro-group which fulfills the conditions in Lemma 2.3 is called poly-ordered.

After all this, it is hardly surprising that the positive cone P of a locally archimedean ro-group behaves like the multiplicative semigroup of a locally invariant chain ring.

LEMMA 2.6. *Let* G *be a locally archimedean* ro-*group, and* P *its positive cone. Let* $a \in P$.

(i) *If* $a=ras$ *with* $r, s \in P$, *then* $s^n<a$, $r^n<a$ *for all* $n \in \mathbb{N}$.

(ii) $Pa^2 \subseteq aP$, $a^2P \subseteq Pa$.

(iii) *Let* C *and* C' *be convex subgroups of* G *such that* C' *covers* C *and* $a \in P \cap (C' \backslash C)$. *Set* $Q = P \cap (G \backslash C')$, *then we have* $Qa = aQ$.

(iv) *Let* C *be a subgroup of* G. *Then* C *is convex with respect to the right order iff* C *is convex with respect to the left order.*

Proof. (i) Let $a=ras$, $r, s \in P$, so $a=r^nas^n$ for all $n \in \mathbb{N}$. Suppose $a<s^n$ for some n. Then $s^n=pa$ with $p \in P$, and $a=r^napa$ which implies $1=r^nap \in P$: contradiction. If $a<r^n$ for some n, then $r^n=qa$ with $q \in P$, and we get $a=qa^2s^n=q^ka(as^n)^k$ for all $k \in \mathbb{N}$. As G is locally archimedean, there exists k with $(as^n)^k>s^na$, so $(as^n)^k=ts^na$ with $t \in P$, which again leads to a contradiction.

(ii) Let $p \in P$. If $a=pa^2q$ with $q \in P$, then $a=p^ka(aq)^k$ for all $k \in \mathbb{N}$. But for a suitable k we have $(aq)^k>qa>a$, contradicting (i). So $Pa^2 \subseteq aP$. Similarly, one obtains $a^2P \subseteq Pa$.

(iii) is left to the reader.

(iv) It suffices to prove $\Rightarrow$. Let C be a (right) convex subgroup, $y \in C$ and $1 <_l x <_l y$, where $<_l$ denotes the corresponding left order. Hence $xr=y$ with $r \in P$. If we have $x <_r y$, we are done. Otherwise we obtain

$sy = x$ with $s \in P$ which implies $sxr = x$ and $syr = y$. By (i) we know $s, r <_r y$, hence $sy = x \in C$.

The proof above shows that Conrad's condition (see Lemma 2.3(i)) can be sharpened.

COROLLARY 2.7. *Let G be a* ro-*group which is locally archimedian. Then we have for all $a, b \in P$ $(ab)^3 > ba$.*

Proof. $(ab)^3 = a(ba)^2 b = ab'(ba)$ by 2.6(ii), hence $ba < (ab)^3$.

Now we have the tools to give new characterizations of locally archimedian right-ordered groups. For convenience we introduce the following notation:

$$C(a) = \{1\} \cup \{x \in G \mid x \in P \text{ and } x < a^n \text{ for some } n,$$
$$\text{or } x^{-1} \in P \text{ and } x^{-1} < a^n \text{ for some } n\}$$

is the convex segment generated by the powers of a. Furthermore, we say that x is *infinitely smaller* than y (and write: $x \ll y$) if $x^n < y$ for all $n \in \mathbb{N}$ or $x^n < y^{-1}$ for all $n \in \mathbb{N}$.

THEOREM 2.8. *Let G be a* ro-*group with positive cone. Then the following conditions are equivalent*:

(i) *$(G, <)$ is locally archimedian.*

(ii) *For all $a \in P$: $a^2P \subseteq Pa$.*

(iii) *For all $a \in P$: $Pa^2 \subseteq aP$.*

(iv) *$C(a)$ is a convex subgroup for all $a \in P$.*

(v) *For all $a, b \in P$: $C(ab) = C(ba)$.*

(vi) *For all $x, y \in G$: $[x, y] \ll x$ or $[x, y] \ll y$.*

Remark. The equivalence of (vi) and (i) answers a problem of Ault [2] whether condition (vi) is sufficient for being locally archimedian.

