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Right Chain Rings, Part 1

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Most of the results presented in this first volume have been obtained during the last fifteen to twenty years. However, many of the problems that led to the investigation of this particular kind of rings are of much earlier date. New problems arise as an attempt is made to understand and treat right chain rings systematically.

A *right chain ring* R is a ring with unit element whose right ideals are linearly ordered by set inclusion. These rings are obvious generalizations of commutative valuation domains as well as of division rings.

Such rings appear as coordinate rings of Hjelmslev planes (Klingenberg [54], [55]), as building blocks for the localizations of non-commutative Dedekind rings (Gwynne/-Robson [71]) or of FPF-rings (Faith/Page [84]) and as 'valuation domains' of ordered non-commutative division rings (Schröder [86], Morandi/Wadsworth [89]). Domains with a distributive lattice of right ideals (i.e. Prüfer domains in the commutative case) are characterized by the fact that their localizations at maximal right ideals are right chain rings (Brungs [76]). Osofsky showed in 1968 that local rings whose cyclic modules have cyclic injective hulls are right and left chain rings. Roughly speaking right chain rings are often the 'atoms' in structure theorems of noncommutative ring theory, in particular for right semihereditary rings.

Right and left chain domains are exactly the *valuation rings* considered by Mathiak ([77], [86]) or the *total rings* in Cohn ([89], p. 3). Schilling [50] used the term *valuation ring* for total *and* invariant subrings of a skew field. Such rings occur in the construction of division rings (Amitsur [72], Cohn [61] or B. Jacob/A.R. Wadsworth [86] and in many other papers) and in the computation of SK_1 for skew fields (Draxl/Kneser [80]). A right chain domain R such that its skew field $Q(R) = D$ of quotients is finite dimensional over its center K is also a left chain domain, but not necessarily invariant.

More recently, Dubrovin introduced a generalization of chain rings for which matrix rings over chain rings are the easiest eamples and which have a richer extension theory than the previously mentioned types of valuation rings (Dubrovin [84], [85]; Brungs/Gräter [90]; Wadsworth [89]).

In these notes we will attempt to treat one-sided as well as noninvariant chain rings.

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This *Part 1* deals with the basic definitions and general facts. It is subdivided into 9 chapters. Main topics are right ideals, two-sided ideals, prime ideals, zero-divisors, annihilators, prime segments, chain conditions, localizations, various degrees of non-commutativity, chain conditions in right chain rings and overrings of right chain rings. For example, it is proved in Chapter 1 that the intersection of the powers of any non-nilpotent ideal is a completely prime ideal in a right chain ring. A rather intriguing topic is the still open question whether or not prime ideals that are not completely prime can exist in right chain rings. We call this type of prime ideal *exceptional* (see Chapter 6). We describe conditions necessary for the existence of such ideals and its implications. In particular, an exceptional prime ideal Q is always paired with a completely prime ideal P as its upper neighbour in the lattice of prime ideals such that there is no two-sided ideal properly between P and Q.

Fundamental for the understanding of right chain rings is also the knowledge of the prime segments or equivalently the rank-1-case. Examples show that it is not possible at this time to treat the most general case. It is therefore necessary to introduce various conditions, like *right invariance* (all right ideals are two-sided) or the condition that there exists another two-sided ideal between any two prime ideals (then R is called *locally archimedean*)(see Chapter 7). Right noetherian right chain rings are right invariant (see Chapter 6) and a chain ring R in a division ring D is locally archimedean if D is finite dimensional over its center. Finally, the consequences of various chain conditions on prime ideals in R are investigated (Chapter 8). Chapter 9 is devoted to the study of overrings of right chain domains in their quotient fields where we restrict ourselves to the case where R is of rank 1.

Throughout the sections examples are given that also provide an introduction to construction methods which are discussed in detail in later chapters.

In the following chapters of the next parts we will deal with chain rings in finite dimensional division algebras and Dubrovin valuation rings; with the structure of rank-1-discrete valuation rings with ordered fields and the geometric structures over right chain rings.

There will be a chapter on modules over chain rings and a chapter on completions.

The semigroup of the principal right ideals of a right invariant right chain ring forms a holoid which is a generalization of the positive cone of an ordered group. We will prove a structure theorem for a certain class of holoids.

Throughout we will consider examples and construction methods for right chain rings.

Contents

Part I
Fundamental terminology and basic results

1 Some properties of right chain rings and first theorems

After introducing the basic definitions and notations we prove some elementary facts about right ideals, especially the so-called standard right ideals. Although a detailed discussion of prime ideals in right chain rings will be presented in Chapter 6 we prove here in a first theorem (Theorem 1.15) that idempotent ideals are completely prime and that the intersection of powers of a non-nilpotent ideal I is a completely prime ideal. Of particular interest is the situation in which there exists an additional two-sided ideal between any two neigbouring prime ideals. Then $\bigcap_{n\in\mathbb{N}} x^n R$ equals Q for all $x \in P\backslash Q$ (Theorem 1.21).

1.1 Terminology and notations

All rings are associative, but in general not commutative. Every ring has a unit element, denoted by 1, which is inherited by subrings, preserved by homomorphisms, and acts as the identity operator on modules. Rings may have zero-divisors. Rings without zero-divisors will be called *domains*. Furthermore, for a ring R we set $R^* = R\backslash\{0\}$.

DEFINITION 1.1 (i) *A ring R is called a* right (left) chain ring *if either $aR \subseteq bR$ or $bR \subseteq aR$ ($Ra \subseteq Rb$ or $Rb \subseteq Ra$) for any elements $a, b \in R$. If R is both a right and a left chain ring then R is called a* chain ring.

(ii) *A right chain ring R is called a* right Hjelmslev ring *(sometimes called affine Hjelmslev ring) if every non-unit in R is a right and left zero-divisor in R. If in addition R is a chain ring, we speak of a (projective)* Hjelmslev ring.

It follows immediately from the definition that right chain rings are *local* rings (see Lambek [66], p. 75), that is, R has exactly one *maximal right ideal*, the *Jacobson radical* of R which is denoted by $J(R) = J$. Trivially, $J(R)$ is a two-sided ideal consisting of the non-units of R; the *group of units* $R\backslash J$ is denoted by $U(R) = U$.

Rings of this type occur in various circumstances, but often under different names. In the commutative case the term *generalized valuation ring* is used for rings which are not necessarily a domain (see Warfield [69a], [70], Shores [74]). Schilling's *valuation rings* (see [50]) are not necessarily commutative chain domains, however all one-sided ideals are assumed to be two-sided, in other words, the ring is invariant,i.e. $Ra = aR$ for all $a \in R$. (*Right*) *Valuation ring* is a name used by several authors (see Posner [63], Beauregard [73], Koehler [76], Jain [78], [84], Goel/Jain [78], Jain/Saleh [87], Faith [79], van Geel [81], Chacron [85], Mathiak [77], [81], [86] and others), whereas Dubrovin [78], [84] applies the term valuation ring to a larger class of non-commutative rings. Gilmer ([72], p. 184) and Froeschl [76] speak of a *chained ring*, whereas Clark/Liang [73] and Clark/Drake [73] restrict the word *chain ring* to the finite case. However, the term chain ring can also be found in a more general context (see Skornyakov [64], [66], Brameret [63], Dubrovin [78], [80], [82], [83] and others). Chain rings in a geometrical context were first considered by Klingenberg [54], [55] in several papers since 1954. (Right) *Hjelmslev rings* are used as coordinate rings of affine as well as projective desarguesian Hjelmslev planes (see a forthcoming part). In particular, their algebraic structure contains valuable informations for the geometrical context.

Many examples of right chain rings will be given in the following chapters and starting in Chapter 3, several construction methods will be discussed in Part 3. A class of right chain rings will be introduced in Chapter 3 which illustrates various phenomena that do not occur for chain rings.

The next lemma lists some observations that follow directly from the definition. First a further notation: Let $A \supset B$ be right ideals which are *neighbours*, that is, $A \supseteq C \supseteq B$ for a right ideal C implies $A = C$ or $B = C$, we write $A \succ B$ or $B \prec A$ for short. If we include the case of equality we set $A \succeq B$ resp. $B \preceq A$.

LEMMA 1.2 *Let R be a right chain ring. Then the following is true:*

(i) *The lattice of right ideals of R is linearly ordered by inclusion.*

(ii) *All finitely generated right ideals are right principal.*

(iii) *Let R be in addition a domain. Then any two principal left ideals with nonzero-intersection are comparable.*

(iv) *Let R be in addition a domain with the property that the intersection of any two principal left ideals is $\neq (0)$. Then R is a chain domain.*

(v) *Let A, B be right ideals with $A \succ B$, then $A = aR$ and $B = aJ$ for some $a \neq 0$ in R.*

By statement (v) right ideals which are lower neighbours are of type aJ for some $a \in R$. This is the only case where a right ideal B is not equal to the intersection of the principal right ideals strictly containing B.

PROOF: (i) Take right ideals A, B and assume $A \not\subseteq B$. Thus an element $a \in A \backslash B$ exists showing $x \in aR$ for all $x \in B$. So we obtain $B \subseteq aR \subseteq A$.

(ii) Obviously all finitely generated right ideals of a right chain ring are principal.

(iii) Let $0 \neq Ra \cap Rb$, hence $r_1 a = r_2 b$ for some $r_1, r_2 \in R$. We may assume $r_1 t = r_2$ and we obtain $a = tb$, that is, $Ra \subseteq Rb$.

(iv) follows from (iii).

(v) Let $a \in A \backslash B$, hence $A \supseteq aR \succ aJ \supseteq B$, thus $A = aR$ and $B = aJ$. ∎

We remark that by (iv) right chain domains which are not left chain rings are not left Ore.

Let us recall some terminology which will be used in this context. A ring R is said to be a *right Bezout ring*, if for any two principal right ideals, their sum and intersection are again principal. *Left Bezout rings* are defined similarly, and a ring which is both left and right Bezout is called a *Bezout ring* (see Cohn [63]). If one restricts this property to principal right ideals having nonzero intersection we obtain the definition of a *weak Bezout domain* resp. *weak Bezout ring*. It is shown in Cohn [63] that the definition of a *weak* Bezout domain is left-right symmetric. By Beauregard ([73], Prop. 1) local weak Bezout domains are the so-called *weak valuation domains*. Let us say that R is a *weak valuation domain* if $aR \cap bR \neq 0$ implies either $aR \subseteq bR$ or $bR \subseteq aR$. By the above cited result the left-right symmetry of this definition is obvious. Since a right chain domain R is a weak valuation domain, the statements (iii) and (iv) follow immediately.

We say that a ring R is *right distributive* if

$$A \cap (B + C) = (A \cap B) + (A \cap C)$$

for any right ideals $A, B, C \subseteq R$.

The next result describes local right Bezout or right distributive rings as right chain rings.

PROPOSITION 1.3 *For a ring R the following assertions are equivalent:*

(a) *R is a right chain ring.*

(b) *R is a local right Bezout ring.*

(c) *R is a local right distributive ring.*

PROOF: (a) implies (b) as well as (c).

To show that a local right Bezout ring R is a right chain ring let $0 \neq a, b \in J$ and we have $aR + bR = cR$ for some $c \in R$, hence $aR, bR \subseteq cR$. If a and b are both in cJ, then $cR = cJ$ and $c = 0$, a contradiction. Hence we have w.l.o.g. $aR = cR$ and thus $bR \subseteq aR = cR$.

To prove that (a) follows from (c) choose any $a, b \in R$ and $aR = aR \cap (bR + (a - b)R) = (aR \cap bR) + (aR \cap (a - b)R)$ implies $a = (a - b)t + r$ for $r \in aR \cap bR$. It follows that $a(1 - t) \in bR$, $bt \in aR$. If $t \notin J$, $bR \subseteq aR$. If $t \in J$ then $1 - t$ is a unit in R and $aR \subseteq bR$ follows, i.e. R is a right chain ring. ∎

1.2 Right ideals in right chain rings

In this section we summarize a few basic results on right ideals in right chain rings. The following useful lemma shows that a right ideal $I \subseteq R$ is a left ideal if $UI = I$.

LEMMA 1.4 (Test-units-Lemma) *Let R be a right chain ring and $a \in R$. Then $Ra \subseteq UaR$. More precisely, for any $x \in R$ there exist $y \in R$, $u \in U$ with $x = uy = yu$ and $ya \in aR$.*

PROOF: If x is a unit or xa is in aR, we are done. Otherwise we have $xas = a$ for some $s \in J$, hence $1 + s \in U$. Then $xa(1 + s) = xa + a = (1 + x)a$ and we conclude $xa = (1+x)a(1+s)^{-1} \in UaR$ and the statement follows with $u = 1+x,\ y = (1+x)^{-1}x$. ∎

The next lemma lists several facts on right ideals:

LEMMA 1.5 *Let R be a right chain ring.*

(i) *Let A be a right ideal of the multiplicative semigroup of the ring R. Then A is also a right ideal of R.*

(ii) *An additive subgroup A of R is a right ideal exactly if $Au \subseteq A$ for all $u \in U$.*

(iii) *A right ideal A of R is a two-sided ideal exactly if $uA \subseteq A$ for all $u \in U$.*

(iv) *Let A be a right ideal and B be a two-sided ideal. Then the complex multiplication $A \cdot B$ reduces to ring multiplication, that is*

$$A \cdot B = \{\sum a_i b_i \mid a_i \in A, b_i \in B\} = \{ab \mid a \in A, b \in B\}.$$

PROOF: (i) It is sufficient to prove that A is an additive subgroup of the ring R. Let $a, b \in A$ and w.l.o.g. $ar = b$. Then $a - b = a(1 - r)$ which is in A by assumption.

To prove (ii) let b be in A and $r \in J$. Then $1 + r \in U$, so $b(1 + r) \in A$ and hence $br \in A$.

(iii) This follows immediately from Lemma 1.4.

(iv) Obviously $\{ab \mid a \in A, b \in B\}$ is contained in $A{\cdot}B$. Let $\sum a_i b_i \in A{\cdot}B$. W.l.o.g. assume $a_1 R \supseteq a_i R$ and $a_1 r_i = a_i$. Then $\sum a_i b_i = a_1 \sum r_i b_i \in \{ab \mid a \in A, b \in B\}$. ∎

An immediate consequence of Lemma 1.5 is the following observation.

LEMMA 1.6 *Let I be a right ideal in a right chain ring R.*

(i) *Then $\overline{I} = \bigcup_{u \in U} uI$ is the minimal two-sided ideal containing I and $\underline{I} = \bigcap_{u \in U} uI$ is the maximal two-sided ideal contained in I.*

(ii) *$\overline{I}$ is nil if and only if the right ideal I is nil.*

Note the a right ideal I is called *nil* if all elements in I are *nilpotent*, that is, for each $a \in I$ there exists $n \in \mathbb{N}$ satisfying $a^n = 0$.

PROOF: (i) Lemma 1.4 and 1.5 show that $RI = \bigcup_{u \in U} uI$ holds. This is clearly the minimal two-sided ideal over I. Any two-sided ideal contained in I is contained in uI for every $u \in U$. That $\underline{I}$ is two-sided follows using Lemma 1.4.

(ii) obvious. ∎

Hence in the class of right chain rings there is no counterexample to the *Koethe-Conjecture* which asks for the existence of nil ideals provided the ring contains a nil right ideal.

If we take an arbitrary right ideal I the next two-sided ideals $\underline{I}$, $\overline{I}$ may differ substantially and may possibly coincide with the next neighbouring prime ideals. This special situation is analysed in detail in Section 1.5 and Chapter 6.

For the sake of completeness we recall the definition of a prime ideal. Although we give the definition for one-sided prime ideals, we will usually consider two-sided prime ideals if it is not mentioned explicitly.

DEFINITION 1.7 *Let $P \neq R$ be a right ideal of the ring R.*

(i) *P is called* completely prime *if and only if $xy \in P$ implies $x \in P$ or $y \in P$ where $x, y \in R$.*

(ii) *P is called* prime *if and only if $xRy \subseteq P$ implies $x \in P$ or $y \in P$ for $x, y \in R$.*

(iii) *A prime ideal P which is not completely prime is called* exceptional.

(iv) *The intersection of all (two-sided) prime ideals of R is called the* prime radical *denoted by $Rad(R)$.*

It is clear that in a right chain ring the intersection of prime (completely prime) ideals is again prime (completely prime). The union of (completely) prime ideals lying in a chain is (completely) prime. Note that for a completely prime ideal P in a right chain ring we have $P = sP$ for any $s \notin P$.

Let A be a right ideal, then there exists a minimal prime ideal P containing A, namely the intersection of all prime ideals containing A. If A does not lie in the *prime radical*, then we have also a maximal prime ideal Q contained in A by taking the union of all prime ideals below A. So if A is not itself a prime ideal, it makes of makes sense to speak of the *prime segment* containing A consisting of all right ideals between the two neighbouring prime ideals P and Q. A precise definition will be given later (see Definition 1.17).

The next lemma states some helpful observations:

LEMMA 1.8 (Test-squares-Lemma) *Let R be a right chain ring and P a right ideal of R. Then*

(i) *P is prime if and only if $xRx \subseteq P$ implies $x \in P$.*

(ii) *P is prime if and only if $X^2 \subseteq P$ implies $X \subseteq P$ for every right ideal X.*

Moreover, let P be two-sided, then

(iii) *P is a completely prime ideal if and only if $x^2 \in P$ implies $x \in P$.*

(iv) *Let $A \neq R$ be a one-sided ideal with $R \backslash A$ multiplicatively closed. Then A is a two-sided completely prime ideal.*

PROOF: (i) If P is prime, the assertion follows directly by the definition. Now assume $xRy \subseteq P$. We consider the two cases $xR \subseteq yR$ or $yR \subseteq xR$ and obtain $xRxR \subseteq P$ in the first case and $yRyR \subseteq P$ in the second case. Thus $x \in P$ or $y \in P$ and we are done.

(ii) Let P be a prime right ideal and $X^2 \subseteq P$ for some right ideal X. For any $x \in X$ we have $xRx \subseteq X^2 \subseteq P$, hence $x \in P$ and so $X \subseteq P$. Now suppose $X^2 \subseteq P$ implies $X \subseteq P$. Taking $X = xR$ it is obvious that the condition in (i) is satisfied.

(iii) Let P be a completely prime ideal, then $x^2 \in P$ implies $x \in P$. It suffices to prove that P is completely prime if $x^2 \in P$ yields $x \in P$. Let $x, y \in R$ with $xy \in P$. We have either $x = ys_1$ or $y = xs_2$ for some $s_1, s_2 \in R$. In the first case $x^2 = x \cdot ys_1 \in P$ and hence $x \in P$. In the second case we first note that $y(xy)x \in P$ as P is two-sided, thus $(yx)^2 \in P$ and therefore $yx \in P$. Further $y^2 = yxs_2 \in P$ which implies $y \in P$.

(iv) If A is a right ideal, then by Lemma 1.5(iii) it is two-sided, and clearly A is completely prime. The same arguments hold in the case of a left ideal A using Lemma 1.5(ii). ■

See Thierrin [57] for a related result.

Note that by definition *semiprime* ideals are exactly the intersections of prime ideals. By a result of Levitzki and Nagata (see Goodearl/Warfield [89], p. 27) semiprime ideals P are exactly those with $xRx \subseteq P$ implying $x \in P$. Thus in a right chain ring each semiprime ideal must be prime proving again statement (i) in Lemma 1.8 in the case where P is two-sided.

1.3 Some standard right ideals

We will now define some sets and ideals associated with a given right ideal. In the following, R will always be a right chain ring and I a right ideal of R.

For $s \in R$ we set

$$Is^{-1} = \{x \in R | xs \in I\}$$

This set contains I and is closed under addition, but in general it is not a right ideal. Obviously, if I is a two-sided ideal, then Is^{-1} is a left ideal. By the following construction we do get a right ideal associated with I:

LEMMA 1.9 *Let R be a right chain ring, I a right ideal of R, P a left ideal of R and $S = R \backslash P$. Then $I \subseteq IS^{-1} = \bigcup_{s \in S} Is^{-1}$ is a right ideal of R. Clearly, if I is a two-sided ideal, then so is IS^{-1}.*

PROOF: Take $x \in IS^{-1}$, so $xs \in I$ for some $s \in S$. Let $r \in R$. If $rs \in sR$, then $xrs \in xsR \subseteq I$, so $xr \in IS^{-1}$. In the other case, $s = rst$ for some $t \in R$, and then $st \in S$ because P is a left ideal. Thus $xs = xr(st) \in I$ implies $xr \in IS^{-1}$. ■

We make a few easy observations concerning the right ideal IS^{-1} above. If $I \cap S \neq \emptyset$, then $1 \in IS^{-1}$ and $IS^{-1} = R$. Therefore we will usually assume that I is contained in $P = R \backslash S$. Furthermore, one can easily check that for any right ideals I_1, I_2 of R with $I_1 \subseteq I_2 S^{-1}$ we have $I_1 S^{-1} \subseteq I_2 S^{-1}$. The most important situation is the one where P is a completely prime ideal.

COROLLARY 1.10 *Let R be a right chain ring, $P = R \backslash S$ a completely prime ideal and I a two-sided ideal contained in P. Then IS^{-1} is again a two-sided ideal and closed under S-quotients, that is, $(IS^{-1})S^{-1} = IS^{-1}$.*

For the following investigations we will always take for P a completely prime ideal of R, and we set again $S = R\backslash P$. Of course, given a right ideal I and a completely prime ideal $P \supseteq I$ as above we can also consider the associated right ideal IP. To clarify the relationship between IS^{-1} and IP we introduce one further right ideal associated with I and P:

$$\mathcal{I}(I) = \bigcap_{x \in P}(IP)x^{-1} = \{r \in R | rP \subseteq IP\}$$

In other words, if I is a two-sided ideal, $\mathcal{I}(I)$ is the annihilator of the left R-module P/IP.

LEMMA 1.11 *Let R be a right chain ring, $P = R\backslash S$ a completely prime ideal and I a right ideal.*

(i) *For $I \subseteq aR \subseteq IS^{-1}$ we have $(aR)S^{-1} = IS^{-1}$.*

(ii) *$(IS^{-1})P = IP$.*

(iii) *If $IP \subset I$, then $\mathcal{I}(I) = IS^{-1}$.*

(iv) *If $IP \subset I$ holds, then for all $a \in I\backslash IP$ we have $aP = IP$.*

(v) *IP is a two-sided ideal if and only if IS^{-1} is a two-sided ideal.*

PROOF: (i) If we apply Corollary 1.10 to $I \subseteq aR \subseteq IS^{-1}$ we obtain $IS^{-1} \subseteq (aR)S^{-1} \subseteq IS^{-1}$.