Proof of 2.8. (i) $\Rightarrow$ (ii) follows from Lemma 2.6. On the other hand with (ii) we get $(ab)^3 = a(baba)\,b = ab'(ba)$, hence $ba < (ab)^3$ which implies that G is locally archimedian. Thus (i) $\Leftrightarrow$ (ii). (ii) $\Leftrightarrow$ (iii) is obvious.

To prove (iii)(resp. (ii)) $\Rightarrow$ (iv) we show that $C = C(a)$ is multiplicatively closed for an arbitrary element $a \in P$. Let $x, y \in C$.

Case 1. If $x, y \in P$, we have $x < a^n$, $y < a^m$ for some $n, m \in \mathbb{N}$, so $sx = a^n$, $ty = a^m$. Using (ii) we get $a^n a^n a^m = sxsxty = t'sxy$, hence $xy < a^{2n+m}$ and we are done.

Case 2. Now suppose $x^{-1}, y \in P$. If we assume $xy \in P$, we obtain $xy < x^{-1}xy = y$ which implies $xy \in C$. Otherwise we have $y^{-1}x^{-1} \in P$. Then $y^{-1} < 1$ implies $y^{-1}x^{-1} < x^{-1}$ and we conclude $xy \in C$.

Case 3. In case of $sx = a^n$, $ty^{-1} = a^m$ for suitable $m, n \in \mathbb{N}$ and $s, t \in P$ we distinguish between the two possibilities: $xy \in P$ and $y^{-1}x^{-1} \in P$. If $xy = q_l a^l$ holds for all $l \in \mathbb{N}$ with suitable $q_l \in P$ we have $a^n = sq_l a^l y^{-1}$ for all l, however for $l \geqslant 2n$ we get a contradiction with (iii). The case $y^{-1}x^{-1} \in P$ is treated similarly.

Case 4. $x^{-1}, y^{-1} \in P$, use the same arguments as in Case 1.

Clearly the definition of $C(a)$ implies that for $x \in C(a)$ we also have $x^{-1} \in C(a)$. Also, it is obvious that $C(a)$ is convex.

(iv)$\Rightarrow$(ii) First we prove $x^2P \subseteq Px$ for all $x \in P$. Suppose not, so $x = px^2q$ with $p, q \in P$. Clearly $q < x$, so $q < q^2 < xq$. On the other hand, $x = p^n x(xq)^n$ for all $n \in \mathbb{N}$, so $x \in \bigcap_n P(xq)^n$ and thus $x \notin C(xq)$. But q and xq belong to $C(xq)$ and this is a subgroup, so $x = xqq^{-1} \in C(xq)$: contradiction.

(i)$\Rightarrow$(vi) see Ault [2, p. 663] or use Conrad [5, Theorem 4.1].

(vi)$\Rightarrow$(ii) Let $x, y \in P$. We have to show $y^2P \subseteq Py$; set $s = [y, x] = yxy^{-1}x^{-1}$. If $s \in P$, we have $y^2x = y(yx) = ysxy \in Py$. Otherwise we know $s^{-1} = [x, y] \in P$. First we assume $s^{-1} \ll y$. In particular $rs^{-1} = y$ for some $r \in P$. Hence $y^2x = y(yx) = rs^{-1}yx = rxy \in Py$. So it remains to consider the case $s^{-1} \ll x$. Then $ts^{-1} = x$ for some $t \in P$. If we assume $y^2P \nsubseteq Py$, we would get $y = qy^2x$ for some $q \in P$, then $y = q^2y^2xyx = q^2y^2ts^{-1}yx = q^2y^2txy \in Py$: contradiction. Thus we must have $y^2x \in Py$.

For examples of locally archimedian ro-groups we refer the reader to [5, 2, 1].

3. Locally Invariant Valuations and Locally Archimedian Right Pre-ordered Groups

We want to continue in the direction of Proposition 3.1, here in regard to its valuation-theoretical part. For the convenience of the reader, we first recall some terminology introduced by Mathiak [11]. If K is a field (not necessarily commutative) and W a linearly ordered set with minimal element 0, then a surjective map v: $K \to W$ is called a *valuation* of K, if for all $x, y, z \in K$ the following holds:

(i) $v(x)=0 \Leftrightarrow x=0$.