(ii) Let $x \in IS^{-1}$, say $xs \in I$ for $s \in S$. Hence $xP = xsP \subseteq IP$. The other inclusion is obvious.

(iii) Suppose $x \in \mathcal{I}(I)$, so $xP \subseteq IP$. Since $IP \subset I$, this immediately implies that $xs \in I$ for some s, since $xR \subseteq aR$ or $aR \subseteq xR$ for some aS with $IP \subset aR \subseteq I$, i.e. $x \in IS^{-1}$. The inclusion $IS^{-1} \subseteq \mathcal{I}(I)$ follows from (ii).

(iv) Obviously we have $aP \subseteq IP$ for any $a \in I\backslash IP$. Take $xp' \in IP$ and assume $xs = a$, thus $s \in S$ and we obtain $p' \in P$ with $sp' = p$ which shows $xp = xsp' = ap' \in aP$.

(v) If IS^{-1} is two-sided, then so is $IP = (IS^{-1})P$. If IP is a two-sided ideal, then obviously so is $\mathcal{I}(I)$. Hence IS^{-1} is also a two-sided ideal. ∎

COROLLARY 1.12 *Let R be a right chain ring, $0 \neq a \in R$, $P = R\backslash S$ a completely prime ideal. Then the following assertions are equivalent:*

(a) *$(aR)S^{-1}$ is a two-sided ideal of R.*

(b) *aP is a two-sided ideal of R.*

(c) *$a \notin RaP$.*

PROOF: By Lemma 1.11(ii) assertion (a) implies (b), whereas the implication (b) to (c) is obvious. We only have to show that (c) implies (a). Let $x \in (aR)S^{-1}$, $r \in R$. Take $s \in S$ with $xs \in aR$, say $xs = ab$. If $rxs \notin aR$, then for some $v \in R$: $a = rxsv = rabv$, and hence $bv \in S$ by (c). Thus $v \in S$ and so $sv \in S$, which proves that $rx \in (aR)S^{-1}$. ∎

In the corollary above we have described the relationship between two rather special right ideals associated with a principal right ideal aR. As we will see in the following these ideals play an important role. Also then the reason for the term *P-associated* will become clear.

We will call $(aR)S^{-1}$ and aP *standard P-associated right ideals.* If the completely prime ideal P is not idempotent, we have $P = (pR)S^{-1}$ for each $p \in P \backslash P^2$ and each right ideal aP equals $(apR)S^{-1}$.

COROLLARY 1.13 *Let R be a right chain ring and $P = R \backslash S$ a completely prime ideal. Then we have:*

(i) $(aR)S^{-1} = aR \cup \{x \in R \mid xs = a \text{ for some } s \in S\}$.

(ii) *Let $0 \neq aR \subseteq bR$. Then the following assertions are equivalent:*

(a) $a = bs$ *for some* $s \in R$.

(b) $(aR)S^{-1} = (bR)S^{-1}$.

(c) $aP = bP$.

PROOF: (i) Obviously the right-hand side is contained in the left-hand side of the equation. To prove the opposite inclusion we may assume $a \neq 0$, otherwise $0 \cdot S^{-1} = \{x \mid xs = 0 \text{ for some } s \in S\}$ and we are done. Let $x \in (aR)S^{-1} \backslash aR$, hence $xs = ar$ with $s \in S$ and $a = xt$ for some $t \in R$. We assume $t \in P$. Then $xs = xtr$, hence $x(s - tr) = 0$ leading to $xs = 0$ as $sR \supset trR$. This implies $xt = a = 0$, a contradiction.

(ii) To prove that (a) implies (b) apply Lemma 1.11(i) for $I = aR$. The condition (c) follows from (b) by Lemma 1.11(ii). If we assume (c) and $a = bs$ for $s \in P$ we obtain $aR = bsR \subseteq bP = aP$ and $a = 0$, a contradiction. ∎

Elements $a, b \in R^*$ satisfying the conditions above are called *right-S-associated*, and we abbreviate this as $a \sim_S b$. *Left-S-associated* is defined similarly.

1.4 First results on prime ideals

We prove a first theorem on prime ideals. Note that by the Test-squares-Lemma 1.8 a two-sided ideal is completely prime if its complement is closed under squares.

LEMMA 1.14 *Let R be a right chain ring.*

(i) *If A is a right ideal of R, then there is no prime right ideal P with $\bigcap_{n \in \mathbb{N}} A^n \subset P \subset A$.*

(ii) *If $t \in R$, then there is no completely prime ideal P with $\bigcap_{n\in\mathbb{N}} t^nR \subset P \subset tR$.*

(iii) *Let P be a right ideal whose complement is multiplicatively closed. Then P is a completely prime ideal.*

PROOF: (i) Let n be minimal with $A^n \subseteq P$. As P is prime, we must have $n = 1$ leading to the contradiction $P = A$.

(ii) follows by using Lemma 1.8(iii).

We have $U \cdot P \subseteq P$, thus by Lemma 1.5 P is two-sided. ∎

This result provides examples of segments of right ideals which do not contain prime ideals. The following theorem shows that under additional conditions the limits for these segments are the best possible.

THEOREM 1.15 *Let R be a right chain ring.*

(i) *Nonzero idempotent ideals are completely prime.*

(ii) *If A is an ideal which is not nilpotent, then $P = \bigcap_{n\in\mathbb{N}} A^n$ is a completely prime ideal.*

(iii) *If $t \in J$ is not nilpotent, then $P = \bigcap_{n\in\mathbb{N}} t^nR$ is a prime right ideal. Moreover, if P is a two-sided ideal, then P is completely prime.*

PROOF: (i) Let $(0) \neq A = A^2$ be an ideal of R. Suppose $a \notin A$ but $a^2 \in A$. Then $A \subseteq aJ$ and thus $A = A^2 \subseteq aA \subseteq a^2J \subset a^2R \subseteq A$. This contradiction shows that A is completely prime.

(ii) Set $P = \bigcap_{n\in\mathbb{N}} A^n$. If $P = A^n$ for some $n \in \mathbb{N}$, then $A^{2n} = A^n$, hence P is idempotent and by (i) the assertion follows. If $P \subset A^n$ for all $n \in \mathbb{N}$, take any $t \notin P$, and there exists $n \in \mathbb{N}$ with $A^n \subseteq tR$. Then we obtain $P \subset A^{2n} \subseteq tA^n \subseteq t^2R$. Hence $t^2 \notin P$.

(iii) As t is not nilpotent, we have $t^{n+1}R \subset t^nR$ for all n and thus $\bigcap_{n\in\mathbb{N}} t^nR \subset t^nR$. Let $x \notin P$. We have to show that $xRx \nsubseteq P$ holds. Note that there is an $n \in \mathbb{N}$ with $t^nR \subseteq xR$, hence $t^{2n}R \subseteq t^nxR \subseteq xRxR$. Thus $\bigcap_{n\in\mathbb{N}} t^nR \subset t^{2n}R \subseteq xRx$ and we are done.

To show that the prime right ideal $P = \bigcap_{n\in\mathbb{N}} t^nR$ is completely prime if it is two-sided, take $x \notin P$ and we have to show that $x^2 \notin P$. We have $t^n = xa$ for some $n \in \mathbb{N}$, $a \in R$ and hence $t^{2n} = xaxa$. In the case $ax \in xR$, we are done. Otherwise there exists $r \in J$ with $axr = x$. As P is two-sided, we conclude $xr \notin P$, thus $xrq = t^m$ for some $q \in R$, $m \in \mathbb{N}$. We obtain $x^2q = x(axr)q = (xa)xrq = t^nt^m = t^{n+m}$, which implies $x^2 \notin P$. ∎

We remark that idempotent *right ideals* are not necessarily completely prime (see Theorem 1.15(i)). There exist right chain domains R with exactly two prime ideals, namely J and (0) such that (0), J and R are the only two-sided ideals of R. In such a ring all right ideals A with $(0) \subset A \subset J$ which are not right principal, are idempotent; assume $A^2 \subset A$. Take any $a \in A \backslash A^2$ and $aR \subseteq A$ implies $aRA \subseteq A^2$

where RA is a two-sided ideal. Hence, $RA = J$, $aJ \subseteq A^2$. As the element $a \in A \backslash A^2$ was chosen arbitrarily, $A = aR$ follows. For an example, see Section 6.5.

We consider finitely generated prime ideals. If $J \supset J^2$ holds, the maximal ideal J is finitely generated. Take $m \in J \backslash J^2$ and let $J \supseteq m'R \supseteq mR$, hence $m'r = m$ for some $r \in U$ which proves the assertion. Otherwise $m'r \in J^2 \subset J$, a contradiction. Note that in contrast to the commutative situation, there exist right noetherian right chain rings with more than one prime ideal which by definition are right principal. Examples are given in Chapter 3. On the other hand, we easily deduce that in chain rings a completely prime ideal $\neq 0$ which is finitely generated equals the maximal ideal J.

LEMMA 1.16 *Let R be a chain ring and P a completely prime ideal $\neq 0$ which is right principal. Then P equals the maximal ideal J.*

PROOF: Assume $P = pR \subset J$ and take any $a \in J \backslash P$.Then $p \in P = Pa = pRa \subseteq pJ$ and hence $p = 0$ follows. ■

1.5 Two-sided ideals and prime segments

We consider the right ideals between two prime ideals. We fix the following notations.

DEFINITION 1.17 *Let R be a right chain ring and $P \supset Q$ neighbouring prime ideals resp. set $Q = 0$ provided $P \neq (0)$ is the prime radical. We call the lattice interval $[P, Q[$ of right ideals the* prime segment *$[P, Q[$, that is,*

$$[P, Q[= \{X \mid P \supseteq X \supset Q,\ X \text{ a right ideal of} R\}.$$

The prime ideal P will be called the leader *of the segment. In the case of $[P, 0[$ and P the prime radical we sometimes speak of the* radical prime segment.

If A is an arbitrary right ideal, P the intersection of the prime ideals containing A, and Q the union of the prime ideals strictly contained in A, we say that $[P, Q[$ is the prime segment generated by A. *Set $Q = 0$ if A does not contain any prime ideals.*

The next results give some informations on the structure of the right ideal lattice between two neighbour prime ideals. Of particular interest is the situation in which there is a further (two-sided) ideal in a prime segment.

DEFINITION 1.18 *A prime segment $[P, Q[$ is called* simple, *if there is no two-sided ideal properly between P and Q.*

Next we describe quite natural situations where prime segments are never simple.

LEMMA 1.19 *Let R be a right chain ring and $[P, Q[$ a prime segment. Assume P is not the prime radical, then*

(i) *If P is a non-idempotent prime ideal, then $[P, Q]$ is not simple and Q is completely prime.*

(ii) *If $P = aR$ is a right principal (two-sided) prime ideal, then $[P, Q[$ is not simple and Q is completely prime.*

Assume P is the prime radical, then we have:

(iii) *If $P^2 \neq (0)$, then $[P, 0[$ is not simple.*

PROOF: (i) By assumption $Q \subset P^2 \subset P$ is a two-sided ideal and Q completely prime using Theorem 1.15(ii).

(ii) Note $Q \subset P^2 = aRaR \subseteq aJ \subset aR$, since RaR is a two-sided ideal. The rest follows from (i).

(iii) In this case we have either $P \neq P^2$ and the statement follows immediately, or $P = P^2$. If we assume $P = P^2 \neq (0)$, then there exists an element $x \in P$ with $xP \neq 0$. We show that xP is a two-sided ideal which is clear if xR is two-sided. Otherwise, let u be a unit with $uxr = x$ and $r \in J$. Since $u^n x r^n = x$ and the elements in P are nilpotent, it follows that $r \notin P$ and hence $rP = P$. Therefore, $uxP = uxrP = xP$. It follows that $uxP \subseteq xP$ for all units in R and $0 \neq xP \neq P$, is a two-sided ideal by Lemma 1.4. ∎

It will be shown that the existence of a further two-sided ideal in a prime segment $[P, Q[$ has strong consequences. The situation then resembles the commutative case.

DEFINITION 1.20 *Let R be a right chain ring and A a right ideal with $[P, Q[$, the prime segment generated by A, not simple. Then the* radical *of A, $\sqrt{A}$ for short, is defined by*

$$\sqrt{A} = \{x \in R \mid \exists k \in \mathbb{N} : x^k \in A\}.$$

As it will be shown in Theorem 1.21 the radical $\sqrt{A}$ in a non-simple segment $[P, Q[$ will be the leader P of that prime segment, hence it equals the intersection of all prime ideals containing A.

The next theorem provides some information on P and Q in the case where the prime segment $[P, Q[$ is not simple.

THEOREM 1.21 *Let R be a right chain ring with the prime segment $[P, Q[$ not simple. Then we have:*

(i) *Q is completely prime if $[P, Q[$ is not the radical segment. In this case there exists for every $x \in P \backslash Q$ a two-sided ideal $X \in [P, Q[$ with $X \subset xR$ and $\bigcap_{n \in \mathbb{N}} X^n = Q$.*

(ii) *$Q = \bigcap_{n \in \mathbb{N}} I^n$ for all two-sided ideals $I \in [P, Q[$, $I \neq P$.*

(iii) *Let P be an idempotent prime ideal. Then P is the union of all two-sided ideals properly contained in P.*

(iv) *For every* $x \in P\backslash Q$ *there exists a two-sided ideal* $X \in [P,Q[$ *with* $xR \subseteq X$, *and* $\bigcap_{n\in\mathbb{N}} X^n = Q$.

(v) $Q = \bigcap_{n\in\mathbb{N}} x^n R$ *for all* $x \in P\backslash Q$.

(vi) *Let* P *be a completely prime ideal. Then* $P = \sqrt{I}$ *for any right ideal* $I \in [P,Q[$.

(vii) $Q = \bigcap_{n\in\mathbb{N}} I^n$ *for all right ideals* $I \in [P,Q[$, $I \neq P$.

PROOF: (i) We take some two-sided ideal $X \in [P,Q[$, $X \neq P$. By Lemma 1.14(i) we have $Q \subseteq \bigcap_{n\in\mathbb{N}} X^n$. Since $[P,Q[$ is not the radical segment, X is never nilpotent. Thus Theorem 1.15(ii) implies that $\bigcap_{n\in\mathbb{N}} X^n$ is completely prime which leads to $Q = \bigcap_{n\in\mathbb{N}} X^n$ and proves the assertion.

(ii) follows from the proof (i) if $[P,Q[$ is not the radical segment. In the remaining case I must be nilpotent since otherwise $\bigcap_{n\in\mathbb{N}} I^n$ would be a prime ideal.

(iii) By Theorem 1.15(i) the prime ideal P is completely prime. Let A be the union of all two-sided ideals I strictly contained in P. Since R is a right chain ring, A is an ideal of R and $Q \subset A$ as $[P,Q[$ is not simple. If $A \subset P$, then A is not a prime ideal, so there exists a two-sided ideal $X \supset A$ with $X^2 \subseteq A$. But then $A \subset X \subset P$, contradicting the definition of A.

(iv) If P is not idempotent, we choose $X = P$ and apply (ii) to P^2. Otherwise, by (iii) there exists a two-sided ideal $X \subset P$, $xR \subseteq X$, and the intersection $\bigcap_{n\in\mathbb{N}} X^n$ equals Q by (ii).

(v) If $[P,Q[$ is the radical segment, we are done, since x is nilpotent and $Q = 0$. By (iv) there exists a two-sided ideal X with $x \in X$ and $\bigcap_{n\in\mathbb{N}} X^n = Q$. Therefore $\bigcap_{n\in\mathbb{N}} x^n R \subseteq Q$. By (i) Q is completely prime, so the other inclusion is obvious. The other inclusion follows from (i).

(vi) To prove $P = \sqrt{I}$ it is enough to show $P \subseteq \sqrt{I}$, since P is completely prime. For $x \in P$ we have $x^n \in I$ for large enough n by (v).

(vii) Since $I \neq P$ there exists $x \in P\backslash Q$ with $I \subseteq xR$ and by (iv) we have $xR \subseteq X \subseteq P$ for a two-sided ideal $X \subseteq R$ with $\bigcap_X^n = Q$. Since Q is prime, $Q \subseteq \bigcap I^n$. ■

COROLLARY 1.22 *Let* R *be a right chain ring and* $[P,Q[$ *a prime segment which is either not simple or the radical segment. Then we have* $x \notin RxP$ *for any* $x \in P\backslash Q$. *Furthermore, if* P *is completely prime and not the prime radical, then* $r \notin P$ *implies* $rx \notin xP$.

PROOF: If $[P,Q[$ is the prime radical, $x = rxp = r^n x p^n$ with $p \in P$ leads to $x = 0$, since each element in P is nilpotent. Next assume that P is not the prime radical, $x = uxp \in UxP = RxP$ with $u \in U$, $p \in P$. We take a two-sided ideal $I \in [P,Q[$ with $p \in I$ and $\bigcap_{n\in\mathbb{N}} I^n = Q$. (use Theorem 1.21(iv)) and $x \in Q$ follows contradicting $x \in P\backslash Q$.

To prove the last assertion, assume $rx = xp$ with $r \notin P$, $p \in P$. By Theorem 1.21(i) there exists a two-sided ideal $X \in [P,Q[$ with $X \subset xR$ and $\bigcap_{n\in\mathbb{N}} X^n = Q$. Again by the same theorem there exists n with $p^n \in X$. We have $r^n \notin P$, since P

is completely prime, and $r^n q = x$ for some $q \in P$. We obtain $r^n(x - qp^n) = 0$ and $r^n x = 0$, since $qp^n \in xJ$. By assumption Q is a prime ideal and therefore completely prime by Theorem 1.21(i) which leads to $x \in Q$, a contradiction. ■

We add a further characterization for the situation where $[P, Q[$ is simple.

THEOREM 1.23 *Let R be a right chain ring, $P \supset Q$ neighbour prime ideals. Then the following assertions are equivalent:*

(a) *$[P, Q[$ is simple.*

(b) *There exists a prime right ideal P' with $Q \subset P' \subset P$.*

PROOF: (a) $\Rightarrow$ (b) Let $P \supset Q$ be simple, hence $P^2 = P$ and P is c.prime by Theorem 1.15. We now construct a right ideal P' satisfying $Q \subset P' \subset P$ and we prove that P' is prime. Take any $a \in P \backslash Q$ and set $P' := \bigcup yR$ where y runs through the set of those elements with $ys = a$ for some $s \in R \backslash P$. Let $x \notin P'$ and assume $xRxR \subseteq P'$. As RxR is two-sided, we conclude $xRxR = xP \subseteq P'$, hence $xP \subseteq yR$ for some y with $ys = a$ for some $s \in R \backslash P$. The element x does not lie in P', hence $xt = y$ and thus also $t \in P$; otherwise $x \in P'$. This shows $yR = xtR \subseteq xP$, hence $xP = yR$. As P is idempotent, we have $t = t_1 t_2$ with $t_1, t_2 \in P$. Then $xP = yR = xt_1t_2R \subset xt_1R \subseteq xP$ leads to a contradiction.

(b) $\Rightarrow$ (a) Now we assume that the prime segment $[P, Q[$ is not simple. If P is not idempotent, we have $\bigcap P^n = Q$ and there is no prime right ideal P' with $P \supset P' \supset Q$ by Lemma 1.14(i). In the case where is P idempotent, we apply Theorem 1.21 to find a two-sided ideal I with $P \supset I \supseteq P'$ provided P' is a prime right ideal with $P \supset P' \supset Q$. The ideal I cannot be idempotent by Theorem 1.15, hence $\bigcap_{n \in \mathbb{N}} I^n = Q$. Again Lemma 1.14(i) leads to a contradiction. ■

Right chain rings with an additional two-sided ideal between each pair of prime ideals will be discussed in detail in Chapter 7.

1.6 Three classes of classical examples

The ring R of integral elements in a number field F over $\mathbb{Q}$ is a Dedekind domain and the localizations R_M of R at a maximal ideal M are therefore valuation rings, that are commutative chain domains. In this way one obtains an interesting class of examples of discrete valuation rings.

Krull [32] pointed out that given an ordered commutative group G one can form the group ring FG over any commutative field F and FG contains the subring R_0 of elements

$$R_0 = \{\sum g a_g \in FG \mid e \leq g \text{ if } a_g \neq 0\}.$$

The subset $S = \{\sum g a_g \in R_0 \mid a_e \neq 0\}$ of R_0 is multiplicatively closed and $R_0 S^{-1} = R$ is a valuation ring. The set of nonzero principal ideals is given by $\{gR \mid e \leq g,\ g \in G\}$.

This construction cannot be extended to noncommutative linearly ordered groups G, since the group ring is not necessarily ordered (see Passmann [85]). However, let

k be a division ring, G an ordered group, then by $k[[G]]$ we denote the set of all generalized power series $a = \sum g a_g$ over k and G which have well-ordered support, that is

$$supp(a) = \{g \in G \mid a_g \neq 0\}.$$

Further we assume the commutation rule

$$ag = ga \ , \ a \in k \ , \ g \in G \ .$$

Since the supports are well-ordered, this leads to a multiplication of $a, b \in k[[G]]$ in which the sums

$$(ab)_g = \sum_{h \in G} a_h b_{h_g^{-1}}$$

are finite. The ring $k[[G]]$ is sometimes called the *Malcev-Neumann-ring* of generalized power series. We will discuss modifications of that construction later.

The Malcev-Neumann theorem asserts that the ring $k[[G]]$ is a skew field: If $a = \sum g a_g$ and $g_0 = min\ supp(a)$ we can write

$$a = g_0(1 - q)a_{g_0}$$

where the support of q only contains elements $> e$. Since

$$(1 - q)^{-1} = 1 + q + q^2 + \ldots \in k[[G]]$$

we have

$$a^{-1} = a_{g_0}^{-1}(1 + q + q^2 + \ldots)g_0^{-1} \ .$$

PROPOSITION 1.24 *Let G be an ordered group, k a skew field. Then the subring*

$$R = \{a = \sum g a_g \in k[[G]] \mid min\ supp(a) \geq e\} \cup \{0\}$$

of the Malcev-Neumann skew field $k[[G]]$ is a chain domain. More precisely we have:

(i) *R is a duo (or invariant) ring, that is $Ra = aR$ for all $a \in R$.*

(ii) *The set of nonzero principal right (left) ideals is given by $\{gR \mid g \geq e\}$ ($\{Rg \mid g \geq e\}$).*

(iii) *The (two-sided) ideals of R correspond to the upper classes of $G^+ = \{g \in G \mid g \geq e\}$, the prime ideals to the convex subsemigroups of G^+.*

(iv) *The residue field of R, that is R/J, is isomorphic to k.*

Homomorphic images of rings constructed by using the above methods will serve to illustrate some of our results, and various generalizations and modifications of these constructions will be discussed later.