(ii) $v(x+y) \leqslant \max\{v(x), v(y)\}$.

(iii) $v(x) \leqslant v(y) \Rightarrow v(zx) \leqslant v(zy)$.

If v is a valuation of K, then we define

$$B_v = \{x \in K \mid v(x) \leqslant v(1)\},$$
$$M_v = \{x \in K \mid v(x) < (1)\}.$$

B_v is the *valuation ring* associated to v, with the *maximal ideal* M_v.

Given a valuation $v\colon K \to W_v$, we have for any $x \in K$ an order-preserving map $\tilde{x}\colon W_v \to W_v$, where $\tilde{x}v(y) = v(xy)$ for all $y \in K$. Then the set $G_v = \{\tilde{x} \mid x \in K^*\}$ is a group of order-preserving bijections of W_v, and is called the *value group of* v. Note that the value group is not W_v as in the classical situation, but is a subgroup of Aut W_v. The valuations in the sense of Mathiak and those in the sense of Schilling coincide if and only if the valuation ring B_v is a duo ring, that is $xB_v = B_v x$ for all $x \in K$.

In this case every element of G_v can be represented by an element of $W_v \backslash \{0\}$. With the definition

$$a \leqslant b \quad \text{iff } \tilde{a}v(x) = v(ax) \leqslant v(bx) = \tilde{b}v(x) \qquad \text{for all } x \in K$$

we have an order relation on G_v which is compatible with its group structure. Some results on these (not necessarily linearly) ordered groups can be found in Rohlfing [14]. The difficulty is, that this order on G_v is linear if and only if B_v is subvariant (or semi-invariant, see Brungs–Törner [4]), that is for all $a \in B_v$ we have $B_v a \subseteq aB_v$ or $aB_v \subseteq B_v a$ (c.f. Mathiak [11, p. 228]). In other words, a linear order on G_v implies that our situation is not too far from the commutative case. However, the linear structure seems to be adequate because of the linear structure of the ideal lattice, so we prefer to study a (linear) pre-order, namely

$$\tilde{a} \preccurlyeq \tilde{b} \Leftrightarrow \exists r \in B_v \backslash \{0\}\colon \tilde{a} = \widetilde{rb}.$$

It is easily seen that this relation is well-defined, transitive, linear and right monotone, but $\tilde{a} \preccurlyeq \tilde{b}$ and $\tilde{b} \preccurlyeq \tilde{a}$ does not imply $\tilde{a} = \tilde{b}$, we only get $B_v a = B_v b$. Nevertheless, this pre-order is important in the situation we want to study, as can be seen from the following result.

THEOREM 3.1. *Let v be a valuation of the (not necessarily commutative) field K and B_v the associated valuation ring. Then the following assertions are equivalent*:

(i) *B_v is a locally invariant chain ring.*

(ii) *$(G_v, \cdot, \preccurlyeq)$ satisfies the condition in Lemma 2.3 for being locally archimedian, that is, for all $\tilde{a}, \tilde{b} \prec 1$ there is a positive integer n with $(\widetilde{ab})^n \preccurlyeq \widetilde{ba}$.*

Note that we have to reverse the order compared to our earlier notation, as our valuation ring is defined to be $B_v = \{x \in K \mid v(x) \leqslant 1\}$.

Proof. (i) $\Rightarrow$ (ii) Let B_v be locally invariant. Then $(ab)^3 = a(ba)^2 b \in B_v(ba)$, so $(\widetilde{ab})^3 \preccurlyeq \widetilde{ba}$.

(ii) $\Rightarrow$ (i) Suppose $B_v a^2 \nsubseteq aB_v$ for some $a \in B_v$, so $a = ra^2 s$ for some $s \in M_v$. Then $a = r^n a(as)^n$ for all $n \in \mathbb{N}$. But $(\widetilde{as})^n \prec \widetilde{sa}$ for some n, so $(as)^n = qsa$ with $q \in M_v$ and hence $a = r^n aqsa$: contradiction.