2 Zero-divisors

A description of the left annihilators of prime ideals in right chain rings is given (Lemma 2.6). Of particular interest is the connection between the size of the ideal N_r of all right zero-divisors in a right chain ring R and the left-right annihilator A^{lr} for a right ideal A of R (Theorem 2.9). We have $P^{lr} = P$ for a prime ideal P in a chain ring R (Proposition 2.10).

2.1 Zero-divisors in right chain rings

We observed earlier that chain domains are closely related with valuations on skew fields. However, other applications motivate the investigation of right chain rings with zero-divisors. An affine Hjelmslev ring is a right chain ring R in which *all* non units are left and right zero-divisors. Such rings occur as coordinate rings of affine Hjelmslev planes. Examples can be obtained as localizations of homomorphic images R/I of right chain domains R provided I satisfies certain annihilator conditions. A criterion whether R/I leads to a Hjelmslev ring can be derived by studying the double annihilators P^{ll} of all prime ideals in R/I.

This condition is satisfied if the prime ideal lattices satisfies certain chain conditions. Further, questions about zero-divisors arise naturally in the discussion of mudules over right chain rings. Occasionally it is of advantage to replace a chain domain R by a homomorphic image R/I with many zero-divisors, since the mapping that associates the left annihilators A^l with the right ideal A provides then a useful link between the lattice of right ideals and the lattice of left ideals.

DEFINITION 2.1 *An element a in a ring R is called a right (left) zero-divisor if $ba = 0$ $(ab = 0)$ for some $b \in R^*$.*

We denote by A^r respectively A^l the right respectively left annihilator of a set $A \neq \emptyset$ in R, i.e. $A^r = \{t \in R \mid At = 0\}$ respectively $A^l = \{t \in R \mid tA = 0\}$. If $t \in R$, we write t^r instead of $\{t\}^r$ if there is no danger of confusion. Hence a is a right zero-divisor if and only if $a^l \neq (0)$.

We list some well known properties of annihilators in the next lemma.

LEMMA 2.2 *Let A, B be right ideals of an arbitrary ring R.*

(i) *If $A \subseteq B$ then $B^l \subseteq A^l$.*

(ii) $A \subseteq A^{lr}$.

(iii) $A^l = A^{lrl}$.

(iv) $(A \cup B)^l = A^l \cap B^l$.

For a multiplicatively closed subset $S \subseteq R$ containing U we define

$$N_l(S) = \{t \in R \mid \exists s \in S \backslash \{0\} : ts = 0\}$$

and

$$N_r(S) = \{t \in R \mid \exists s \in S\backslash\{0\} : st = 0\}.$$

Obviously, $N_l(S)$ and $N_r(S)$ are two-sided ideals if R is a right chain ring, by Lemma 1.8 and 1.5(iii). Moreover

$$N_l(R) = N_l = R\backslash T_l = \{t \in R \mid t^r \neq (0)\}$$

respectively

$$N_r(R) = N_r = \{t \in R \mid t^l \neq (0)\}$$

is the set of all left respectively right zero-divisors of R. Their complements are denoted by T_l respectively T_r. Applying Lemma 1.8(iv) it is evident that N_l and N_r are completely prime ideals.

LEMMA 2.3 *Let R be a right chain ring, A a right ideal and T_r the set of non right zero-divisors.*

(i) *N_l and N_r are completely prime ideals.*

(ii) *If $A \subset N_r$, then $A^l \neq (0)$; if $N_r \subset A$, then $A^l = (0)$.*

(iii) *$AT_r^{-1} \subseteq A^{lr}$.*

PROOF: (i) already done

(ii) Let $s \in N_r\backslash A$. Then $A \subseteq sR$ and therefore $(0) \neq s^l \subseteq A^l$. The second statement in (ii) is obvious.

(iii) We take $x \in AT_r^{-1}$ and may assume $xr = a$ for some $r \in T_r$, $a \in A$. Let y be any element of A^l, thus $ya = 0$ which implies $yxr = 0$ and by $r \in T_r$ the assertion $yx = 0$ follows. ■

In Section 2.3 we will give examples for chain rings which show that in the case where $A = N_r$ both possibilities $N_r^l = 0$ or $N_r^l \neq (0)$, can occur. Furthermore, equality need not hold in (iii) (use Remark 3.10(iv)).

We investigate the relationship between the size of N_r and the set of right ideals A with $B = A^{lr}$ for some fixed right ideal B. The next lemma gives some partial results.

LEMMA 2.4 *Let R be a right chain ring. Then we have:*

(i) *If $(aR)^{lr} = aR$ holds for all $a \in R$, we have $B \preceq B^{lr}$ for each right ideal B. Furthermore, $N_r = J$ is valid.*

(ii) *The following assertions are equivalent:*
 (a) *$(aR)^{lr} = aR$ for all $a \in R^*$.*
 (b) *$r^l \cap Ry \neq (0)$ for $y \notin r^l$.*

PROOF: (i) Assume $B \subset bR \subseteq B^{lr}$. We obtain $B^{lr} \subseteq (bR)^{lr} = bR \subseteq B^{lr}$ which proves $B \prec B^{lr}$. We have $(aR)^l \neq (0)$ for $a \in J$, since $(aR)^{lr} = aR$.

(ii) (a) $\Rightarrow$ (b) We set $yr = a$ and observe $(yR)^l \subset (aR)^l$. Choose $x \in (aR)^l \backslash (yR)^l$, thus $xy \neq 0$ and $xyr = 0$ follow.

(ii) (b) $\Rightarrow$ (a) Suppose $(aR)^{lr} \supset aR$ for some $a \in R^*$. Let $y \in (aR)^{lr} \backslash aR$, hence $yr = a$ for a suitable $r \in J$. The inclusion $(aR)^l \subseteq (aR)^{lrl}$ implies $xy = 0$ for all $x \in R$ with $xa = 0$. Hence $xyr = 0$ leads to $xy = 0$, contradicting the assumption. ∎

For *right* chain rings the results above are as good as possible. For example, the fact $N_r = J$ does not necessarily imply that $(aR)^{lr} = aR$ holds (see Remark 3.10(iii)).

The following statement is an observation of Mazurek ([89], Prop. 6).

PROPOSITION 2.5 *Let R be a right chain ring. Then $N_l \subseteq N_r^{ll}$ and $N_r^l \subseteq N_l^r$. If R is a chain ring, then $N_r^l = N_l^r$.*

PROOF: Assume there exists an element $a \in N_l \backslash N_r^{ll}$. Hence $ab = 0$ for some $b \in R^*$ and $ac \neq 0$ for some $c \in N_r^l$. As $c \notin bR$ holds, we obtain $b = cr$ with $r \in R \backslash N_r$ and so $0 = ab = acr$. But now $r \notin N_r(R)$ implies $ac = 0$, a contradiction. Thus we have $N_l \subseteq N_r^{ll}$ and $N_r^l \subseteq N_r^{llr} \subseteq N_l^r$ follows. Analogously we prove the statement if R is a left chain ring. ∎

Especially fruitful in the case of prime ideals is the following description of left annihilators of prime ideals.

LEMMA 2.6 *Let R be a right chain ring and $P = R \backslash S$ a completely prime ideal. Then for all $0 \neq a \in P^l$ we have $(aR)S^{-1} = P^l$.*

PROOF: For $0 \neq a, b \in P^l$ we have $aP = bP$ and by Corollary 1.13 $(aR)S^{-1} = (bR)S^{-1}$. Therefore, $P^l \subseteq (aR)S^{-1}$ and by Lemma 1.11(ii), we obtain $(aR)S^{-1} \subseteq P^l$. ∎

We observe that in case $P^l = (0)$ the conclusion of the lemma remains valid for $a = 0$. If $xs = 0$ for $x \neq 0,\ s \in S$, then $x \in P^l$, a contradiction.

If P is properly contained in N_r, we obtain an alternative description of P^l.

COROLLARY 2.7 *Let R be a right chain ring and $P = R \backslash S$ be a completely prime ideal. Then the following are equivalent:*

(a) $P \subset N_r$.

(b) $N_l(S) \neq 0$

If the equivalent conditions (a) *or* (b) *hold then $N_l(S) = P^l$.*

PROOF: (a) $\Rightarrow$ (b) By definition $s^l \neq (0)$ for any $s \in N_r \backslash P$, thus $N_l(S) \neq (0)$.

(b) $\Rightarrow$ (a) By $N_l(S) \neq (0)$ we get some $s \in S \cap N_r$, thus $P \subset N_r$.

Let $0 \neq a \in N_l(S)$, hence $as = 0$ for some $s \in S$. We obtain $0 = asR \supseteq aP$ and $a \in P^l$. Thus it remains to prove $P^l \subseteq N_l(S)$. Applying Lemma 2.6 we have

$P^l = (aR)S^{-1}$. Now take any $x \in (aR)S^{-1}$. Hence $xt \in aR$, say $xt = ar$, for some $t \in S$. If $r \in sR$, then $xt = ar = 0$ and $x \in N_l(S)$ follows directly. If not, there exists $u \in S$ with $s = ru$, otherwise $su = r$ would lead to $xt = ar = 0$ and we are done. Thus we get $xtu = aru = as = 0$. As $tu \in S$, $x \in N_l(S)$ follows. ∎

2.2 Zero-divisors in chain rings

In the following we will turn to the case of *chain rings* where zero-divisors can be dealt with easier.

The next lemma can be found in Mazurek [89].

LEMMA 2.8 *Let R be a chain ring. Then $N_l \subset N_r$ if and only if there exists an ideal A of R with $A^r = (0)$ and $A^l \neq (0)$.*

PROOF: Let us assume that $N_l \subset N_r$. We note that $N_r^r = (0)$ and $N_l^l \neq (0)$, using Lemma 2.3(ii) and its right left symmetric version since R is a chain ring. If $N_r^l \neq (0)$, then $A = N_r$ satisfies the statement of the Corollary 2.5 and we can put $A = N_l$.

Now assume that $A^r = (0)$ and $A^l \neq (0)$ for some two-sided ideal A. By the symmetric version of Lemma 2.3(ii) we have $N_l \subseteq A \subseteq N_r$. If $N_l = N_r$, then $N_l = N_r = A$. Hence $N_l^r = A^r = (0)$ and $N_r^l = A^l \neq (0)$, which is impossible by Proposition 2.5. ∎

Lemma 2.4 suggests a certain relationship between the size of N_r on the one hand and the annihilator right ideals A^{lr} on the other hand. In the case of chain rings we obtain the more precise result (ii) of the following theorem. Furthermore, an exact description is given for the relationship between A and A^{lr} for any right ideal A of the chain ring A.

THEOREM 2.9 *Let R be a chain ring and set $T_r = R \backslash N_r$. Then we have:*

(i) *For all $a \in R : (aR)T_r^{-1} = (aR)^{lr} = ((aR)T_r^{-1})^{lr}$.*

(ii) *$N_r = J$ if and only if $(aR)^{lr} = aR$ for some element $a \in J, a \neq 0$.*

(iii) *Let A be a right ideal and $A^l \neq (0)$. Then $A^{lr} = AT_r^{-1}$ or $A^{lr}N_r = A$.*

PROOF: (i) By Lemma 2.3(iii) we know $(aR)T_r^{-1} \subseteq (aR)^{lr}$. If $(aR)^{lr} = aR$, we are done. Otherwise take any $0 \neq x \in (aR)^{lr} \backslash (aR)$, hence $aR \subseteq xR \subseteq (aR)^{lr}$ and $(xR)^{lr} = (aR)^{lr}$, $(xR)^l = (aR)^l$ follow (use Lemma 2.2). Set $xs = a$. Next we show that $(sR)^l \cap Rx = (0)$ which implies $(sR)^l = 0$ since R is a left chain ring and $s \in T_r$ proving $(aR)^{lr} = (aR)T_r^{-1}$. Let $yx \in (sR)^l \cap xR$ and $yxs = ya = 0$ implies $y \in (aR)^l$. By $(aR)^l = (xR)^l$ we obtain $yx = 0$. It remains to show $(aR)T_r^{-1} = ((aR)T_r^{-1})^{lr}$. We observe $((aR)T_r^{-1})^{lr} = ((aR)^{lr})^{lr} = (aR)^{lr}$ which finishes the proof of (i).

(ii) If $N_r = J$, we have $U = T_r$ and the statement follows from (i). To prove the converse assume $aR = (aR)^{lr} = (aR)T_r^{-1}$ (by (i)) and $j \in T_r$. Hence, there exists $x \in (aR)T_r^{-1}$ with $xj = a$, a contradiction.

(iii) For an arbitrary right ideal we have $A \subseteq A^{lr}$. First we consider the case $A = A^{lr}$. Obviously $(AT_r^{-1})^l = A^l$, hence $(AT_r^{-1})^{lr} = A^{lr} = A$. On the other hand $A \subseteq AT_r^{-1} \subseteq (AT_r^{-1})^{lr}$ and $A^{lr} = AT_r^{-1}$ follows. Now assume $A \subset A^{lr}$. Take any a with $A \subseteq aR \subseteq A^{lr}$ and one obtains $(aR)^{lr} = A^{lr}$ and by (i) $A^{lr} = (aR)^{lr} = (aR)T_r^{-1}$. Hence, $AT_l^{-1} \subseteq A^{lr}$.

Assume $AT_r^{-1} \subset A^{lr}$. Hence for all $x \in A$ we find an element $a \in A^{lr} \backslash A$ with $ap = x$ for some $p \in N_r$ which proves $A \subseteq A^{lr}N_r$.

To prove $A^{lr}N_r \subseteq A$ we recall from above that $A^{lr} = (aR)^{lr}$ for any $a \in A^{lr} \backslash A$. It is therefore enough to show that $(aR)^{lr}N_r \subseteq aJ$ which follows from $(aR)^{lr}N_r = ((aR)T_r^{-1})N_r = aN_r$ by (i) and 1.11(ii), since N_r is a completely prime ideal contained in J. ∎

We point out that it follows from (iii) of the above theorem that either $A^{lr} = A$ or $A \prec A^{lr}$ if $N_r = J$ which is the case if R is a Hjelmslev ring (see Törner [74]).

The last theorem has shown that the standard N_r-associated right ideals $(aR)T_r^{-1}$ are *stable* under the left-right-annihilating process. The same applies for prime ideals.

PROPOSITION 2.10 *Let R be a chain ring and P a completely prime ideal with $P^l \neq (0)$. Then the following holds:*

(i) $P^{lr} = P$.

(ii) $P^{llr} = P^l$.

(iii) $P^{llrr} = P$.

PROOF: (i) We can assume that $P \neq (0)$. By Lemma 2.6 we have $(aR)S^{-1} = P^l$ for $0 \neq a \in P^l$. If $P \subset P^{lr} = ((aR)S^{-1})^r$, then there exists $s \notin P$ with $xs = 0$ for all $x \in (aR)S^{-1}$. If there exists $y \in R$ with $ys = a$ we have $y \in (aR)S^{-1}$ and $a = ys = 0$, a contradiction. Otherwise $ya = s \notin P$ implies $a \in P^l \backslash P$ and $P = aP = (0)$, again a contradiction. Hence, $P = ((aR)S^{-1})^r = P^{lr}$.

(ii) If $P = N_r$, we apply Proposition 2.5: $P^l = N_r^l = N_l^r$, hence $P^{llr} = N_l^r = N_r^l = P^l$ follows.

If $P \subset N_r$ then by Theorem 2.9(iii) applied to $A = P^l$ we have $P^lT_r^{-1} = P^{llr}$ of $P^{llr}N_r = P^l$. In the first case we show that $P^lT_r^{-1} = P^l$ where $P^l \subseteq P^lT_r^{-1}$ is obvious. If $x \in P^lT_r^{-1}$ then $xs \in P^l$ for some $s \in T_r = R \backslash N_r$, in particular $s \notin P$. Hence, $0 = xsP = xP$, since P is completely prime, and $x \in P^l$. In the second case we must show $P^{llr} \subseteq P^l$. Assume otherwise there exists $a \in P^{llr} \backslash P^l$, hence $ap \neq 0$ for a suitable $p \in P$. Now take any $t \in N_r \backslash P$ and $tp' = p$ with $p' \in P$ follows leading to $ap = atp' = 0$ since $at \in P^{llr}N^r = P^l$, a contradiction.

By (i) and (ii) we obtain $P^{llrr} = (P^{llr})^r = P^{lr} = P$. ∎

We remark that by Proposition 2.10(i) the mapping $P \rightarrow P^l$ on completely prime ideals in chain rings is injective provided $P^l \neq (0)$ holds. However, in contrast to the case of chain rings, in right chain rings this property does not hold in general (see Remark 3.10).

2.3 Examples

We close this chapter with two examples.

First we show that, as mentioned after Lemma 2.3, the annihilator of the maximal ideal J can be zero as well as nonzero.

EXAMPLE 2.11 *Let* $\mathbf{R}$ *be the real numbers and* k *any skew field. As described in Proposition 1.24 we construct a chain ring* R *whose set of nonzero principal right ideals is given by* $\{gR \mid g \in \mathbf{R}^+\}$ *where* $\mathbf{R}^+$ *denotes the set of nonnegative numbers. Let* $I_1 = \bigcup_{g \geq 1} gR$ *respectively* $I_2 = \bigcup_{g>1} gR$ *be (two-sided) right ideals.*

Then the left (right) annihilator of the maximal ideal in R/I_1 *is zero whereas* $J(R/I_2)^r$ *respectively* $J(R/I_2)^l$ *is nonzero.*

The next example shows that even in the case of duo chain domains zero-divisors are not permutable, i.e. $xy = 0$ does not always imply $yx = 0$ for $x, y \in R$ and R a chain ring.

EXAMPLE 2.12 *Take the linearly ordered group* Γ *of rank 2 given by*

$$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \alpha_2, \alpha_2 e^{\beta_1} + \beta_2)$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2$ *real numbers and* e *the Euler number and lexicographic ordering. Again build the Malcev-Neumann ring over* Γ *and any skew field* k*. Let* R *be the corresponding chain domain in* $k[[\Gamma]]$*. By* I *we denote the ideal of elements in* R *whose minimum of the support is larger or equal than* $(1,1) \in \Gamma$*. Let* x *be an element in* R *with min supp*$(x) = (0,1)$, y *an element with min supp*$(y) = (1, 1-e)$*. Hence*

$$\textit{min supp}(xy) = (0,1) \neq (1, 1-e) = (1, e+1-e) = (1,1)$$

whereas

$$\textit{min supp}(yx) = (1, 1-e) + (0,1) = (1, (1-e)e^0 + 1) = (1, 2-e) < (1,1).$$

This shows that in the homomorphic image $R/I = \overline{R}$ *the product of* $(x+I)(y+I)$ *equals zero, however* $(y+I)(x+I) \neq 0$*. We remark that* $\overline{R}$ *is a Hjelmslev ring.*

3 Distributive rings and right noetherian right chain rings

Right noetherian right chain rings are exactly the local principal right ideal rings. Jategaonkar [69], [70] showed that this class of rings contains interesting counterexamples to various conjectures in ring theory and in Brungs [69] und [76] it was shown that these rings play for right noetherian right distributive domains the role discrete valuation rings play for Dedekind domains in the noncommutative case.

3.1 Right noetherian right chain rings

A ring R is called *right noetherian* if every non-empty family of right ideals contains a maximal element. If a right chain ring is right noetherian, then all its right ideals are principal and, as we will show below, two-sided. The commutative noetherian valuation rings are the rank 1, discrete valuation rings whose semigroup of non-zero ideals is isomorphic to the natural numbers under addition. For each segment $H_I = \{\alpha \mid \alpha < \omega^I\}$ of ordinals less than a power ω^I of ω, the order type of $\mathbb{N}$, there exists a right noetherian right chain domain R whose semigroup $H(R)$ of non-zero ideals is isomorphic to H_I. Conversely, $H(R)$ is isomorphic to some H_I if R is a right noetherian right chain domain.

PROPOSITION 3.1 *The following conditions are equivalent for a ring R:*

(a) *R is right noetherian right chain ring.*

(b) *R is a local principal right ideal ring.*

(c) *R is a right noetherian local right distributive ring.*

(d) *R is a ring in which every non-empty family of right ideals contains exactly one maximal element.*

(e) *R is a ring in which the lattice of right ideals is inversely well-ordered by inclusion.*

PROOF: The equivalence of (a), (b) und (c) follows from Proposition 1.3. Condition (a) implies (d) since the noetherian condition implies that every non-empty family of right ideals has a maximal element and there is only maximal element since R is a right chain ring. Condition (e) is just a rephrasing of conditions (d) and (a). ∎

The next result shows that a right noetherian right chain ring R is *right invariant* i.e. for $r, a \in R$ there exists $r' \in R$ with $ra = ar'$.

LEMMA 3.2 *Let R be a right noetherian right chain ring. Every right ideal of R is a two-sided ideal.*

PROOF: Every right ideal A of R is right principal that is $A = aR$ for some $a \in R$. By Lemma 1.5(iii) we are done if Ua is contained in aR. Otherwise there is an element $u \in U$ with $aR \subset uaR$. Obviously, $u^n aR$, $n \in \mathbb{N}$, defines a strictly ascending chain of right ideals which leads to a contradiction. Hence $UaR \subseteq aR$ follows. ■

It follows that all prime ideals in such rings are completely prime and make use the fact that the chain of prime ideals is inversely well ordered by inclusion to index them by ordinal members.

Hence, let

$$J = P_0 = p_0R \supset P_1 = p_1R \supset \ldots \supset P_\alpha = p_\alpha R \supset P_{\alpha+1} = p_{\alpha+1}R \supset \ldots \qquad (1)$$

be the chain of prime ideals P_α in R with $J = P_0$ and p_α a generator of the principal ideal P_α. We have $P_\alpha^2 = p_\alpha^2 R \neq P_\alpha$ if $P_\alpha \neq (0)$. It follows by Theorem 1.21 that $P_{\alpha+1} = \bigcap_{n\in\mathbb{N}} p_\alpha^n R$, and $P_\alpha = \bigcap_{\beta<\alpha} P_\beta$ for a limit ordinal α.

We prove in the next result that $PQ = Q$ for prime ideals $P \supset Q$ in R.

LEMMA 3.3 *Let R be a right noetherian right chain ring and let $P = pR, Q = qR$ be prime ideals with $P \supset Q$. Then $pq = q\varepsilon$ for some unit $\varepsilon \in U$.*

For $p_\alpha R = P$ and $Q = p_\beta R$ we have $p_\alpha p_\beta = p_\beta \varepsilon_{\alpha,\beta}$ with $\varepsilon_{\alpha,\beta} \in U$.

PROOF: There exists $r_1 \in R$ with $pr_1 = q$ and $r_1 \in qR$ since Q is completely prime. Hence, $r_1 = qr_2$ and $q = pr_1 = pqr_2 = qp'r_2$ for $r_2, p' \in R$. It follows that $p'r_2 \in U$ and hence r_2 must be units in R if $q \neq 0$ as R is local. ■

If $0 \neq a$ is an element of J, let $p_\alpha R$ be the minimal prime ideal containing aR. Then (by Theorem 1.21) $\bigcap_{m\in\mathbb{N}} p_\alpha^m R$ is either (0) or a prime ideal $p_{\alpha+1}R$ and there exists a maximal with $aR \subseteq p_\alpha^n R$. Hence, $a = p_\alpha^n a_1$ for $a_1 \in R$ with $a_1 R \supset p_\alpha R$. The element a determines uniquely the index α and the exponent n; the element a_1 is determined up to factors from the right.

Using induction and the assumption that R is right noetherian one obtains the following result:

LEMMA 3.4 *Let R be a right noetherian right chain ring with Eqn. (1) as its chain of prime ideals. Then every right ideal $aR \neq R, (0)$ can be written in the form $aR = p_{\alpha_1}^{n_1} \cdots P_{\alpha_s}^{n_s} R$ with $\alpha_1 > \alpha_2 > \ldots > \alpha_s$ and the α_i and n_i are uniquely determined by a.*

If τ denotes *the order type of the chain of right ideals* of R, we say τ is the *type* of R and if

$$\tau = \omega^{\rho_1} n_1 + \omega^{\rho_2} n_2 + \ldots + \omega n_{k-1} + n_k + 1 = \tau' + 1 \qquad (2)$$

with $\rho_1 > \rho_2 > \ldots \rho_{k-2} > 1$, then the p_α are indexed by $\alpha \leq \rho_1$ unless τ has the special form $\tau = \omega^{\rho_1} + 1$ in which case the p_α are indexed by $\alpha < \rho_1$ and R is a domain. (see Hausdorff [35] for the terminology)

In particular, R is a domain if and only if τ is of the form $\tau = \omega^{\rho_1} + 1$ in which case the semigroup $H(R)$ of non-zero ideals of R is isomorphic to $H_{\rho_1} = \{\alpha \mid \alpha < \omega^{\rho_1}\}$. Let the type τ of the right noetherian right chain R be given by

$$\tau = \omega^{\rho_1} n_1 + \ldots + \omega n_{k-1} + n_k + 1 = \tau' + 1$$

as in Eqn. (2). Let $H_{\rho_1+1} = \{\alpha \mid< \omega^{\rho_1+1}\}$ and $H = H_{\rho_1+1}/ < \tau' >$ as the ordered semigroup with the set $\{\alpha \mid \alpha < \tau'\} \cup \{\infty\}$ and with the operation

$$\alpha \circ \beta = \begin{cases} \alpha + \beta & \text{for } \alpha + \beta < \tau' \\ \infty & \text{otherwise} \end{cases}$$

If we map $aR = p_{\alpha_1}^{n_1} \dots p_{\alpha_s}^{n_s} R$ onto $\omega^{\alpha_1} n_1 + \dots + \omega^{\alpha_s} n_s$ for $aR \neq (0)$, R onto (0) and the ideal (0) to ∞ we obtain an isomorphism between the semigroup of all ideals of R and $H = H_{\rho_1+1}/ < \tau' >$. H is called the *Rees factor semigroup* of H_{ρ_1+1}, see Fuchs [66].

We can reformulate these results as follows:

THEOREM 3.5 *Let R be a right noetherian right chain domain ring with*

$$R = I_0 \supset I_1 = J \supset I_2 \supset \dots \supset I_n \supset \dots \supset I_\omega \supset \dots (0) = I_{\tau'},$$

as the chain of right ideals. Then

$$I_\rho I_\sigma = \begin{cases} I_{\rho+\sigma} & \textit{if} \quad \rho + \sigma \leq \tau' \\ I_{\tau'} & \textit{for} \quad \rho + \sigma \geq \tau' \end{cases}$$

The ideal I_σ is a prime ideal if and only if $\sigma = \omega^\kappa$ for some κ.

All right ideals I are *left primary* in the following sense: $x, y \in R$, $xy \in I$, $y \notin I$ implies $x^n \in I$ for $n \in \mathbb{N}$ large enough.

LEMMA 3.6 *Let R be a right noetherian right chain ring. Then each ideal is left primary.*

PROOF: Let $I = aR$ and $x, y \in R$ with $xy \in R$, $y \notin aR$. If no power x^n of x is contained in aR we obtain an ascending sequence of ideals A_n with $A_n = \{r \in R \mid x^n r \in aR\}$ and hence $A_m = A_{m+1}$ for some m. If $x^n \in yR$ for some $n \in \mathbb{N}$, then $x^n = yr$, $x^{n+1} = xyr \in aR$, a contradiction. Hence, $y = x^m r$ for some $r \in R$ and $xy = x^{m+1} r \in aR$ and $y = x^m r \in aR$ follows, a contradiction. ∎

We conclude this section with some observations about right noetherian right chain rings that satisfy additional conditions.

PROPOSITION 3.7 *Let R be a right noetherian right chain ring of type $\tau > \omega + 1$.*

(a) *R is not left noetherian.*

(b) *R is not left Ore.*

PROOF: If R has type $\tau > \omega + 1$ then there exist two prime ideals $P_0 = p_0 R$ and $P_1 = p_1 R$ different from (0) and $p_0 p_1 = p_1 \varepsilon$ for a unit ε in R by Lemma 3.3. To prove (a) one observes that $I_n = R p_1 \varepsilon^{-n}$ defines a strictly ascending sequence of left ideals of R.

To prove (b) it follows from the factorization of elements in R as discussed before Theorem 3.5 that $Rp_1 \cap Rp_2 = (0)$. ∎

3.2 Examples

Jategaonkar [69] constructed right noetherian right chain rings R for arbitrary type $\tau = \tau' + 1$. These examples showed in particular that for right noetherian rings R there is no common bound ρ with $J^\rho = (0)$ where J is the Jacobson radical of R (*Jacobson conjecture*). We will construct a right noetherian right chain ring of type $\tau = \omega^2 + 1$.

EXAMPLE 3.8 *Consider the function field $K = k(t_1, t_2, \ldots)$ over a commutative field k in infinitely many indeterminates t_n. The localization A of $K[x]$, the commutative polynomial ring in one indeterminate x over K, at the ideal $xK[x]$, is a commutative noetherian valuation ring. We define a monomorphism σ by $q^\sigma = q$ for $q \in k$, $x^\sigma = t_1$ and $t_i^\sigma = t_{i+1}$ for $i = 1, 2, \ldots$.*

Let R be the skew power series ring $R = A[[y,\sigma]] = \{\sum_{i=0}^{\infty} y^i a_i \mid a_i \in A\}$ where $ay = ya^\sigma$ defines the multiplication. The right ideals of R have the form $y^n x^m R$ and $p_0 = x$, $p_1 = y$ with $xy = yt_1, t_1 \in U(R)$ and R has type $\tau = \omega^2 + 1$.

EXAMPLE 3.9 *We define $A[[y,\sigma]]$ as in the previous ecample, but $R = A[[y,\sigma]]/(y^2x^2)$ where $y^2x^2A[[y,\sigma]] = (y^2x^2)$ is a two-sided ideal of this ring. Then R is a right noetherian right chain ring of type $\tau = (\omega 2 + 2) + 1$ with two prime ideals $xR \supset yR \neq (0)$ where we wrote x again for the image of x in R.*

REMARK 3.10 *For this ring R we can make the following observations:*

(i) *$N_r(R) = J$ since $yx, x \neq 0$ but $(y^2x)x = 0$.*

(ii) *$N_l(R) = yR$ since $y(yx^2) = 0$, but $x^n a = 0$ if and only if $a = 0$.*

(iii) *For $I = yxR$ we have $I \subset I^{lr}$, since $I^l = (yxR)^l = y^2R$ and $I^{lr} = (y^2R)^r = x^2R$.*

(iv) *Since $N_r = J$ we have $T_r = U$ and hence $I = IT_r^{-1} \subset I^{lr}$ which shows that Theorem 2.9(iii) cannot be extended to right chain rings.*

(v) *For $P = yR$ a prime ideal we obtain $(yR)^{lr} = (y^2R)^r = x^2R \subset R$; in particular, P^{lr} is not a prime ideal.*

Since $P \neq P^{lr}$ is possible for prime ideals in right chain rings (in contrast to Proposition 2.10(i) for chain rings) one cannot conclude that the mapping which assigns P^l to P for prime ideals P in a right chain ring R is injective. The next example illustrates this point.

EXAMPLE 3.11 *We define $A[[y,\sigma]]$ as in Example 3.8 and consider $R = A[[y,\sigma]]/(yx)$. Then J and $yR = P$ are distinct prime ideals in R, but $J^l = yR = P^l \neq (0)$.*

3.3 Right distributive rings

We begin by recalling a definition made earlier.

DEFINITION 3.12 *A ring R is called* right distributive *if and only if*

$$A \cap (B + C) = (A \cap B) + (A \cap C)$$

for any right ideals A, B, C of R.

Proposition 1.3 shows that a local ring is right distributive if and only if R is a right chain ring. The commutative distributive domains are the Prüfer domains (see Gilmer [72]) and such rings R are also characterized by the fact that their localizations R_M at a maximal ideal M are valuation domains. The noetherian Prüfer domains are exactly the Dedekind domains. We have the following result in the general case:

THEOREM 3.13 *A domain R is a right distributive ring if and only if $S = R\backslash N$ is a right Ore system and $R_N = RS^{-1}$ is a right chain ring for every maximal right ideal N of R.*

PROOF: We show first that a maximal right ideal N in a right distributive ring R is two-sided. Otherwise, there exists $s \in R$ with $sN \not\subseteq N$ and thus $sN + N = R$; so there exists $n_1, n_2 \in N$ with $sn_1 + n_2 = 1$. Hence, $n_2 s = s(1 - n_1 s)$ and $1 - n_1 s \in s^{-1}N = \{r \in R \mid sr \in N\} = K$ and $1 - n_1 s \notin N$. This implies $R = K + N$ and for $D = K \cap N$ we obtain

$$R/D \cong R/K \oplus R/N$$

as right R-modules. Moreover, by the definition of K, we have a monomorphism $\alpha : R/K \to R/N$ given by $\alpha(r + K) = sr + N$. The module R/D has a distributive lattice of submodules. However, for $M_1 = R/K, M_2 = R/N$ and $M_3 = \{m_1 + \alpha m_1 \mid m_1 \in M_1\}$ we obtain the contradiction $M_3 = M_3 \cap (M_1 + M_2) \neq (M_3 \cap M_1) + (M_3 \cap M_2) = (0)$ since α is a monomorphism.

We prove next that $R\backslash N = S$ is a right Ore system. If $s_1, s_2 \in S$ we have $s_1 r_1 + n_1 = 1$ and $s_2 r_2 + n_2 = 1$ for certain $r_i \in R$, $n_i \in N$. Hence $1 = s_2(s_1 r_1 + n_1) r_2 + n_2 = s_2 s_1 r_1 r_2 + s_2 n_1 r_2 + n_2$, so S is multiplicatively closed.

Let $r \in R, s \in S$, then we have

$$rR = rR \cap (sR + (r - s)R) = (rR \cap sR) + (rR \cap (r - s)R)$$

and $r = (r - s)t + a$ for some $t \in R, a \in rR \cap sR$ follows.

We obtain $r(1 - t), st \in rR \cap sR$. If $t \in N$, then $1 - t \in S$ and $r(1 - t) = su$ for some u. If $t \notin N$, then $st = rb \in S$ and $b \in S$ follows. This shows that S is a right Ore system, the ring of quotients $RS^{-1} = \{rs^{-1} \mid r \in R, s \in S\}$ exists and is a right chain ring by Proposition 1.3.

Conversely assume that $R \backslash N = S$ is an Ore system and that $RS^{-1} = R_N$ is a right chain ring for every maximal right ideal N of R. Then we have

$$\begin{aligned}(A(B+C))R_N &= AR_N \cap (BR_N + CR_N) \\ &= (AR_N \cap BR_N) + (AR_N \cap CR_N) \\ &= [(A \cap B) + (A \cap C)]R_N\end{aligned}$$

for any maximal ideal N and any right ideals A, B, C of R. Hence, $A \cap (B+C) = A \cap B) + (A \cap C)$ and R is distributive, since $\bigcap IR_N = I$ any right ideal I of R where the intersection is taken over all maximal right ideals N of R. This last fact holds by the following argument: let $a \in \bigcap IR_N$, then $a = b_N s_N^{-1}$ for $b_N \in I$, $s_N \notin N$. Hence $T(a) = \{r \in R \mid ar \in I\} \not\subseteq N$ for all maximal right ideals N, hence $T(a) = R \ni 1$, so $a \in I$. ∎

COROLLARY 3.14 *A right Bezout domain R is right distributive if and only if all maximal right ideals N of R are two-sided.*

PROOF: It remains to show that $S = R \backslash N$ is an Ore system and AS^{-1} is a right chain ring provided N is a maximal right ideal in the Bezout domain R which is also two-sided. That S is multiplicatively closed follows as in the proof of the previous theorem.

If $s \in S, r \in R$, then $rR + sR = dR, s = ds_1, r = dr_1, s_1 R + r_1 R = R$ for some $d_1 r_1 \in R, s_1 \in S$. Hence there exist $x, y \in R$ with $s_1 x + r_1 y = 1$ and $s_1(xs_1 - 1) = -r_1 x s_1$, $r_1(yr_1 - 1) = -s_1 x r_1$ follows. If $y \in N$, then $yr_1 - 1 \in S$ and if $y \in S$ then $ys_1 \in S$ which proves that S is a right Ore system. By Proposition 1.3 $R_N = RS^{-1}$ is a right chain ring since it is a local Bezout domain. ∎

COROLLARY 3.15 *A right noetherian right distributive domain R is right invariant. All prime ideals of such a ring are completely prime. We have $PQ = Q$ for prime ideals $Q \subset P$ in R and $P_1 P_2 = P_2 P_1$ for two maximal prime ideals P_1, P_2 of R.*

PROOF: Since $R = \bigcap R_N$ and every R_N is invariant by Lemma 3.2, R is invariant. Hence, every prime ideal of R is completely prime. If N is a maximal right ideal, we have either $P \not\subseteq N$ and $PR_N = R_N$, $PQR_N = QR_N$ or $P \subseteq R_N$ and $QR_N \subset PR_N$ and therefore $PQR_N = PR_N QR_N = QR_N$ by Lemma 3.3. Finally, we can assume $P_1 \neq P_2$ and $P_1 + P_2 = R$ follows. Hence, P_1 and P_2 cannot be both contained in the same maximal right ideal N, and $P_1 P_2 R_N = P_1 R_N P_2 R_N = P_2 R_N P_1 R_N = P_2 P_1 R_N$ follows for all N. Hence $P_1 P_2 = P_2 P_1$. ∎

3.4 Right artinian right chain rings

A right artinian ring is right noetherian, hence, if R is also a right chain ring, we have $J = mR$, $m^n R = m^{n+1} R$ for a certain $n \in \mathbb{N}$ and $m^n R = (0)$ follows.

PROPOSITION 3.16 *The following conditions are equivalent for a right chain ring R.*

(a) *R is right artinian.*

(b) *J is nilpotent.*

(c) *$J = mR$ with m nilpotent.*

(d) *Each right ideal is a power of J.*

If R is a chain ring then R is right artinian if and only if it is left artinian.

PROOF: We have shown above that (a) implies (b).

(b) $\Rightarrow$ (c) J is finitely generated since otherwise $J = J^2$.

(c) $\Rightarrow$ (d) Let I be any right ideal $\neq R, (0)$ and $m^k R = J^k \subseteq I$ with k minimal. Hence $I = aR \subseteq m^{k-1}R, a = m^{k-1}r$ for some $r \in R$. It follows that r cannot be a unit, $r = mr_1$ and $I = aR = m^k R$.

(d) $\Rightarrow$ (a) Since $J^n = (0)$ for some n, there are only finitely many right ideals in R.

To prove the final statement of the proposition it is enough to show that $Rm = mR = J$ for $J \neq (0)$. Obviously, we have $Rm \subseteq mR$. The left ideal Rm is two-sided unless there exist $u \in U$, $v \in J$ with $vmu = m$ (see Lemma 1.4). In this case $m = vmu = mv'mu$ for some $v' \in R$ and $m = 0$ follows, a contradiction that shows that for chain ring the conditions (a) - (d) are left right symmetric. ∎

The rings characterized in Proposition 3.16 are well known, see for example Lesieur [67], Jonnson; Monk [69]. Krull [32] calls commutative rings of this type *primary* and structure theorems exist (see Clark; Drake [73], Clark; Liang [73]) for finite right chain rings, even though a classification by invariants has been given only for finite right chain rings satisfying additional conditions.

We conclude this chapter with an example given first by Baer ([42], p. 310f) that shows that a right artinian right chain ring which is left noetherian is not necessarily a left chain ring. This is in contrast to the result about right chain domains proved in Lemma 1.2 (iv). Related examples were given by Camillo ([75], p. 24f), Jain et al. [76] and Courter [82].

EXAMPLE 3.17 *Let F be a field with a monomorphism $\sigma : F \to F$ and $F^\sigma \subset F$. Let $A = F[x, \sigma]$ be the skew polynomial ring in X with elements $r = \sum xa_i, a_i \in F$ and multiplication defined by $ax = xa^\sigma$. Then xA is a two-sided ideal and $R = A/(x^2A)$ is a right chain ring with $J = xR$ and $J^2 = (0)$. However, for $a \in F \backslash F^\sigma$ we have $Rx \not\subseteq Rxa \not\subseteq Rx$, i.e. R is not a left chain ring. (We denote the image of x in R again by x). If $[F : F^\sigma]$ is finite with basis $\{b_1, \ldots, b_n\}$ then $\{1, xb_1, \ldots, xb_n\}$ is a basis for A as left vector space over F. In particular A is left artinian. For $F = \mathbb{Q}(t)$ and σ with $\sigma(t) = t^n$, t an indeterminate over the rationals $\mathbb{Q}$ we have $[F : F^\sigma] = n$.*

4 Associated prime ideals

With every proper right (two-sided) ideal I of a right chain ring R, is associated a completely prime ideal $P_r(I) \supseteq I$ ($P_l(I) \supseteq I$) and $P_r(I)/I$ ($P_l(I)/I$) is the set of right (left) zero-divisors in R/I whenever I is two-sided. For right ideals I the minimal completely prime ideal P containing I is the corresponding right associated prime ideal if and only $P = \sqrt{I}$ holds and I is right P-primary (Proposition 4.8). Associated prime ideals of various types of right ideals are calculated. It is proved that related right ideals A, B (i.e. $s^{-1}A = t^{-1}B$ for some $s \notin A$, $t \notin B$) have the same right associated prime ideals. From this it follows (Theorem 4.19) that $P = \bigcup A_\lambda$ where the union is taken over the class of right ideals related to A and P is the right associated prime ideal of A. The associated prime ideals are also helpful in computing double annihilators in case R is a chain ring: We prove that P^{rr} is again completely prime if P is a completely prime ideal and $P^r \neq (0)$ (Proposition 4.14).

4.1 Preliminaries on associated prime ideals

If R is a commutative valuation ring, I an ideal of R, then the *minimal prime ideal P over I* is the prime ideal of R minimal with the property of containing I. However there are other prime ideals connected with I.

The set

$$P_x(I) = \{a \in R \mid a^n x \in I \text{ for some } n \in \mathbb{N}\}$$

(see Jacobson [80], p. 432) is a prime ideal of R for any $x \in R \backslash I$, since $(ab)^n x \in I$, $b = ra$ for some $r \in R$ say, implies $r^n (ab)^n = b^{2n} x \in I$. It follows that $P_x(I) = P$ for $x \notin P$ and that

$$P(I) = \{p \in R \mid px \in I \text{ for some } x \notin I\} = \bigcup_{x \in R \backslash I} P_x(I)$$

is the prime ideal of all elements in R that induce zero-divisors in R/I.

We consider an example. Let R be a commutative valuation ring with $G = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$ as a group of values and v the corresponding mapping from R^* into G. Then $I = \{r \in R \mid v(r) > (2, 1, a) \text{ for all } a > \sqrt{2}\}$ is an ideal of R. If $R \supset P_1 \supset P_2 \subset P_3 \supset (0)$ is the chain of prime ideals of R then P_3 is equal to P, the minimal prime over I. Let x_1 be an element of R with $v(x_1) = (2, 1, 1)$, say, then $P_{x_1}(I) = P_1$, if $v(x_2) = (2, 1/2, 0)$, then $P_{x_2}(I) = P_2$ and if $v(x_3) = (1, 1, 0)$, then $v(x_3) = P_3 = P$. Finally $P_l(I) = P_1$.

The sets $P_x(I)$ are not necessarily prime ideals for an ideal I in a commutative noetherian ring R (for example $P_2((12\mathbb{Z})) = 6\mathbb{Z}$ in the ring $\mathbb{Z}$ of integers), but the prime ideals among the $P_x(I)$ are associated prime ideals of I. Furthermore, $P(I) = \bigcup_{x \in R \backslash I} P_x(I)$ is the set of elements in R that induce zero-divisors in R/I, but for commutative noetherian rings R in general it is not an ideal. We recall that $\sqrt{I} = \{a \in R \mid a^n \in I \text{ some } n\} = \bigcap_{i=1}^{s} P_i$ where $\{P_i \mid i = 1, \ldots, s\}$ is the set of associated prime ideals of I.