Particularly important is the case that for each class xU (where U is the set of units of B_v), $x \neq 0$, there exists $\alpha(x) \in G_v$ such that $\{\alpha(x) \mid x \in K^*\}$ is a subgroup of G_v. So we have a multiplicatively closed system of representatives for each principal right ideal level. If we assume B_v to be locally invariant then $\{\alpha(x) \mid x \in K^*\}$ is a locally archimedian right-ordered group. Fields with such a locally invariant valuation can be obtained via semigroup rings of right-ordered semigroups. This is the subject of the next section.

4. Locally Invariant Chain Rings via Group Rings over ro-Groups

In this last section we want to construct chain domains using group rings over suitable groups. The following two methods have appeared in the literature so far. In [7], Dubrovin obtains a chain ring which is "associated" to a right-ordered one-relator group, by applying the Lewin–Lewin result [10] that group-rings of torsion-free one-relator groups over skew-fields are embeddable into skew-fields. Albrecht/Törner [1] describe a class of right-ordered groups over which certain group rings are Ore rings; also in this situation a chain ring is a "natural" by-product. In the following analysis we want to pursue both of these results further. First we generalize a definition from [7]:

Definition 4.1. *Let G be an ro-group with positive cone P and R a ring, $S \subseteq R \backslash J(R)$ a multiplicative semigroup with $1 \in S$. The ring R is called S-associated to the group (G, P), if there exists a monomorphism $\mu: P \to J(R)$ such that for any element $0 \neq a \in J(R)$ there exist $g_1, g_2 \in P$, $u_1, u_2, v_1, v_2 \in S$ with $Ru_1 a = Ru_2 \mu(g_1)$, $av_1 R = \mu(g_2) v_2 R$.*

In the following we will always identify P with its image $\mu(P)$. If R is a local ring and $U(R)$-associated to the group (G, P) then R is a chain domain (this is the situation studied by Dubrovin in [7]). As we want to construct locally invariant chain rings, we have to specify the group G. In the preceding section we have seen that locally archimedian groups are characterized by a property similar to that for locally invariant rings. So later on, G will be a locally archimedian ro-group.

But first we fix some general notation. G will always be a ro-group with positive cone P, R a ring without zero-divisors and $\sigma: G \to \mathrm{Aut}(R)$ a monomorphism. By defining $gr = r^{\sigma(g)}g$ $R^{\sigma}[G]$ denotes the skew group ring, and $R^{\sigma}[P \cup \{1\}]$ the skew semigroup ring. Moreover, set

$$S := \left\{ \sum_{g \in P \cup \{1\}} g r_g \mid r_1 \neq 0 \right\}.$$

Then for any $0 \neq a \in R^{\sigma}[G]$ there are uniquely determined elements $u, v \in S$ and $g, h \in G$ with $a = ug = hv$ (see also [1, Lemma 1.1]). Furthermore, we have

$$g \in P \text{ if and only if } h \in P.$$

The above is called the *canonical right* (*left*) *decomposition* of a. In the terms of 4.1, the skew semigroup ring $R^{\sigma}[P \cup \{1\}]$ is S-associated to the group (G, P).

In the case of a locally archimedian group we can say more about the above decomposition.

LEMMA 4.2. *Let G be locally archimedian, $a \in R^{\sigma}[P \cup \{1\}]$. If $a \notin S$ and $a = ug = hv$ is the canonical decomposition of a, then the following holds: There exist $f_1, f_2 \in P \cup \{1\}$ such that*

$$f_1 g = h \text{ and } g = h f_2, \text{ and } f_1, f_2 \ll g, h.$$

Proof. Let $a = u_1 g_1 + \cdots + u_n g_n = h_1 v_1 + \cdots + h_n v_n$. We may assume: $g_1 <_r g_2 <_r \cdots <_r g_n$, $h_1 <_l \cdots <_l h_n$. Set $g = g_1$, so $g_i = g_i' g$ for suitable $g_i' \in P$. Similarly, set $h = h_1$, so $g_i = hh_i'$, $h_i' \in P$. Then

$$u = u_1 + u_2 g_2' + \cdots + u_n g_n' \qquad \text{and} \qquad v = v_1 + h_2' v_2 + \cdots + h_n' v_n.$$

Now $h = g_i$ for some i; set $f_1 = g_i'$, so $h = f_1 g$. If $f_1 = 1$, everything is clear. Otherwise, we have $g = h_j$ for some j; set $f_2 = h_j'$, so $g = hf_2$. From $g = f_1 g f_2$ and $h = f_1 h f_2$ we get the assertion by Lemma 2.6 as G is locally archimedian.