We return to a right chain ring R and let $I \neq R$ be a right ideal in R. We use the following notation for the sets of elements relatively prime to I (see Törner [76]):

DEFINITION 4.1 *Let R be a right chain ring. For a right ideal $I \neq R$ we define*

$$S_r(I) = \{s \in R \mid ts \in I \Rightarrow t \in I\},$$

similarly for a two-sided ideal $I \neq R$:

$$S_l(I) = \{s \in R \mid st \in I \Rightarrow t \in I\}.$$

We set $P_r(I) = R \backslash S_r(I)$ $(P_l(I) = R \backslash S_l(I))$ and call this set the right *((* left*)* associated prime ideal *with respect to the right (two-sided) ideal I.*

It will be proved in Theorem 4.2 that $P_r(I)$ is in fact a completely prime ideal. These prime ideals occur in the literature in various contexts. $P_r(I)$ is called *'adjoint à droite'* to the two-sided I provided $P_r(I)$ itself is two-sided (Brameret [63]). Mathiak [86], p. 85) introduces the prime ideals $P_r(I)$ for two-sided ideals I in a *chain domain.* Matlis ([59], p. 66) defines an ideal I in a commutative valuation ring R as *archimedean* if $P_r(I) = P_l(I) = J(R)$.

With the terminology of Section 1.3 we obtain the following equivalent description:

$$P_l(I) = \{p \in R \mid \exists r \notin I : pr \in I\} = \{p \in R \mid p^{-1}I \supset I\} = I(R \backslash I)^{-1}$$
$$P_r(I) = \{p \in R \mid \exists r \notin I : rp \in I\} = \{p \in R \mid Ip^{-1} \supset I\} = (R \backslash I)^{-1}I$$

It is obvious that I is contained in $P_l(I)$ and $P_r(I)$. We note that for any completely prime ideal P we clearly have $P_l(P) = P_r(P) = P$. For arbitrary rings the complement of the set of elements relatively prime to an ideal is not an ideal. For right chain rings however we have the following result:

THEOREM 4.2 *Let R be a right chain ring. If $I \neq R$ is a right ideal then $P_r(I)$ is a completely prime ideal containing I. If I is a two-sided ideal then $P_l(I)$ is also completely prime and contains I. For $I = (0)$, we have $P_r(0) = N_r(R)$ and $P_l(0) = N_l(R)$.*

PROOF: Obviously $P_r(I) = P$ is a right ideal of the semigroup $(R, \cdot)$ and therefore by Lemma 1.5(i) it is a right ideal of the ring R. As the complement $R \backslash P$ is multiplicatively closed and contains U, the right ideal P is a two-sided completely prime ideal by Lemma 1.4. This proves the first part.

Now let I be a two-sided ideal and $P = P_l(I)$. To show that P is a left ideal consider $x_1, x_2 \in P$ with $x_1t_1, x_2t_2 \in I$ and $t_1, t_2 \notin I$. We can assume that $t_1r = t_2$ and $x_1t_1r + x_2t_2 = (x_1 + x_2)t_2 \in I$ follows. It is clear that $R \cdot P$ is in P which proves that P is a left ideal. As the complement of P is multiplicatively closed and contains the set U of units we have $P \cdot U \subseteq P$ and P is a right ideal by Lemma 1.5(iii). ∎

The left and the right associated prime ideals may not coincide as the following example shows:

EXAMPLE 4.3 *Again take R as in Example 3.8. Consider the ideal $I = yxR$. Then $P_l(I) = yR$ and $P_r(I) = xR$.*

This observation leads to the following terminology:

DEFINITION 4.4 *Let R be a right chain ring and I a two-sided ideal. The ideal I is called P-*symmetric *if $P_l(I) = P_r(I) = P$ is satisfied.*

The next result follows from the definition and states that the images of $P_r(I)$ ($P_l(I)$) in R/I are exactly the right (left) zero-divisors in R/I.

LEMMA 4.5 *Let R be a right chain ring, I a two-sided ideal and $\overline{R} = R/I$. Then we have*

(i) *$N_l(\overline{R}) = P_l(I)/I$ and $N_r(\overline{R}) = P_r(I)/I$.*

(ii) *For a two-sided ideal $L \supseteq I$ of R we have $P_l(L/I) = P_l(L)/I$ and $P_r(L/I) = P_r(L)/I$.*

4.2 Associated prime ideals and the radical of an ideal

We investigate the relationship between the radical of I and the associated prime ideals of I for a two-sided ideal.

LEMMA 4.6 *Let I be a right ideal of the right chain ring R with $P_r(I) = P$ where P is the minimal prime ideal containing I. Further assume that the segment generated by I is not simple. Then I is a two-sided ideal. If in addition $PI \neq 0$ holds, we have $P_l(I) = P$.*

PROOF: Let $[P, Q[$ be the segment generated by I and $P_r(I) = P$ completely prime. To show that I is two-sided assume $ux \notin I, u \in U, x \in I$. Hence, $uxv = x$ for some $v \in R$ and $v \in R \backslash P$ by Corollary 1.22. It follows from the definition of $P_r(I)$ that $ux \in I$, a contradiction.

Obviously we have $P = P_r(I) \subseteq P_l(I)$. To prove $P_l(I) = P$ take $s \in R \backslash P$ and $sx \in I$ for some $x \in R$. We have to show that x is in I. This is obvious if $sxt = x$ for some $t \in R$. Otherwise $sx = xt$. If t is not in $P_r(I)$, then $x \in I$. It remains to consider the case $t \in P$, $x \notin I$. If P is the radical then $t^n = 0$ for some n and $s^n x = 0$. However in this case $(0) = s^n xR \supseteq s^n I \supseteq PI$ which contradicts $PI \neq (0)$. If P is not the radical of R, then $sx = xt$ with $s \notin P$, $t \in P$ contradicts Corollary 1.22. ∎

In Example 4.3 we have for $I = yxR$ that $P = P_l(I) = yR$ is the minimal prime ideal containing I, but $P \subset P_r(I) = xR$. This shows that the second statement in Lemma 4.6 is not left-right-symmetric.

We recall that the set

$$\sqrt{I} = \{x \in R \mid \exists n \in \mathbf{N} : x^n \in I\}$$

is called the *radical* of the two-sided ideal I. It follows from Theorem 1.21(vi) that $P = \sqrt{I}$ where P is the minimal prime ideal containing I, provided P is completely prime. If this additional condition is not satisfied, i.e. P is an exceptional prime, then $\sqrt{I}$ is neither a left nor a right ideal (see Chapter 6).

This leads us to the following definition specializing the notion in Lemma 3.6.

DEFINITION 4.7 *Let R be a right chain ring and I a two-sided ideal in the prime segment $[P, Q[$ with P completely prime. I is called* right P-primary *if $ab \in I, a \notin I$ implies $b^n \in I$ for some $n \in \mathbb{N}$ where $a, b \in R$. Dually, we define* left P-primary.

By Theorem 1.21(vi) we have $\sqrt{I} = P$.

PROPOSITION 4.8 *Let R be a right chain domain and I a two-sided ideal in the prime segment $[P, Q[$. Then the following conditions are equivalent:*

(a) *I is right P-primary.*

(b) *$P_r(I) = P$.*

PROOF: (a) $\Rightarrow$ (b) By Definition 4.7 P is the minimal prime ideal containing I and $P \subseteq P_r(I)$. To prove $R \backslash P \subseteq R \backslash P_r(I)$ assume $xs \in I$, $s \in R \backslash P$ and $x \notin I$. Hence, $s^n \in I \subseteq P$, which contradicts P a completely prime ideal by definition.

(b) $\Rightarrow$ (a) Assume $P_r(I) = P$. By Theorem 4.2 P is a completely prime ideal. Suppose $ab \in I$ and $a \notin I$. If $b \notin P$, the assumption $P = P_r(I)$ implies $a \in I$. Hence we can assume $b \in P$ and by Theorem 1.21(v) we obtain $b^n \in I$ for some $n \in \mathbb{N}$. ∎

With Lemma 4.6 we obtain the following result:

COROLLARY 4.9 *Let R be a right chain domain and I a right P-primary two-sided ideal. Then I is also left P-primary.*

4.3 Associated prime ideals of right ideals of special types

Next we describe explicitly the associated prime ideals for various types of right ideals in right chain rings, in particular of right ideals of the form IP and IS^{-1} for $P = R \backslash S$ a completely prime ideal of R and I a right ideal.

PROPOSITION 4.10 *Let R be a right chain ring, $P = R \backslash S$ a completely prime ideal and $I \neq R$ a right ideal of R. Then we have:*

(i) *$P_r(P) = P_l(P) = P$.*

(ii) *For all $a \in R$ with $aP \neq (0)$ we have $P_r(aP) = P$. In particular, $J = P_r(aJ)$ for all $0 \neq a \in J$.*

(iii) *Let $IP \neq (0)$. Then $P_r(IP) \subseteq P$.*

(iv) *$P_r(I) \subseteq P$ if and only if $I = IS^{-1}$.*

(v) *For all $0 \neq a \in P$ we have $P_r((aR)S^{-1}) \subseteq P$.*

(vi) *Let $P_r(I) = P$ and $Q = R \backslash T$ be a completely prime ideal with $Q \subseteq P$ and $IQ \neq (0)$. Then $P_r(IQ) = Q$.*

PROOF: (i) follows from the definition.

(ii) Since we have $a \notin aP$, we conclude $P \subseteq P_r(aP)$. Assume $P \subset P_r(aP)$ and $s \notin P$ exists with $xs \in aP$, $x \notin aP$. Let $xs = ap$ for some $p \in P$. If $xs = 0$, then $(0) = xP = xsP = aP$, a contradiction since $s \notin P$. Hence, $xs = ap \neq 0$. We compare aR and xR. If $ar_1 = x$ with $r_1 \notin P$, then $ap = xs = ar_1 s$ implies $xs = 0$ since $p = r_1 sj$ for some $j \in J$. If $a = xr_2$, we obtain $xs = ap = xr_2 p$ and again $xs = 0$ follows, a contradiction. Hence, $P \subset P_r(aP)$ is impossible proving $P = P_r(aP)$ and (ii).

(iii) If $IP \subseteq I$ holds, then $IP = aP$ for some $a \in I \backslash IP$ by Lemma 1.11(iv) and (iii) follows from (ii). Hence assume $I = IP$ and we show $R \backslash P \subseteq R \backslash P_r(IP)$. Take $s \in R \backslash P$ and $x \in R$ with $xs \in IP$. We have $xs = yp$ for some $y \in I$, $p \in P$. If $x \notin I = IP$, then $y \in xR$ and hence $xs \in xP$ leading to $xs = 0$ and further $IP \subseteq xP = xsP = (0)$, a contradiction. Hence, $x \in I$, and we are done.

(iv) Assume $P_r(I) \subseteq P$ and take $x \in IS^{-1}$. Then $xs \in I$ for some $s \in S = R \backslash P$, hence $x \in I$. We obtain $IS^{-1} \subset I$ and so $I = IS^{-1}$. Conversely, assume $I = IS^{-1}$. Let $x \in P_r(I) \backslash P$, then $tx \in I$ for some $t \notin I$ and $t \in IS^{-1} = I$, a contradiction.

(v) Note that by Corollary 1.10 we have $((aR)S^{-1})S^{-1} = (aR)S^{-1}$, hence the assertion follows by (iv).

(vi) Assume first that $P_r(I) = P = Q$. If $IQ \subset I$, we have $IQ = aQ$ for some $a \in I \backslash IQ$ (Lemma 1.11(iv)) and $P_r(IQ) = P_r(aQ) = Q$ follows by (ii). The case $IQ = I$ is obvious. We now turn to the case $Q \subset P$. Applying (iii) it remains to prove that $Q \subseteq P_r(IQ)$. If $IQ \subset I$, then $IQ = aQ$ and $P_r(aQ) = Q$ for some $a \in I \backslash IQ$.

We assume $I = IQ$ and choose $p \in P \backslash Q$. Then there exists $t \notin I$ such that $tp = x \in I$, since $P_r(I) = P$. Since $x \in I = IQ$ we have $x = tp = ab, a \in I, b \in Q$. Therefore $x = 0$ since $a = tj$, $j \in J$ and $p \notin Q$, $jb \in Q$. We obtain $IQ \subseteq tQ = (0)$, a contradiction. ■

In the case of chain rings the statements become smoother.

LEMMA 4.11 *Let R be a chain ring, then we have:*

(i) $P_r((aR)S^{-1}) = P$ *for all* $0 \neq a \in P$.

(ii) $P_r(aR) = J$ *for all* $0 \neq a \in J$.

(iii) *Let $(0) \neq I$ be right ideal, $P_r(I) = P$ and $Q = R \backslash T$ a completely prime ideal satisfying $I \subseteq Q \subseteq P$. Then $P_r(IT^{-1}) = Q$.*

PROOF: (i) By Proposition 4.10 (v) it remains to prove $P \subseteq P_r((aR)S^{-1})$. If $P = (aR)S^{-1}$ we are done (Proposition 4.10(i)). Now we assume $(aR)S^{-1} \subset P$ and take $p \in P \backslash (aR)S^{-1}$. We have either $b_1 p = a$ or $p = b_2 a$. If in the first $b_1 \notin (aR)S^{-1}$, then $p \in P_r((aR)S^{-1})$. If $b_1 \in (aR)S^{-1}$, say $b_1 s = a$ for some $s \in S$, we have $a = b_1 p = b_1 sp' = ap'$ for $p = sp'$ and $p' \in P$ since $sP = P$, hence $a = 0$, a contradiction. In the other case, $b_2 \in (aR)S^{-1}$, we may assume $b_2 s = a$ for some $s \in S$, otherwise if $b_2 = as$ holds, $p = b_2 a = asa \in (aR)S^{-1}$ would follow. Hence, $a = sa'$ for $a' \in R$, since $(aR)S^{-1} \subset P \subset sR$. We obtain $p = b_2 a = b_2 sa' = aa' \in (aR)S^{-1}$, a contradiction.

(ii) follows from (i)

(iii) If $I \subset IT^{-1}$ we are done by (i) and Lemma 1.11(i). Hence we may assume $I = IT^{-1}$ which implies $P = P_r(I) \subseteq Q$ by Proposition 4.10(iv), hence $P = Q$ and $P_r(IT^{-1}) = P_r(I) = P = Q$. ■

Next we are asking for the *left* associated prime ideals of two-sided P-standard right ideals in case of chain rings. By understanding this relation we are able to control a shifting process on prime ideals which will also be studied for right chain rings under additional conditions in Chapter 7 and 8.

Let P be again a completely prime ideal in a chain ring R and $Pa \neq (0)$ a left ideal. Then $Pa \neq (0)$ is a right ideal if and only if $Pa \subseteq aJ$. To see this, first let Pa be a right ideal and assume $par = a$ for some $r \in R$, $p \in P$. Hence, $a \in Pa$ leading to $a = 0$, a contradiction. Hence we have $Pa \subseteq aJ$.

Now let $Pa \subseteq aJ$ and take $pa \in Pa$, $u \in U$. If $au = ra$ for some $r \in R$ we are done. We are also finished if $rau = a$ for some $r \notin P$. However, $rau = a$ with $r \in P$ leads to $a = 0$ using $Pa \subseteq aJ$.

PROPOSITION 4.12 *Let R be a chain ring, $P = R\backslash S$ a completely prime ideal and $a \in R$.*

(i) *If $Pa \neq (0)$ is a two-sided ideal, then there exists a completely prime ideal $P_1 = R\backslash S_1$ with $Pa = aP_1$. We have $P = P_1$ if and only if $P_l(Pa) = P_r(Pa)$.*

(ii) *If $S^{-1}(Ra) \neq (0)$ is a two-sided ideal, then there exists a completely prime ideal $P_1 = R\backslash S_1$ with $S^{-1}(Ra) = (aR)S_1^{-1}$. We have $P = P_1$ if and only if $P_l(S^{-1}(Ra)) = P_r(S^{-1}(Ra))$.*

(iii) *Assume $Pa = aP_1$ as in* (i) *and let Q be the lower neighbour of P as a prime ideal. Assume in addition that Q is completely prime and $Qa \neq (0)$. Then we have $Qa = aQ_1$ where Q_1 is the lower neighbour of P_1 as a prime ideal which is again completely prime.*

PROOF: (i) Set $P_1 = \{x \in R \mid ax \in Pa\}$. First we show that P_1 is a right ideal. Let $r \in R$ and $x \in P_1$, so $ax = pa$ with $p \in P$. If $ar = s_1 a$ for some s_1, then $axr = par = ps_1 a \in Pa$, hence $xr \in P_1$. Otherwise we have $s_2 ar = a$ with $s_2 \in J$. If $s_2 \in P$, then $a = s_2 ar \in Par \subseteq aJ$, so $a = 0$ - a contradiction. Thus $s_2 \notin P$ and so $p = qs_2$ with $q \in P$. In this case $axr = par = qs_2 ar = qa \in Pa$, so $xr \in P_1$.

Now we want to prove that P_1 is a left ideal. Take $u \in U$. If $au = va$ for a suitable $v \in R$ we are done. Otherwise we have $vau = a$ with $v \notin P$ because of $Pa \neq (0)$. The proof then proceeds with the same arguments as above.

Finally it remains to show that P_1 is completely prime. If $x \notin P_1$ then $ax = q_1 a$ or $q_2 ax = a$ with $q_1, q_2 \notin P$. As q_1^2, $q_2^2 \notin P$, we obtain $x^2 \notin P_1$ in both cases.

(ii) Let $P_r(S^{-1}(Ra)) = P_1 = R\backslash S_1$ with P_1 a completely prime ideal. Hence, $(aR)S_1^{-1} \subseteq S^{-1}(Ra)$ since $S^{-1}(Ra)$ is two-sided and therefore $aR \subseteq S^{-1}(Ra)$. Then $xs_1 \in aR$ with $s_1 \in S_1$ implies $x \in S^{-1}(Ra)$. To prove equality, assume $(aR)S_1^{-1} \subset S^{-1}(Ra)$ with $y \in S^{-1}(Ra)\backslash(aR)S_1^{-1}$. We have $yp = a$ for some $p \in P_1$, otherwise

$y \in (aR)S_1^{-1}$ would follow. Since $P_r(S^{-1}(Ra)) = P_1$ we find $y' \notin S^{-1}(Ra)$ with $y'p \in S^{-1}(Ra)$. Hence, $sy'p = a$ with $sy' = y$ for some s and $s \in S$ follows showing $y' \in S^{-1}(Ra)$, a contradiction.

(iii) Obviously, by (i) there is a completely prime ideal $Qa = aQ_1$. There cannot be any completely prime ideal Q_1' with $Q_1 \subset Q_1' \subset P_1$, otherwise (i) would induce a completely prime ideal Q' with $Q'a = aQ_1'$ and $Q \subset Q' \subset P$. It remains to show that there cannot be an exceptional prime ideal Q_1' with $Q_1 \subset Q_1' \subset P_1$. As it will be shown in Theorem 6.2 we have $P_1^2 = P_1$ and $P^2 = P$ follows. Note that by $Qa = aQ_1 \subseteq aJ$ an equation $uas = a$ for some unit $u \in U$ implies $s \notin Q$, hence $Q' = \{x \in R \mid xa \in aQ_1'\}$ is a two-sided ideal with $Q \subseteq Q' \subseteq P$. Since $P^2 = P$ is idempotent, $[P, Q[$ is not simple, by Theorem 1.21 a two-sided ideal I with $Q \subset I \subset P$ must exist causing again a two-sided ideal I_1 to exist (use symmetrical arguments) with $Ia = aI_1$ and $Q_1' \subset I_1 \subset P_1$, a contradiction. ∎

The symmetric case when $P = P_1$ in the last proposition is of particular interest and will be investigated later.

We recall that $P^l = (aR)S^{-1}$ for some $a \in P^l$ if P is completely prime by Lemma 2.6. Applying Lemma 4.10(v) and 4.11(i) we obtain Lemma 4.13.

LEMMA 4.13 *Let R be a right chain ring and P a completely prime ideal with $P^l \neq (0)$. Then $P_r(P^l) \subseteq P$. If R is, in addition, a left chain ring, then equality holds.*

It is easy to show that in right chain rings the equality does not hold in general; take the ring R in Example 3.9. Then $P_r((xR)^l) = P_r(yR) = yR \subset xR = P$.

As mentioned in Proposition 2.10(i) we saw that $P^{rl} = P$ if P is a completely prime ideal and $P^r \neq (0)$. The next result gives information about *right-right annihilator ideal* P^{rr} for completely prime ideals P with $P^r \neq (0)$.

PROPOSITION 4.14 *Let R be a chain ring and $P \neq 0$ a completely prime ideal with $P^r \neq (0)$. Then*

(i) $P_r(P^r) = P^{rr}$.

(ii) P^{rr} *is completely prime.*

PROOF: (i) First we show: $R \backslash P_r(P^r) \subseteq R \backslash P^{rr}$. Let x be in $R \backslash P_r(P^r)$ and assume $x \in P^{rr}$. Take $0 \neq z \in P^r$. Since $P^r \subseteq P_r(P^r)$, $x \notin P^r$, so $z = sx$ for some $s \in R$. We obtain $s \in P^r$, hence $z = 0$, since $x \in P^{rr}$, a contradiction. Thus $P^{rr} \subseteq P_r(P^r)$.
Conversely, let $x \in P_r(P^r)$. Then there exists $t \notin P^r$ with $tx \in P^r$. We must show: $zx = 0$ for any $z \in P^r$. Let $z \in P^r$, then $ut = z$ for some $u \in R$ since $t \notin P^r$. If $u \notin P$, we apply the symmetric version of Lemma 4.13 to conclude that $u \in R \backslash P_l(P^r)$ and obtain $t \in P^r$. Therefore $u \in P$, and hence $zx = utx = 0$. Thus $P_r(P^r) \subseteq P^{rr}$, which proves (i).

The statement (ii) follows from (i) and Theorem 4.2. ∎

4.4 The set of right ideals with a fixed right associated prime ideal

We pointed out earlier that the prime ideals $P_r(I)$ and $P_l(I)$ reflect essential properties of the two-sided ideal I in a right chain ring. We will obtain some information about the distribution of right ideals with the same associated prime ideal in the lattice of right ideals of R. The next result shows that for $P_r(I) = P$ and for a completely prime ideal $Q = R\backslash T \subset P$ we have $P_r(IQ) = Q = P_r(IT^{-1})$, but $P_r(L) \supset Q$ for all right ideals L with $IQ \subset L \subset IT^{-1}$.

THEOREM 4.15 *Let R be a chain domain whose prime ideals are completely prime and $(0) \neq Q = R\backslash T \subset P = R\backslash S$ be prime ideals. Let $I \neq (0)$ be a right ideal with $I \subseteq Q$ and $P_r(I) = P$. Then we have:*

(i) *In the prime segment generated by I there exists a right ideal I' with Q as its right associated prime ideal. If I is two-sided, I' can be chosen to be two-sided.*

(ii) *IT^{-1} is the smallest right ideal containing I with Q as its right associated prime ideal. We have $IT^{-1} = (aR)T^{-1}$ for any $a \in IT^{-1}\backslash I$.*

(iii) *$IQ \neq (0)$ is the largest right ideal contained in I with Q as its right associated prime ideal.*

(iv) *If L is any right ideal with $IQ \subset L \subset IT^{-1}$, then we have $P_r(L) \supset Q$.*

PROOF: (i) Take IT^{-1} and apply Lemma 4.11(iii). If I is two-sided, so is IT^{-1} (use Lemma 1.9).

(ii) By Lemma 4.11(iii) we know that $P_r(IT^{-1}) = Q$ holds. Since $P_r(I) \neq P_r(IT^{-1})$ we have $I \subset IT^{-1}$ and $IT^{-1} = (aR)T^{-1}$ for any $a \in IT^{-1}\backslash I$. If $L \supseteq I$, $P_r(L) = Q$ for a right ideal L, then $I \subset L$ and $(aR)T^{-1} \subseteq L$ for any $a \in L\backslash I$.

(iii) Again we have $P_r(IQ) = Q$ using Proposition 4.10(vi). This implies that $IQ \subset I$, hence $IQ = aQ$ for some $a \in I\backslash IQ$ by Lemma 1.11(iv). For $a_1, a_2 \in I\backslash IQ$ with $a_1R \supseteq a_2R$ we have $a_1r = a_2$ with $r \in T$. Any ideal $L \supset aQ$ with $P_r(L) = Q$ would contain at least one element $a \in I\backslash IQ$ and hence all such elements, since $(aR)T^{-1} \subseteq L$. This would imply $I = L$, however $P_r(I) = P_r(L) = Q$ is contradicting our assumption.

(iv) follows from (ii) and (iii). Note that there cannot be an ideal L with $IQ \subset L \subset IT^{-1}$ with $P_r(L) = Q' = R\backslash T' \subset Q$. Otherwise $L = LT'^{-1}$ would imply $IT^{-1} \subseteq L$. ∎

In the next result we describe a right ideal I with associated prime ideal $P = R\backslash S$ in terms of the special right ideals $(aR)S^{-1}$ and aP. As a consequence we show that if I with $P_r(I) = P$ is two-sided, then there exist standard P-associated ideals in a neighbourhood of I which are also two-sided.

THEOREM 4.16 *Let R be a chain domain, $P = R\backslash S \supset Q = R\backslash T$ completely prime ideals with $[P, Q[$ not simple. Further let $(0) \neq I$ be a right ideal with $P_r(I) = P$. Then we have:*

(i) $I = \bigcup_{a \in I}(aR)S^{-1}$.

(ii) $I = \bigcap_{a \notin I} aP$.

(iii) *Assume that I is a two-sided ideal. Then we have $I = \bigcup_{a \in I' \subset I}(aR)S^{-1}$ with I' a subset of I such that $(aR)S^{-1}$ is two-sided for any $a \in I'$.*

(iv) *Assume that I is a two-sided ideal. Then we have $I = \bigcap_{a \in I' \subset R \backslash I} aP$ with I' a subset of $R \backslash I$ such that aP is two-sided for any $a \in I'$.*

PROOF: (i) Since $(aR)S^{-1} \subseteq I$ holds for all $a \in P$, the equation is obvious.

(ii) First, $aP \subset I$ for some $a \notin I$ could never hold. Otherwise take any $b \in I \backslash aP$ and $b = as$ follows for some $s \in S$, since $b \notin aP$. However, $s \in P_r(I) = P$ since $a \notin I, b \in I$, a contradiction. So, $I \subseteq \bigcap_{a \notin I} aP$. Assume there exists $b \in (\bigcap_{a \notin I} aP) \backslash I$, hence $b \in bP$, again a contradiction.

(iii) We consider $\bigcup(aR)S^{-1}_{a \in I \backslash IQ}$ and prove $(aR)S^{-1}$ is two-sided for $a \in I \backslash IQ$. Then we restrict the elements over which the union of $(aR)S^{-1}$ is taken to those elements of aR lying between IQ and I. It suffices to show that $(aR)S^{-1}$ is two-sided. Let $p \in P \backslash Q$, hence there exists $x \notin I$ with $xp = z \in I$ for some $z \in I$. We can assume that $zR \subseteq aR$ holds, otherwise multiply $z = xp$ by some $p' \in P \backslash Q$. If $(aR)S^{-1}$ is not two-sided we have $uaq = a$ for some $q \in P$. Without loss of generality $Rq \subseteq Rp$. We obtain $z = ar = uaqr = uar'p = xp$ for $r' \in R$, hence $x = uar' \in I$, a contradiction since I is assumed to be two-sided.

(iv) with similar arguments as in (iii) ∎

We consider right ideals A_λ, $\lambda \in \Lambda$ related to a right ideal A and show that they have the same right associated prime ideals $P = P_r(A)$. In addition we show that $P = \bigcup_{\lambda \in \Lambda} A_\lambda$ and obtain some information about $\bigcap_{\lambda \in \Lambda} A_\lambda$.

DEFINITION 4.17 *Let R be an arbitrary ring and A, B right ideals. A, B are called* related *provided there exist $s \notin A$, $t \notin B$ satisfying $s^{-1}A = t^{-1}B$.*

We recall that $s^{-1}A = \{x \in R \mid sx \in A\}$. The relation defined above is an equivalence relation. We only prove its transitivity, the rest is obvious. Let $s^{-1}A = t^{-1}B$ and $v^{-1}B = w^{-1}C$. W.l.o.g. assume $tr = v$. By a straightforward calculation one shows that $(sr)^{-1}A = w^{-1}C$ holds with $sr \notin A$. The following results will be used in different situations, for example in the classification of injective modules over right chain rings.

LEMMA 4.18 *Let A, B be right ideals of a right chain ring R. If $s^{-1}A = t^{-1}B$ with $s \notin A$, $t \notin B$, then $P_r(A) = P_r(B)$.*

PROOF: Let $s^{-1}A = t^{-1}B$. Take $p \in S_r(A) = R \backslash P_r(A)$. Suppose $xp \in B$. First we assume $tz_1 = x$ for some $z_1 \in R$. Then $xp = tz_1p \in B$, so $sz_1p \in A$. As $p \in S_r(A)$ we get $sz_1 \in A$ and thus $tz_1 = x \in B$, hence $p \in S_r(B) = R \backslash P_r(B)$. Now assume $t = xz_2$ for some $z_2 \in R \backslash P$. But then $z_2z_2' = p$ for some $z_2' \in S_r(A)$ and

$xp = xz_2z_2' = tz_2' \in B$, so $sz_2' \in A$. Thus $s \in A$, as $z_2' \in S_r(A)$, and this is again a contradiction. ■

We note that $xA \neq 0$ for some right ideal implies A related to xA. To prove this we claim that $x^{-1}(xA) = A$ where $A \subseteq x^{-1}(xA)$ is obvious. If, conversely, an element $z \in x^{-1}(xA) \backslash A$ exists then z is not a right zero-divisor, but $xz = xa \in xA$ for some $a \in A \subset zR$. Then $a = zj, j \in J$ and $xz = 0$ follows, a contradiction which shows that $x^{-1}(xA) = A$ and xA is related to A for every $0 \neq x \in R$.

THEOREM 4.19 *Let R be a right chain ring, $A \neq R$ a right ideal in R. Further, let A_λ, $\lambda \in \Lambda$ be the set of all right ideals related to A; with $P_r(A) = P = R \backslash S$ and $D = \bigcap_{\lambda \in \Lambda} A_\lambda$ we obtain:*

(i) $P = \bigcup_{\lambda \in \Lambda} A_\lambda$.

(ii) $P_r(D) \subseteq P$.

(iii) (a) *If $N_r(R) \subset P$, then $D = (0)$.*

(b) *If $P \subseteq N_r(R)$, then either $D = (0)$ and $P = P_r(0) = N_r(R)$ or $D \neq (0)$ and $D = P^l$.*

PROOF: (i) Obviously, $\bigcup_{\lambda \in \Lambda} A_\lambda = (R \backslash A)^{-1} A = P_r(A)$ (see the remarks after Definition 3.1).

(ii) We show that $S = S_r(A_\lambda) \subseteq S_r(D)$. Let $s \in S$ and assume $xs \in D$. Then $xs \in A_\lambda$ for all $\lambda \in \Lambda$ follows. Hence, $x \in A_\lambda$ for all $\lambda \in \Lambda$ and $x \in D, s \in S_r(D)$ as claimed.

(iii)(a) By (i) there exists a right ideal A_λ with $P \supseteq A_\lambda \supset N_r(R)$. As A_λ contains regular elements the right ideal xA_λ is never the zero-ideal, thus $xR \supseteq xA_\lambda \neq (0)$ for any $x \in R^*$. As mentioned above A and xA are related. If $d \in D$ then $d \in dA_\lambda \subseteq dJ$ and so $d = 0$. Hence $D = (0)$.

(iii)(b) If $D = (0)$ we obtain by (ii) that $P_r(D) = P_r(0) \subseteq P$ and by assumption $P \subseteq N_r(R)$. Since $N_r(R) = P_r(0)$ we conclude $P = N_r(R) = P_r(0)$. We consider next the case $D \neq (0)$ and must show $D = P^l$. Let $0 \neq x \in D$. If we assume $xA_\lambda \neq (0)$ for some $\lambda \in \Lambda$, then $xA_\lambda \subset xR \subseteq D$ and as xA_λ is related to A_λ, also $D \subseteq xA_\lambda$ follows, a contradiction. Thus we have $xA_\lambda = (0)$ for all $\lambda \in \Lambda$, hence by (i) $xP = (0)$ and so $D \subseteq P^l$. By (ii) $P^l \subseteq D$ follows and we have shown $D = P^l$. ■

5 Localization of right chain rings

Right chain rings can be localized at every completely prime ideal (Proposition 5.5). In the case of chain domains R each overring in the quotient field of $Q(R)$ is obtained as a localization of R and again a chain domain (Proposition 5.3). This is not true for right chain domains (Example 5.7). For arbitrary right chain rings we describe the left (right) zero-divisors of the localized ring in terms of the left (right) associated prime ideal of the kernel which itself is zero or the annihilator of a completely prime ideal (Proposition 5.6).

5.1 Quotient rings of right chain domains

If R is a right chain domain, R is a right Ore ring and the right quotient ring $Q(R)$ exists. If, in addition, R is a chain domain, each element of $Q(R)$ has the form a or a^{-1} with $a \in R$. Conversely, if the quotient ring of a domain R has this property, then R is a chain domain. This was noticed by Brameret [63]. Radó [70] calls a subring R of a skew field D *total* if $x \in D \backslash R$ implies $x^{-1} \in R$ (see also Cohn [89], p. 3).

We summarize these observations:

PROPOSITION 5.1 (i) *A right chain domain R is a right Ore domain and its skew field of quotients exists.*

(ii) *If a right chain domain R is left Ore, then R is a chain domain.*

(iii) *R is a chain domain if and only if R is a total subring of its skew field D of quotients.*

Statement (ii) was already mentioned in Lemma 1.2(iv).

We consider the localization of R at a completely prime ideal.

LEMMA 5.2 *Let R be a right chain domain and $P = R\backslash S$ a completely prime ideal. Then the localization R_S exists and is again a right chain domain.*

PROOF: S is multiplicatively closed since P is completely prime and $r = sr_1$ or $rs_1 = s$ for $s_1 \in S, r_1 \in R$ if $r \in R, s \in S$ is given; $R_S = \{rs^{-1} \mid r \in R, s \in S\}$. ∎

Each localization R_S is an overring of R in $Q(R)$. The next result shows that for a chain domain R, all overrings of R in $Q(R)$ are obtained by localization. However, this does not remain true for right chain domains (see Section 5.3 for an example).

PROPOSITION 5.3 *Let R be a chain domain, $K = Q(R)$ its skew field of quotients. Then there is a one-to-one correspondence between the set of rings T between R and K and the set of completely prime ideals of R given by $P \to R_S$ where $S = R\backslash P$ is an Ore system, $T \to P = R\backslash S$ with $S = \{s \in R \mid s \in U(T)\}$. R_S is the ring of quotients of R with respect to the Ore-system S and again a chain domain.*

PROOF: Let T be an overring of R and let $P = R\backslash S$ be defined as in the proposition. Then, $p \in P, r \in R$ implies $pr, rp \in P$, $s_1, s_2 \in S$ implies $s_1 s_2 \in S$, $U(R) \subseteq S$ and P is a completely prime ideal of R with $R_S = T$. ∎

5.2 Localization of right chain rings

Let S be a multiplicatively closed subset of a right chain ring R that contains the set $U(R)$ of units of R and is saturated, i.e. $s_1s_2 \in S$ implies $s_1, s_2 \in S$. Then $P = R\backslash S$ is a completely prime ideal of R by Lemma 1.5 and 1.8. Conversely, if P is a completely prime ideal in the ring R, then $S = R\backslash P$ is a multiplicatively closed set that is saturatetd and contains $U(R)$.

For the right chain ring R and the completely prime ideal $P = R\backslash S$ consider the set

$$I = \{a \in R \mid \exists s, s' \in S : sas' = 0\}.$$

We recall the following definitions

$$N_l(S) = \{a \in R \mid as = 0 \text{ for some } s \in S\}$$

and

$$N_r(S) = \{a \in R \mid sa = 0 \text{ for some } s \in S\}.$$

Both sets are two-sided ideals (see Section 2.1). The next result shows that I is a two-sided ideal with the property that as or $sa \in I, s \in S, a \in R$ implies $a \in I$.

LEMMA 5.4 *Let R be a right chain ring and $R\backslash S = P$ a completely prime ideal. Then the following is valid:*

(i) *The set $I = \{a \in R \mid \exists s, s' \in S : sas' = 0\} = N_l(S) \cup N_r(S)$ is a two-sided ideal.*

(ii) $N_l(S) \cup N_r(S) = \{a \mid \exists s, s' \in S : sas' \in I\}$.

(iii) $P_l(I), P_r(I) \subseteq P$.

PROOF: (i) $N_l(S) \cup N_r(S) \subseteq \{a \mid \exists s, s' \in S : sas' = 0\}$. The converse inclusion also holds: As $N_l(S), N_r(S)$ are two-sided ideals, we have to consider two cases: $N_l(S) \subseteq N_r(S)$ or $N_r(S) \subseteq N_l(S)$. Let $N_l(S) \subseteq N_r(S)$ and $sas' = 0$. We have $sa \in N_l(S)$ which implies $sa \in N_r(S)$, hence $a \in N_r(S)$. The case $N_r(S) \subseteq N_l(S)$ is treated similarly and $I = N_l(S) \cup N_r(S)$.

(ii) Obviously we have $\{a \mid \exists s, s' \in S : sas' = 0\} \subseteq \{a \mid \exists s, s' \in S : sas' \in I\}$. Now take any $a \in R$ with $sas' \in I$ for suitable $s, s' \in S$ and by (i) we obtain $tsas't' = 0$ for some $t, t' \in S$.

(iii) Assume that for some $t \in S$ the element ta lies in I, thus $stas' \in I$ with $s, s' \in S$ which shows $a \in I$. The other inclusion can be checked with the same arguments. ■

The set $\overline{S}$ of images in $\overline{R} = R/I$ of elements in S is a right Ore set of $\overline{S}$ is multiplicatively closed. In addition, $\overline{S}$ consists of non-zero divisors of $\overline{R}$ by Lemma 5.4(ii) and the ring $\overline{R}[\overline{S}^{-1}] = \{\overline{rs}^{-1} \mid \overline{r} \in R, \overline{s} \in \overline{S}\}$ exists and is denoted by R_S or $R[S^{-1}]$. We summarize these observations:

PROPOSITION 5.5 *Let R be a right chain ring and $R\backslash S = P$ a completely prime ideal. Let $I = N_l(S) \cup N_r(S) = \{a \in R \mid \exists s \in S : sa = 0$ or $as = 0\}$. Then the following holds:*

(i) *I is a two-sided ideal of R and no element of $\overline{S} = S/I$ is a zero-divisor in $R/I = \overline{R}$.*

(ii) *$\overline{R}$ is a right chain ring and $\overline{S}$ a right Ore system.*

(iii) *$\overline{R}[\overline{S}^{-1}] = \{\overline{rs}^{-1} \mid \overline{r} \in \overline{R}, \overline{s} \in \overline{S}\}$ is a right chain ring with $P[S^{-1}]$ as its maximal ideal.*

We will usually denote the ring $\overline{R}[\overline{S}^{-1}]$ by $R[S^{-1}]$ and its maximal ideal by $PR[S^{-1}]$ and we will use the following terminology:

For a right ideal A of R we denote by $A^e = A[S^{-1}] = AR[S^{-1}]$ the *extended* right ideal and by $B^c = \phi^{-1}(B \cap \overline{R})$ the *contraction* of a right ideal B of $R[S^{-1}]$ in R where ϕ is the canonical mapping from R to $\overline{R} \subseteq \overline{R}[\overline{S}^{-1}]$.

We are interested in results which describe the left and right zero-divisors in $R[S^{-1}]$.

PROPOSITION 5.6 *Let R be a right chain ring and $P = R\backslash S$ a completely prime ideal. Let $I = N_l(S) \cup N_r(S)$. Then we have:*

(i) *$N_l(R[S^{-1}]) = P_l(I)[S^{-1}]$ and $N_r(R[S^{-1}]) = P_r(I)[S^{-1}]$.*

(ii) *If R is, in addition a chain ring and $I \neq (0)$, then we have: $N_r(R[S^{-1}]) = P[S^{-1}]$ or $N_l(R[S^{-1}]) = P[S^{-1}]$. Hence, each element of $R[S^{-1}]$ which is not a unit is either a left or right zero-divisor.*

PROOF: (i) If $\overline{a} \in \overline{R}$ is a left zero-divisor in $\overline{R}$, hence $\overline{a}\overline{b} \neq \overline{0}$, then so is $\overline{as}^{-1}$ for any $\overline{s} \in \overline{S}$. The opposite implication is also true: Let $\overline{as}^{-1}$ be a left zero-divisor in $\overline{R}[\overline{S}^{-1}$, hence $\overline{as}^{-1}\overline{bt}^{-1} = 0$ for some $\overline{0} \neq \overline{bt}^{-1} \in \overline{R}[\overline{S}^{-1}]$ and $\overline{as}^{-1}\overline{b} = 0$ follows. Finally we have $\overline{s}^{-1}\overline{b} = \overline{b}_1\overline{s}_1^{-1}$, so $\overline{a}\overline{b}_1 = (0)$. The rest follows by Lemma 4.5, since $N_(R/I) = P_l(I)/I$ and $N_r(R/I)/I$.

(ii) It suffices to prove $P_(I) = P$ or $P_r(I) = P$ provided R is a chain ring. Without loss of generality we assume $I = N_r(S) \neq (0)$. Using the symmetric version of Corollary 2.7 we obtain $P^r = N_r(S) = I$. Lemma 4.14 then implies $P_l(P^r) = P$. ∎

5.3 Examples

The next example shows that overrings of right chain rings are not necessarily right chain rings again.

EXAMPLE 5.7 *Take the ring R as constructed in Example 3.8. The ring R is a right invariant right chain domain which is not a left chain domain. In $L = Q(R)$ we have the ring $R_{-1} = yRy^{-1}$ which can be described as $A_{-1}[[y, \sigma]]$ where $A_{-1} = k(x, t_1, \ldots)[u]_{(u)}$ with $u = yxy^{-1}$ an indeterminate over $k(x, t_1, \ldots)$. The ring $T = \{a \in R_{-1} \mid a(0) \in A\}$ is an overring of R, but not a right chain ring, for example y and yu are not comparable in T since $u \in A_{-1}\backslash A$.*

Whereas in the chain ring case at least one of the left or right zero-divisor sets in a localized ring equals the maximal ideal (provided the kernel $\neq (0)$) the following example shows that localization in a right chain ring which consists of elements which are either units or left-/right zero-divisors may lead to a domain which is not a division ring.

EXAMPLE 5.8 *Let R be a right noetherian right chain ring as described in Chapter 3, in particular let R possess the prime ideals $J = xR \supset P = yR \supset Q = zR$ satisfying $zx = 0$. Set $R \backslash P = S$. Then $N_l(S) = Q$, $N_r(S) = (0)$, and hence $I = N_l(S) \cup N_r(S) = Q$. Thus $P_l(Q) = P_r(Q) = Q$ and $R[S^{-1}] = R/Q[(S/Q)^{-1}]$.*

6 Prime ideals in right chain rings

Central for the investigation of prime ideals in right chain rings is the open question whether exceptional prime ideals, i.e. prime ideals which are not completely prime, can exist in such rings. In Theorem 6.2 we describe the idealtheoretical prerequisites in right chain rings for the existence of prime ideals that are exceptional and list more information about such a situation (Proposition 6.7). In particular, an exceptional prime Q is always paired with a completely prime ideal $P = P^2$ as its upper neighbour such that $[P, Q[$ is simple. These results lead to a division of right chain domains with exactly two completely prime ideals into three specific classes (Corollary 6.3). Finitely generated prime ideals $P \neq (0)$ in a chain ring R are either J or the maximal prime ideal below J in which case P is an exceptional prime. Simple segments are investigated further and an example of a right chain domain is given where $R \supset J \supset (0)$ are the only two-sided ideals, i.e. R is almost simple.

6.1 Exceptional prime ideals - a first characterization

We consider two questions concerning prime ideals in right chain rings R.

(i) *Do there exist prime ideals P, Q in R with $[P, Q[$ simple?*

Dubrovin [80] presented the first example of such a ring which will be referred in Section 6.5. Further examples were given by Mathiak [81] and Brungs/Törner [84b]. These rings are nearly simple in the following sense.