Lemma 4.2 shows that the "leading elements" g and h are not too far from each other, they are in the same convex segment and can be compared by elements which are infinitely smaller. To be more precise, we have

$$C(g) = C(h) \qquad \text{and} \qquad C(hg^{-1}),\ C(h^{-1}g) \subsetneq C(g).$$

We will write $g \sim h$ iff the above is satisfied. It is easy to see that in locally archimedian ro-groups $\sim$ is an equivalence relation on P, if for $x \in P^{-1}$ we define $C(x) := C(x^{-1})$.

Now we assume that S is a left and right Ore set in $R^\sigma[P \cup \{1\}]$. Then every $0 \neq a \in R^\sigma[P \cup \{1\}]_S$ can be written in the form $a = u_2^{-1} u_1 g = hv_1 v_2^{-1}$, where $g, h \in P$, $u_1, u_2, v_1, v_2 \in S$. Then $u_1 g v_2 = u_2 h v_1$. Now $gv_2 = v_2' g'$ and $u_2 h = h' u_2'$ for some $g', h' \in P$, $v_2', u_2' \in S$, so $u_1 v_2' g' = h' u_2' v_1$. Using 4.2 we obtain $g \sim g'$, $h \sim h'$, and $g' \sim h'$, hence $g \sim h$.

Now we have all the tools to prove:

THEOREM 4.3. *Let K be a skew-field, G a* ro-*group with positive cone P and S a left and right Ore set in $K^\sigma[P \cup \{1\}]$. Then the following are equivalent*:

(i) *$R = K^\sigma[P \cup \{1\}]_S$ is locally invariant.*

(ii) *G is locally archimedian.*

If these conditions are satisfied, then the chain ring R is $U(R)$-associated to (G, P).

Proof. (i) $\Rightarrow$ (ii) By Proposition 1.3 we know that for any $f, g \in P$ we have $fg^2 \in gR$. Hence $Pg^2 \subseteq gP$.

(ii) $\Rightarrow$ (i) We show $Ra^3 \subseteq aR$ for any $a \in R$. Let $a = hv_1 v_2^{-1}$ and $r = gu_1 u_2^{-1}$. Then $ra^3 = gu_1 u_2^{-1} (hv_1 v_2^{-1})^3 = gh_1 h_2 h_3 w$, $w \in U(R)$, where $h_i \sim h$ for all i. In the "worst" case we have $h_1 f_1 = h$ with $f_1 \in P$, $f_1 \ll h$. Then also $f_1 \ll h_2$, so $h_2 = f_1 f_2'$, $h_2' \in P$. Assuming $h = h_2' f_2$, we have $f_2 h_3' = h_3$, which implies $ra^3 = gh_1 f_1 h_2' h_3 w = ghh_2' f_2 h_3' w = gh^2 h_3' w = hg'w$, as G is locally archimedian. Thus $ra^3 \in aR$.

COROLLARY 4.4. *Let K be a skew-field, G a* ro-*group with positive cone P. If G has a subnormal series with torsion-free abelian factors, then $R = K^\sigma[P \cup \{1\}]_S$ is locally invariant.*

Proof. By [1, 2.2] G is locally archimedian and by [1, 3.4] S is a left and right Ore set, so by Theorem 4.3 R is locally invariant.

For examples we refer the reader to [1].

For our method of constructing a locally invariant ring R it was essential that we could obtain R as a classical localization of a skew semigroup ring, as in this case certain properties of the group are inherited by the ring R.

Dubrovin [7] has shown that one can construct a chain ring R associated to a given group G even if it is only known that the group ring is embeddable into a division ring. However, in this situation it is difficult to decide whether, for example, G locally archimedian implies that R is locally invariant.

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