DEFINITION 6.1 *A local ring R is called* nearly simple *if J and (0) are the only two-sided ideals of R.*

A second question seems to be more challenging:

(ii) *Does there exist a right chain ring R with a prime ideal which is not completely prime?*

Posner [63] hinted that such ideals might exist in right chain rings without giving further evidence. A classification of hypercyclic rings by Osofsky [68] is complete only if exceptional prime ideals do not exist in chain rings. V.K. Goel and S. K. Jain encountered this existence problem in [78] and it was mentioned explicitely in [78] and [84] by Jain. A geometric version of this problem was discovered by Törner in [74] and it was formulated along with (i) as a ringtheoretical problem in [76] by Brungs and Törner. Dubrovin [83] announced an example of a chain ring with an exceptional prime ideal, but there is a gap in the proof; see Schröder [90].

We will describe conditions that are necessary for the existence of such exceptional prime ideals.

THEOREM 6.2 *Let R be a right chain ring, Q an exceptional prime ideal and P the intersection of all completely prime ideals containing Q. Then $P = P^2$ and there are no two-sided ideals properly between P and Q, that is, the segment $[P, Q[$ is simple. Moreover, $Q \neq (0)$ implies $Q^2 \neq Q$ and Q is nilpotent or $\bigcap_{n \in \mathbb{N}} Q^n$ is completely prime.*

PROOF: Let I be a two-sided ideal with $Q \subset I \subseteq P$. As Q is prime, $Q \subset I^n$ for all n, hence $Q \subseteq \bigcap_{n\in\mathbb{N}} I^n$. But $\bigcap_{n\in\mathbb{N}} I^n$ is a completely prime ideal by Theorem 1.15(ii), hence $P = \bigcap_{n\in\mathbb{N}} I^n \subseteq I^2 \subseteq I \subseteq P$. This implies $I = P$ and $P = P^2$. The last assertion follows from Theorem 1.15. ■

Using Theorem 6.2 we are able to give an idealtheoretical characterization of right chain domains of rank 1.

COROLLARY 6.3 *If R is a right chain domain with exactly two completely prime ideals J and (0), then one of the following cases occurs:*

(i) *R is a right invariant right chain domain.*

(ii) *R is nearly simple.*

(iii) *R has an idempotent maximal ideal and possesses an exceptional prime ideal Q with $\bigcap_{n\in\mathbb{N}} Q^n = (0)$.*

In the cases (ii) or (iii) the maximal ideal is idempotent.

PROOF: First we assume that all prime ideals are completely prime. If J is not idempotent, then $J = aR$ and $(aR)^n = a^n R$ is a chain of two-sided ideals intersecting in the zero ideal (Theorem 1.15(ii)). Hence for any $x \in R\backslash\{0\}$ there is some $n \in \mathbb{N}$ and a unit $u \in U$ with $x = a^n u$, and thus $xR = a^n R$ is a two-sided ideal, i.e. R is right invariant.

Now assume that J is idempotent. If there is no further two-sided ideal $\neq (0), J$, we are in Case (ii). Hence assume the existence of a two-sided ideal $I \neq J, (0)$. If $I = Q$ is prime, it must be an exceptional prime and by Theorem 6.2 assertion (iii) follows.

In the remaining case we may assume that there exists a two-sided ideal I which is not prime, hence $[P, (0)[$ is not simple. Corollary 1.22 can be applied showing $ux \in xR$ for all $x \in P\backslash(0)$, $u \in U$ and by the second statement in Corollary 1.22 $Ux \subseteq xU$ follows leading to $Rx \subseteq xR$ (use Lemma 1.5). ■

6.2 Arithmetic in simple prime segments

In Theorem 1.21 we proved several results for non-simple prime segments. We will show by an example that simple prime segments do exist. We will obtain several results for the case when the prime segment $[P, Q[$ with Q an exceptional prime is simple (Theorem 6.2) and additional conditions must be satisfied for this situation to occur.

We begin with a technical result.

PROPOSITION 6.4 *Let R be a right chain ring, $A \subseteq B$ two-sided ideals of R with no two-sided ideal X between A and B. Let $x \in B\backslash A$ with $x^2 \notin A$ and $xR \neq B$. Then there exists a unit u with $A \subset xR \subseteq \bigcap_{n\in\mathbb{N}}(ux)^n R$, in particular $A \subset \bigcap_{n\in\mathbb{N}}(xu)^n R$.*

PROOF: Let $x \in B\backslash A$. By Lemma 1.5 there exists $u \in U$ with $x = uxw, w \in J$. We consider the following cases:

Case 1: $w = xq$ with $q \in R$; then $x = uxw = ux^2q = ux(ux^2q)q = (ux)^2xq^2 = (ux)^nxq^n$ for all $n \in \mathbb{N}$. Thus $x \in \bigcap_{n\in\mathbb{N}}(ux)^nR$. Hence in this case $A \subset xR \subseteq \bigcap_{n\in\mathbb{N}}(ux)^nR$ and in particular, as A is a two-sided ideal $A \subset \bigcap_{n\in\mathbb{N}}(xu)^nR$ follows.

Case 2: $x = wq$ with $q \in J$; then $ux^2 = uxwq = xq \in xR$. Note further that for a unit $v \in U$ with $vx \in xR$ also $vx^2 \in xR$ follows. Hence, either there exists a $u \in U$ for which the condition of Case 1 is satisfied or we have $ux^2 \in xR$ for any $u \in U$, hence $Rx^2R \subseteq xR$. By assumption $x^2 \notin A$ and $xR \neq B$, hence $A \subset Rx^2R \subseteq xR \neq B$ contradicting the the assumption that there is no two-sided ideal between A and B. Thus, for any $x \in B\backslash A$ there exists a unit u such that the condition in Case 1 is satisfied and the proposition is proved. ∎

COROLLARY 6.5 *Let R be a right chain ring, P, Q neighbouring prime ideals with $[P, Q[$ simple. Then for any $x \in P\backslash Q$ there exist units $u, v \in U$ with $Q \subset xR \subseteq \bigcap_{n\in\mathbb{N}}(uxv)^nR$ and $Q \subset \bigcap_{n\in\mathbb{N}}(xvu)^nR$. If Q is completely prime we can choose $v = 1$.*

PROOF: Note that $P^2 = P$, since otherwise $[P, Q[$ is not simple. This implies that $xR \subset P$ for any $x \in P$. If Q is completely prime, then clearly $x^2 \notin Q$ for any $x \in P\backslash Q$. If Q is an exceptional prime ideal, then for any $x \in P\backslash Q$ there exists a unit v with $(xv)^2 \notin Q$, since otherwise $xRx \subseteq Q$ (by Lemma 1.4), contradicting the fact that Q is prime. Now apply Proposition 6.4 to xv. ∎

The *nil radical* $Nil(R)$ of a ring R is defined as the sum of all nil ideals (see for example Wisbauer [88]). With the last corollary we obtain the following result:

COROLLARY 6.6 *Let R be a right chain ring. Then the prime radical Rad(R) equals the nil radical. Hence, Rad(R) $= \sqrt{(0)}$.*

PROOF: The prime radical is nil, hence contained in the nil radical. We are done if the prime radical is completely prime and can therefore assume that the prime radical equals the exceptional prime ideal Q with P completely prime as its upper neighbour in the lattice of prime ideals and $[P, Q[$ simple. However, Corollary 6.5 shows that in such a case the prime ideal P is never nil, so the nil radical equals the prime radical in both cases. ∎

It follows from Corollary 6.5 that for any $x \in P\backslash Q$ with $[P, Q[$ simple, there exists a unit $w \in U$ with $(xw)^n \notin Q$ for any n. The next result shows that for Q exceptional there also exists a unit v with $(xv)^2 \in Q$. We say that $x \in R$ is *Q-nilpotent* if $x^n \in Q$ for some $n \in \mathbb{N}$.

PROPOSITION 6.7 *Let R be a right chain ring, $P \supset Q$ neighbouring prime ideals with Q exceptional prime.*

(i) *Let $x \in P\backslash Q$ be not Q-nilpotent, then $Q \subset \bigcap_{n\in\mathbb{N}} x^nR$.*

(ii) *For each $x \in P \backslash Q$ there exists at least one unit v with $(xv)^2 \in Q$.*

(iii) *Let $x \in P \backslash Q$ be Q-nilpotent. If Q is nilpotent, then $\bigcap_{n \in \mathbb{N}} x^n R = (0)$. If Q is not nilpotent, then $\bigcap_{n \in \mathbb{N}} x^n R = \bigcap_{n \in \mathbb{N}} Q^n$ is a completely prime ideal.*

(iv) *For any $a \in Q$ there are $x, y \in P \backslash Q$ with $aR \subseteq xyR \subseteq Q$.*

(v) *If $Q^2 \neq 0$, then $xy \notin Q^2$ for any $x, y \in P \backslash Q$.*

PROOF: (i) As x is not Q-nilpotent, $Q \subseteq \bigcap_{n \in \mathbb{N}} x^n R$. If $Q = \bigcap_{n \in \mathbb{N}} x^n R$, then Theorem 1.15(iii) would imply Q completely prime contradicting our assumption on Q.

(ii) Let x be in $P \backslash Q$. Obviously $\bigcap_{u \in U} uxR$ is a two-sided ideal containing Q, hence by Theorem 6.2 we have $Q = \bigcap_{u \in U} uxR$. If we assume that xUx is contained in $P \backslash Q$ we will show that Q is completely prime, a contradiction to our assumption. By Lemma 1.8 it suffices to prove that $y \notin Q$ implies $y^2 \notin Q$ for any $y \in P \backslash Q$. As $Q = \bigcap_{u \in U} uxR$ we have $ys = ux$ for some $s \in J, u \in U$. By Lemma 1.4 we find $s_1 \in U$ such that $s_1 s y = y s'$ for some $s' \in R$ and $s_1 s = s s_1$. From $ux \in P \backslash Q, s_1 \in U$ we conclude $uxs_1ux = yss_1ys = y^2 s' s$ which lies again in $P \backslash Q$, since $xs, ux \in xUx \subseteq P \backslash Q$ by assumption. Hence y^2 is not in Q, a contradiction that shows that xvx is in Q for some $v \in U$ and proves (i).

(iii) If x is Q-nilpotent then $\bigcap_{n \in \mathbb{N}} x^n R \subseteq \bigcap_{n \in \mathbb{N}} Q^n$ and Q nil implies $\bigcap_{n \in \mathbb{N}} x^n R = (0)$. In the other case $P_1 = \bigcap_{n \in \mathbb{N}} Q^n$ is completely prime by Theorem 6.2 and $\bigcap_{n \in \mathbb{N}} x^n R = P_1$ by Theorem 1.21(v).

(iv) By (ii) there exists $x \in P \backslash Q$ with $x^2 \in Q$. Let $a \in Q$ and $a = xy$ follows for some $y \in R$. If we assume $y \notin P$, we obtain $xP \subseteq aR$ which contradicts the fact that Q is prime, hence $y \in P$ and we are done if $y \notin Q$. Otherwise, $y \in Q$, we have $y = xs$ for some $s \in J$ and we obtain $aR = xyR = x^2 sR \subseteq x^2 R$, so $y = x$ satisfies the assertion (iv).

(v) Assume $xy = ab$ with $a, b \in Q$. Then $a = xr$ for some $r \in R$, and $xy = xrb$. Since $y \notin Q$, $b \in Q$ we have $y - rb \notin Q$. Hence $Q^2 \subseteq xQ \subseteq x(y - rb)R = (0)$, a contradiction. ■

6.3 More on prime ideals in right chain rings: finitely generated prime ideals

In this section, Q will always denote an exceptional prime ideal in the right chain ring R and P the minimal completely prime ideal containing Q.

We will show that $P_r(Q) = P$ and that for a chain ring $PQ = QP$. The result of Lemma 1.16 will be extended: A finitely generated prime ideal $\neq (0)$ in a chain ring is either equal to J or it is exceptional in which case it is the prime ideal just below J.

PROPOSITION 6.8 *Let R be a right chain ring. Then we have*

(i) $P_r(Q) = P$.

(ii) $PQ \subseteq QP$

PROOF: (i) If $s \notin P$ and $ts \in Q$, then $tP \subseteq tsR \subseteq Q$. Now Q is prime and $P \not\subseteq Q$, hence $t \in Q$. On the other hand $P \subseteq P_r(Q)$ because $P_r(Q)$ is completely prime.

(ii) Set $I = \{x \in P \mid xQ \subseteq QP\}$. As Q is a two-sided ideal, so is I. Obviously $Q \subseteq I \subseteq P$. As $[P, Q[$ is simple (by Theorem 6.2) we must have $I = Q$ or $I = P$. By Proposition 6.7(ii) there exists $x \in P \backslash Q$ with $x^2 \in Q$. For $q \in Q$, there exists $r \in R$ with $q = xr$ and by (i) it follows that $r \in P$ and $Q \subseteq xP$. Therefore $xQ \subseteq x^2 P \subseteq QP$. Thus $I = P$. ∎

In the case of a chain ring we obtain by symmetric arguments:

COROLLARY 6.9 *Let R be a chain ring. Then we have:*

(i) $P_l(Q) = P = P_r(Q)$.

(ii) $PQ = QP$.

We consider right principal prime ideals.

THEOREM 6.10 *Let R be a chain ring, $I \neq 0$ a prime ideal of R. If I is finitely generated as a right ideal, we have one of the following situations:*

(i) *$I = J$ and $I = Ra = aR$ for some $a \in R$.*

(ii) *I is the maximal prime ideal below J, further I is exceptional prime and $I = Ra = aR$ for some $a \in R$.*

PROOF: By Lemma 1.16 we are left with the case that $I = aR$ is an exceptional prime ideal. Now let P be the minimal prime ideal above I; this is completely prime by Theorem 6.2. Assume $P \neq J$. Let $x \in J \backslash P$. Then $a = rx$ for some $r \in R$. By Proposition 6.8 (i), $r \in I$, say $r = as$. Hence $a = asx$ and $sx \in J$ which implies $a = 0$ - a contradiction. Thus $P = J$. It remains to show that $I = Ra$. Clearly, $Ra \subseteq aR$. If $ar \notin Ra$ for some $r \in R$, then $a \in Jar$ and hence by Proposition 6.8(ii) $a \in Jar \subseteq aJ$, a contradiction. ∎

This theorem is obviously not true for right chain rings as the examples in Section 3 show.

6.4 Investigation of $Q/\bigcap Q^n$

Our goal is to obtain informations on the lattice of two-sided right ideals in the prime segment $[Q, \bigcap Q^n[$, where Q is an exceptional prime ideal. It is obvious that one should assume $P = J$ where P the minimal completely prime ideal containing Q, otherwise any two-sided ideals lying above P induce two-sided ideals in the segment under discussion. Finally we further assume $\bigcap Q^n$ to be completely prime and $= (0)$.

PROPOSITION 6.11 *Let R be a chain ring, Q an exceptional prime ideal with J the minimal completely prime ideal containing Q. Assume further $(0) \subset Q^{n-1}$. Then we have:*

(i) *Let I be a two-sided ideal with $Q^n \subset I \subset Q^{n-1}$ or $(0) \subset I \subset Q^{n-1}$ with maximal n. Then* (i1) $I = Q^{n-1}J$ *or* (i2) $I = Rs = sR$ *for some $s \in R$ with $Js = sJ = Q^n$ or $Js = sJ = (0)$.*

(ii) *There is an abundance of non-two-sided right ideals in the lattice interval* $(0) \neq Q^k \subset Q^{k-1}$.

PROOF: First note that $J^2 = J$ by Theorem 6.2. We can assume that $Q^n = (0)$, otherwise rename R/Q^n as R. Suppose $I \neq Q^{n-1}J$. Now the right annihilator I^r of I is a two-sided ideal with $Q \subseteq I^r \subseteq J$. By Theorem 6.2 we get $I^r = Q$ (Case 1) or $I^r = J$ (Case 2).

Case 1: $I^r = Q$. Let $a \in Q^{n-1}J \backslash I$. As $I \subset Ra \subset Q^{n-1}$ we have $Q \subseteq (Q^{n-1})^r \subseteq (Ra)^r \subseteq I^r = Q$, hence $Q = (Ra)^r = I^r$. Now let $x \in J \backslash Q$. By Proposition 6.6 (ii) there exists $u \in U$ with $(xu)^2 \in Q$. If $ax \notin I$, then also $axu \notin I$ and by the above $Q = (Raxu)^r$. Thus $axuxu = 0$ since $(xu)^2 \in Q = (Ra)^r$, but $xu \notin Q$. Contradiction. Hence $ax \in I$ for any $x \in J \backslash Q$ and so $aJ \subseteq I \subset aR$ which implies $I = aJ$. By the choice of the element a we get $Q^{n-1}J = aR$ leading to $Q^{n-1}J \cdot J = aJ = Q^{n-1}J = aR$, a contradiction.

Case 2: $I^r = J$. Suppose I is not finitely generated as right ideal. Let $0 \neq r \in I$, then there exists $s \in I$ with $rR \subset sR$, hence $r = st$ with $t \in J$ and thus $r = 0$, contradiction. Therefore I is finitely generated as right ideal, say $I = sR$, and, of course, $sJ = (0)$. In particular, since I is assumed to be two-sided, $Rs \subseteq sR$ follows and by Lemma 7.12 (can be checked directly, see the proof) we have $sJ \subseteq Js$. Obviously $Us \subseteq sU$ is valid. Next we show $Js \subseteq sJ$ which implies $sR \subseteq Rs$ (use again Lemma 7.12). Let $x \in J \backslash Q$. Then there exists a unit $w \in U$ with $(xw)^2 \in Q$. If $xs = sy$ with $y \in U$, then $xwxws \in sU$, contradicting the fact that $Qs = (0)$. Thus $Js \subseteq sJ$ showing at least $Rs = sR$.

(ii) By (i) in the best possible case there are the following two-sided ideals in the lattice interval $Q^k \prec sR = Rs \subseteq Q^{k-1}J \prec Q^{k-1}$. Note that sR can never equal $Q^{k-1}J$, since $J = J^2$ and $sJ = Q^{k-1}J^2 = Q^{k-1}J = sR$ would follow. The same arguments show that $Q^{k-1}J$ has no lower neighbour as a right ideal. Hence by (i), the interval $[Q^{k-1}J, sR[$ must contain an abundance of right ideals which are not two-sided. ∎

6.5 An example of a nearly simple chain domain

The following example originates in a idea of Dubrovin [80].

Let G be the group of affine linear functions on the rational number field $\mathbb{Q}$, i.e.

$$G = \{\alpha : t \to at + b \mid a, b \in \mathbb{Q},\ a > 0\}$$

Denoting an element $\alpha \in G$ by the pair (a, b) we obtain the multiplication rule

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1).$$

G is a semi-direct product of the additive group $(\mathbf{Q}, +)$ and the multiplicative group $(\mathbf{Q}_{>0}, \cdot)$. We define a right order on G which is not an order.

A group G with an order relation $\leq_r$ is said to be a *right ordered* group provided $\leq_r$ is a linear order relation satisfying the right monotony law, i.e.

$$a \leq_r b \Longrightarrow ac \leq_r bc \text{ for all } c \in G.$$

A right order on G can also be defined as follows: The set $P \subseteq G$ is called a *generalized positive cone* provided P satisfies the following conditions: (i) $PP \subseteq P$, (ii) $P \cap P^{-1} = \{1\}$ and (iii) $P \cup P^{-1} = G$. Then we set

$$a \leq_r b \Longleftrightarrow ba^{-1} \in P$$

Each right order on G corresponds to a left order by

$$a \leq_l b \Longleftrightarrow a^{-1}b \in P \Longleftrightarrow b^{-1} \leq_r a^{-1}.$$

In the case where $\leq_r$ is a left and right order, $(G, \leq_r)$ is obviously an ordered group. Exactly in this case the generalized positive cone is a normal subsemigroup of G, i.e. $cPc^{-1} \subseteq P$ for all $c \in G$.

Let G be again the above mentioned group of affine linear functions and ε any irrational number in $\mathbf{R}$. It is easy to check that evaluation each function $\alpha \in G$ at ε defines a generalized positive cone P and hence a right (left) order can be defined on G. We set

$$\alpha \in P \Longleftrightarrow \varepsilon \leq \alpha(\varepsilon).$$

The corresponding left order can be described geometrically. Let $\alpha_1 = (a_1, b_1), \alpha_2 = (a_2, b_2)$ and $\alpha_1 \leq_l \alpha_2$ or equivalently $\alpha_1^{-1}\alpha_2 = (a_1^{-1}a_2, a_1^{-1}b_2 - a_1^{-1}b_1) \in P$ leading to

$$\varepsilon \leq a_1^{-1}a_2\varepsilon + a_1^{-1}b_2 - a_1^{-1}b_1$$

finally showing

$$\alpha_1 \leq_l \alpha_2 \Longleftrightarrow a_1\varepsilon + b_1 \leq a_2\varepsilon + b_2.$$

Thus the intersection of the graphs representing the group elements with the line through $(\varepsilon, 0)$ parallel to the second axis describe the right linear order.

Next we want to construct a chain ring associated with G as described above.

G is the semidirect product of its subgroups

$$H = \{(1, q) \mid q \in \mathbf{Q}\} \cong (\mathbf{Q}, +)$$

and the subgroup

$$L = \{(k, 0) \mid 0 < k \in \mathbf{Q}\} \cong (\mathbf{Q}_{>0}, \cdot).$$

In particular we have $(k, 0)(1, b) = (1, kb)(k, 0)$ where $\sigma_g : H \to H$ with $\sigma(1, b) = (1, kb)$ defines an automorphism of H for every $g = (k, 0) \in L$. We want to show that the group ring $K[G]$ is an Ore domain for every skew field K. If $a = \sum c_i h_i, b = \sum d_j h'_j$ are two elements in $K[H]$, then they are elements in $K[H_0]$ where H_0 is a finitely generated subgroup of H generated by the h_i's and h_j's and as such is a direct sum of finitely many infinite cyclic subgroups. Applying the fact that $R[x]$ is Ore if R is Ore shows that $K[H]$ is an Ore domain.

We have $A = K[G] \cong K[H][L, \sigma]$ where σ indicates that we now deal with a skew group ring with the commutation rule

$$(k, 0)a = a(k, 0) \text{ for } a \in K,$$

but

$$(k, 0)(1, b) = (1, kb)(k, 0) \text{ for } (1, b) \in H, (k, 0) \in L.$$

Again, an element $\sum a_i g_i$, $a_i \in K[H]$, $g_i \in L$, is contained in $K[H][L_0, \sigma_0]$ for a finitely generated subgroup $L_0 \subseteq L$ which is a finite product of infinite cyclic subgroups:

$$L_0 = \langle g_1 \rangle \times \ldots \times \langle g_m \rangle$$

and

$$K[H][L_0, \sigma_0] = K[H][\langle g_1 \rangle, \sigma_{g_1}] \ldots [\langle g_m \rangle, \sigma_{g_m}]$$

is again an Ore ring, since it is an iterated skew polynomial ring over an Ore ring. The automorphisms σ_{g_i} are extended from H to $K[H][L_0, \sigma_0] = K[H][\langle g_1 \rangle, \sigma_{g_1}] \ldots [\langle g_{i-1} \rangle, \sigma_{g_{i-1}}]$ where they map the elements of K as well as the $g_j, j < i$, to themselves. Hence, the group ring $K[G]$ will be a right and left Ore domain.

As mentioned earlier G admits a left order and a right order with P the generalized positive cone. From this we conclude that the set of elements of the semigroup ring $K[P]$ which do not lie in the maximal ideal $M = \sum_{g > e} gK[P]$ is a right and left Ore system. The localization of the ring $K[P]$ with respect to this Ore system will be denoted by R. Since the group G is linearly right ordered, it follows that the ring R is a chain domain. As each elements is by construction a product of an element of the positive cone P and a unit, each principal right ideal can be uniquely written as gR for some $g \in P$. Thus the inclusion in the lattice of right ideals inversely corresponds to the left order $\leq_l$ on G, e.g.

$$gR \supseteq hR \Longleftrightarrow g \leq_l h$$

Now we prove that R is a nearly simple ring. Considering the subgroup H of the group G we know that for every element $(a, b) \in P$ there exist elements $(1, x)$ and $(1, y)$ such that $(1, x) > (a, b) > (1, y) \geq (1, 0)$. Therefore it is sufficient to prove the equation $J(R)(1, x)J(R) = J(R)$ for any positive element $(1, x)$. This follows, in turn, from the fact that for any elements $(1, x) > (1, y) > (1, 0)$ there exist positive elements (a_1, b_1), (a_2, b_2) of the group G such that $(a_1, b_1)(1, x)(a_2, b_2) < (1, y)$. To see this, we set $a_1 = a_2^{-1}, b_i = (1 - a_i)/\varepsilon$, $i = 1, 2$, and find that

$$(a_1, b'_1)(1, x)(a_2, b'_2) = (1, a_1 x),$$

if we choose k_1 sufficiently small. It now remains only to take the rational numbers b_1, b_2 so close to b'_1, b'_2 that we satisfy the inequality $a_i \varepsilon + b_i > \varepsilon$, $i = 1, 2$ such that the last equation remains valid.

Our understanding of the left and right order in G enables us to describe the various possibilities of powers of an element x which are mentioned in Proposition 6.7.

Let $x \in R$ be the group element $x = \alpha = (a, b) \in P$, hence $\alpha^n : t \rightarrow a^n t + (a^{n-1} + \ldots + a + 1)b$.

Case 1: $0 < a < 1$. The evaluation of the limit function at ε reads

$$\lim_{n\rightarrow\infty}(a^n\varepsilon + (a^{n-1} + \ldots + a + 1)b) = 0 + \lim_{n\rightarrow\infty}(\sum_0^{n-1} a^i)b = \frac{1}{1-a}b = \beta.$$

Thus the constant function $t \rightarrow \beta$ as a ring element will be a lower bound in the lattice of right ideals for all powers $\alpha^n R$, even more is valid: $\bigcap_{n\in\mathbb{N}} \alpha^n R = \beta R$.

Case 2: $1 < a$. Note $\alpha \in P$ implies $\alpha(\varepsilon) = a\varepsilon + b > \varepsilon$, set $\delta = \alpha(\varepsilon) - \varepsilon$. By induction $\alpha^n(\varepsilon) - \varepsilon = (a^{n-1} + a^{n-2} + \cdots + 1)\delta$, hence

$$\lim_{n\rightarrow\infty}\alpha^n(\varepsilon) = \infty.$$

This means that for any $0 \neq c \in R$ there is a power of α such that $cR \supset \alpha^n R$, hence $\bigcap_{n\in\mathbb{N}} \alpha^n R = (0)$.

7 Degrees of noncommutativity in right chain rings

In this chapter we develop useful notions concerning various degrees of noncommutativity. An important class is formed by the so-called locally archimedean right chain rings whose prime segments are not simple. Equivalent characterizations are given in Theorem 7.1 and 7.6. The class is closed unter localizations (Theorem 7.18), which is not true for right invariant right chain rings. However, this result holds for semiinvariant right chain rings which include the class of right invariant rings. Moreover, in the domain case, each right semiinvariant chain ring is a localization of a right invariant one (Theorem 7.21).An example of a noninvariant, but semiinvariant ring as well as of a locally archimedean ring is given.

7.1 Locally archimedean right chain rings

Since a detailed treatment of general right chain rings does not seem possible at present, we investigate various classes of right chain rings defined through conditions that restrict the degree of noncommutativity in some way. For example, noncommutativity can be measured by the distribution of two-sided ideals within the lattice of right ideals $\mathcal{L}_r(R)$.

Gräter [84a] observed that the valuation rings R of a division algebra finite dimensional over its center are *locally invariant* which can be defined by the condition that between any two prime ideals of R there is a further (two-sided) ideal of R. We consider the corresponding condition for right chain rings.

THEOREM 7.1 *Let R be right chain ring. Then the following properties are equivalent:*

(a) *The right ideal $\bigcap_{n\in\mathbb{N}} a^n R$ is two-sided for any $a \in R$.*

(b) *For any $a \in R$ we have $Ra^2 \subseteq aR$.*

(c) *For any $a \in R$ there exists $n = n(a) \in \mathbb{N}$ such that $Ra^n \subseteq aR$.*

(d) *Any prime segment, except possibly the radical prime segment contains a two-sided ideal.*

We recall from Section 1.5 that the radical prime segment $[P, (0)[$ is never simple provided $P^2 \neq (0)$.

PROOF: (a) $\Rightarrow$ (b) Let $a \in R$, $a \neq 0$. If $Ra^2 \not\subseteq aR$ then $a = ua^2s$ for some $u \in U$, $s \in J$. Thus $a = (ua)^n as^n$ for all $n \in \mathbb{N}$, hence $a \in \bigcap_{n\in\mathbb{N}}(ua)^n R$. By assumption, the intersection is a two-sided ideal, so $ua \in \bigcap_{n\in\mathbb{N}}(ua)^n R$ which implies $ua = 0 = a$ - a contradiction.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d) Let $Q \subset P$ be prime ideals and suppose $[P, Q[$ is simple. Then by Corollary 6.5 we can find $x \in P \backslash Q$ and $u \in U$ such that $Q \subset xR \subseteq \bigcap_{n\in\mathbb{N}}(ux)^n R$. Choose $n \in \mathbb{N}$ with $R(ux)^n \subseteq= uxR$. Hence we obtain $RxR \subseteq R(ux)^n R \subseteq uxR$ which implies $RxR = xR$. Since $x \notin Q$, RxR is a two-sided ideal satisfying $Q \subset RxR = xR \subset P$, a contradiction.

(d) $\Rightarrow$ (a) Let $a \in R$. Clearly, we can assume that $a \in J$ and a is not nilpotent. Then there exists a prime ideal segment $[P, Q[$ with $Q \subset aR \subseteq P$ which is not simple by assumption. Theorem 1.21(v) proves the assertion. ■

DEFINITION 7.2 *A right chain ring R that satisfies the equivalent conditions of Theorem 7.1 is called* locally archimedean.

Note that this is in fact only a condition on the non-units which are not nilpotent. In the following result further properties of locally archimedean right chain rings are listed.

COROLLARY 7.3 *Let R be a right chain ring. Then the following holds:*

(i) *If R is locally archimedean, then each prime ideal is completely prime.*

(ii) *If J is the only prime ideal of R, then R is locally archimedean, moreover R is right invariant.*

(iii) *Let R be locally archimedean. If there is a prime segment $[J, P[$ with $J \supset P$ as neighbouring prime ideals, then for all $a \in J\backslash P$ we have: $Ra \subseteq aR$.*

(iv) *Let $P = R\backslash S$ be the smallest prime ideal containg aR for some $a \in R$. Then aP as well $(aR)S^{-1}$ is two-sided.*

PROOF: (i) follows from Definition 7.2 and Theorem 6.2.

(ii) The conditions of Theorem 7.1 are trivially satisfied since they are void. Note further that J is nil. Note further that J is nil. Let $ra \in Ra \neq (0)$ and assume $ra \notin aR$, henc $ras = a$ for some $s \in J$ leading to $a = r^n as^n = 0$, a contradiction.

(iii) Assume $ra \notin aR$ for $a \in J\backslash P$ and $r \in R$. Then $a = raj \in RaJ$, contradicting Corollary 1.22.

(iv) If $aP \subset UaP$, then there exists $s \in S$ with $as \in UaP = UasP$, contradicting Corollary 1.22, as $x = as \in P\backslash Q$. Hence $UaP \subseteq aP$ and thus aP is a two-sided ideal. That $(aR)S^{-1}$ is two-sided follows from Corollary 1.12. ■

The next result shows that a chain domain R is locally archimedean if and only if R is locally invariant as defined by Gräter in [84a].

PROPOSITION 7.4 *Let R be a chain ring. Then the following conditions are equivalent:*

(a) *R is locally archimedean.*

(b) *For any $a \in R$ with $a^2 \neq 0$ the minimal prime ideal containing a satisfies $Pa = aP$.*

PROOF: (a) $\Rightarrow$ (b) Of course, we can assume $a \neq 0, a \in J$ and denote the prime segment generated by aR by $[P, Q[$. By Corollary 7.3(i) the prime ideal P is completely prime. By assumption (a) $[P, Q[$ is not simple if P is not the radical, otherwise Lemma 1.19 guarantees the existence of a further two-sided ideal since $a^2 \neq (0)$ implies $P^2 \neq (0)$. To prove $aP \subseteq Pa$ we take $ap \in aP$ and have to consider the cases $r_1 ap = a$ (Case 1) respectively $ap = r_2 a$ (Case 2). We apply Corollary 1.22 showing that Case 1 is impossible whereas in Case 2 the element r_2 must lie in P. By symmetric arguments $Pa \subseteq aP$ follows.

(b) $\Rightarrow$ (a) Let $a \in P \backslash Q$ for neighbouring prime ideals Then $Pa = aP$ is a two-sided ideal lying in $[P, Q[$, hence by Theorem 7.1(d) R is locally archimedean. ∎

DEFINITION 7.5 *Let R be a chain domain. If R satisfies one of the equivalent conditions of Proposition 7.4 the ring R is called* locally invariant.

Gräter's terminology *locally invariant* is justified by the fact that localization at an arbitrary prime ideal produces a chain ring which is invariant in its first prime segment (see Corollary 7.3 (iii)).

The next result characterizes locally archimedean right chain rings by a condition on commutator elements.

THEOREM 7.6 *Let R be a right chain ring, then the following conditions are equivalent:*

(a) *R is locally archimedean.*

(b) *For any prime segment $[P, Q[$ which is not the radical segment and $x, y \in P \backslash Q$ we have $xy = yxs$ or $yx = xys$ for $s \in R \backslash P$.*

PROOF: (a) $\Rightarrow$ (b) We will use two results which are proved later, but independently of Theorem 7.6:

($\star$) The localization R_P of a locally archimedean right chain ring at any prime ideal P is again locally archimedean (see Theorem 7.18).

($\star\star$) The semigroup of principal right ideals of a locally archimedean right chain domain with exactly one or two prime ideals is commutative.

The last theorem ($\star\star$) is an application of a result of Hölder (see Fuchs [66], p. 74) and will be proved in a later part.

Let $[P, Q[$ be a prime segment. Let $x, y \in P \backslash Q$ where $[P, Q[$ is a prime segment that is not the radical segment. We know that Q is completely prime since R is locally archimedean and $R_1 S_1^{-1}$ exists for $R_1 = R/Q$, $S_1 = (R/Q) \backslash (S/Q)$, $S = R \backslash P$ which is again locally archimedean by ($\star$). We can apply ($\star\star$) and obtain $x_1 y_1 R_1 S_1^{-1} = y_1 x_1 R_1 S_1^{-1}$, $y_1 x_1 = x_1 y_1 s_1$ or $y_1 x_1 t_1 = x_1 y_1$ for some $s,\ t \in S$ with x_1, y_1, s_1, t_1 denoting the images of x, y, s, t in R_1. If $x_1 y_1 s_1 = y_1 x_1$, say, then $xys = yx + q = yx + yxq' = yx(1 + q')$ for $q, q' \in Q, 1 + q' \in U(R)$ which proves (b).

(b) $\Rightarrow$ (a) We are done if we have proved $Ub^2 \subseteq bR$ using Theorem 7.1. Assume $ubbt = b$ with $t \in J$, $b \neq 0$ and hence $(ub)^n bt^n = b$ holds. If b lies in the prime radical, then ub is nilpotent which leads to the contradiction $b = 0$. Thus we may assume that there are neighbouring prime ideals $P \supset Q$ with $b \in P \backslash Q$ and hence $ub \in P \backslash Q$. We compare the products of ub and b.

Case 1: $ub \cdot b = b \cdot ub \cdot s$ for some $s \notin P$. Thus $b = ub^2 t = b \cdot ubs$, so $b = 0$, a contradiction.

Case 2: $ub \cdot b \cdot s = b \cdot ub$ for some $s \notin P$. Again two cases occur:

Case 2a: $sq = t$ for some $q \in R$. Hence $bubq = ubbsq = ubbt = b$, a contradiction.

Case 2b: $s = tq$ for some $q \in J$ and hence $q \notin P$. We calculate $bub = ubbs = ubbtq = bq$. By $ubR \subset qR$ we have $qv = ub$ for some $v \in J$, hence $bqv = bubv = bub$, so $bub = 0$ leading to $b = (ub)^2 bt^2 = u(bub)bt^2 = 0$, a contradiction. ■

We mention without proof (see also Thierrin [57]) the following related result which characterizes a larger class of right chain rings including locally archimedean right chain rings via commutator relations.

THEOREM 7.7 *Let R be a right chain ring, then the following conditions are equivalent:*

(a) *All prime ideals are completely prime.*

(b) *For all $x, y \in R$ the elements xy and yx are in the same prime ideal segment (i.e. there exist neighbouring prime ideals $P \supset Q$ with $xy, yx \in P \backslash Q$).*

7.2 Right-shifting of prime ideals in locally archimedean right chain rings

In Chapter 4 we had already discussed situations in *chain rings* where completely prime ideals P can be shifted over arbitrary elements a to obtain again completely prime ideals Q satisfying $Pa = aQ$. It does not seem successful to discuss such shiftings in right chain rings in general, however in locally archimedean right chain rings nice results can be obtained. Recall that by Proposition 7.4(b) $Pa = aP$ always holds in locally archimedean chain rings where P is the minimal completely prime ideal containing a. Soon we are able to generalize this result for arbitrary right chain rings by which we will obtain specializations of Corollary 1.22.

Let P be a completely prime ideal, $a \in R$ and $Pa \neq (0)$. Further assume $Pa \subseteq aJ$. This condition is always satisfied if Pa is a two-sided ideal, since otherwise $a \in Pa$, leading to $a = 0$. The converse conclusion holds in chain rings, as mentioned before Proposition 4.12.

By $P^\pi(a)$ we denote the minimal completely prime ideal Q with $Pa \subseteq aQ$, hence $Pa \subseteq aP^\pi(a)$. $I^\pi(P, a)$ stands for the smallest two-sided ideal L satisfying $Pa \subseteq aL$. If there is no danger of confusion, we write $P^\pi(a) = P^\pi$ respectively $I^\pi(P, a) = I^\pi$.

Some observations are summarized in the next proposition.

PROPOSITION 7.8 *Let R be a locally archimedean right chain ring, P a completely prime ideal, $a \in R$ and $Pa \neq (0)$ satisfying $Pa \subseteq aJ$. Then we have:*

(i) *If P is idempotent, then $P^\pi = I^\pi$.*

(ii) *Let $ra = as \neq 0$ with $r \notin P$, then we have $s \notin P^\pi$.*

(iii) *Let $ras = a$ with $r \notin P$, then $s \notin P^\pi$ follows.*

PROOF: (i) Assume $Pa \subseteq aI^\pi$ with $I^\pi \subset P^\pi$. Then $P^n a = Pa \subseteq a(I^\pi)^n$ for any n and so $Pa \subseteq \bigcap_{n \in \mathbb{N}} (I^\pi)^n$. By Theorem 1.15 the intersection of the powers of I^π is again a completely prime ideal $\subset P^\pi$, contradiction.

(ii) Let $ra = as, r \notin P$ and assume $s \in P^\pi$. Hence there exists $p \in P$ with $pa \neq 0$ such that $pa = aq$ and $t^n R \subset qR$ for some n, say $s^n = qv$ for some $v \in R$. Then $r^n a = as^n = aqv = pav$ and $r^n w = p$ for some $w \in P$ leads to $r^n a = pav = r^n wav = r^n aw'v$. Hence $r^n a = 0$ which implies $Pa = (0)$, a contradiction.

(iii) Let $ras = a$ and $r \notin P$. Assume $s \in P^\pi$. Then we find a power of s, say s^n with $ap' = pa$ for some $p'R \supset s^n R$ and $p \in P$. Then $a = r^n as^n \in r^n ap'R = r^n paR \subseteq PaR \subseteq aJ$. So $a = 0$, a contradiction.

■

The next result establishes Proposition 7.4 for arbitrary locally archimedean right chain rings under a weak additional assumption.

LEMMA 7.9 *Let R be a locally archimedean right chain ring, P the minimal completely prime ideal containing $0 \neq a \in R$. Assume further $aP \neq (0)$. Then $P^\pi(a) = P$.*

PROOF: First we consider the case when R is not a domain and P the prime radical. Let $p \in P$. Then the case $par_1 = a$ can never occur. Thus $pa = ar_2$ for any $p \in P$. Assume there exists $p \in P$ with $pa = ar_2$ and $r_2 \notin P$. Then $p^n a = 0 = ar_2^n$ leading to $aP = (0)$, a contradiction.

If P is not the prime radical, we have $a^2 \neq 0$. Again we have to consider the cases $par_1 = a$ (Case 1) and $pa = ar_2$ (Case 2). Since p and a lie in the same prime segment, some power of p is contained in aR, hence $a = p^n ar_1^n = ar'ar_1^n$, again a contradiction. Thus we have $Pa \subseteq aR$ and by $a^2 \in aP$ it follows $P \subseteq P^\pi$. Again assume $pa = ar_2$ with $r_2 \notin P$. For sufficient large n we have $p^n = aq$ with $q \in P$. Thus $p^n a = aqa = a^2 q' = ar_2^n$. Finally for some $r_3 \in P$ we obtain $r_2^n r_3 = a$ leading to a contradiction. Hence $P^\pi = P$. ■

Whereas we don't know whether Pa is a two-sided ideal in general under the condition $Pa \subseteq aJ$, the right ideal aP^π, however, is indeed a two-sided ideal. Its left and right associated ideals are of interest in the next chapter. Here we prove:

PROPOSITION 7.10 *Let R be a locally archimedean right chain ring, P a completely prime ideal in R and $a \in R$ satisfying $(0) \subset Pa \subseteq aJ$. Then we have:*

(i) *aP^π is a two-sided ideal.*

(ii) *$P_l(aP^\pi) = P$.*

(iii) *$P_r(aP^\pi) = P^\pi$.*

PROOF: (i) Let $ap \in aP^\pi$ and $r \in R$. If $ra = ar_1$, we are done. Otherwise $rar_2 = a$ for some $r_2 \notin P^\pi$ using Proposition 7.8. Thus $r_2p_1 = p$ for some $p_1 \in P^\pi$ and $rap = rar_2p_1 = ap_1 \in aP^\pi$ follows.

(ii) Take $s \notin P$ and $sx = ap' \in aP^\pi$. Next we compare xR and aR to obtain either $xr_1 = a$ (Case 1) or $x = ar_2$ (Case 2) with $r_1, r_2 \in R$. In the first case we conclude $sa = sxr_1 = ap'r_1$ with $s \notin P$, $p'r_1 \in P^\pi$ contradicting Proposition 7.8. If in the second case $r_2 \in P^\pi$ we are done. Assume otherwise $r_2 \notin P^\pi$. If $sa = as'$, then $s' \notin P^\pi$ by Proposition 7.8, and so $ap' = sx = sar_2 = as'r_2$ leading to $as'r_2 = 0$ and so $Pa \subseteq aP^\pi = (0)$, a contradiction. Otherwise $sas_2 = a$ and by Proposition 7.8 again $s_2 \notin P^\pi$ and with similar arguments as above a contradiction can be reached. Thus we have proved: $P_l(aP^\pi) \subseteq P$. To prove the converse inclusion, note that a is not in aP^π and we obtain $Pa \subseteq aP^\pi$ showing $P \subseteq P_l(aP^\pi)$.

(iii) Since $a \notin aP^\pi$ we clearly have showing $P^\pi \subseteq P_r(aP^\pi)$. To check the opposite inclusion assume $P \subset P_r(aP^\pi)$ and let $p' \in P_r(aP^\pi) \backslash P^\pi$. Hence there exists $x \notin aP^\pi$ with $xp' = ap_1 \in aP^\pi$ with $p_1 \in P^\pi$. Again we compare aR and xR to obtain either $xs_1 = a$ (Case 1) or $x = as_2$ (Case 2) for some $s_1 \in R$, $s_2 \notin P^\pi$. In the first case we conclude $xp' = ap_1 = xs_1p_1$ with $p' \notin P^\pi$, however $s_1p_1 \in P^\pi$, so $xp' = 0$ follows. If $s_1R \subseteq p'R$, we obtain $a = 0$, otherwise $s_1r = p'$ for some $r \notin P^\pi$ and $aP^\pi = (0)$ follows, a contradiction. Thus we have $x = as_2$ which is also impossible since $xp' = as_1 = ap_1$ holds with $p_1 \in P^\pi$, $s_1 \notin P^\pi$. Hence, $P^\pi = P_r(aP^\pi)$. ■

7.3 Right semiinvariance and stronger conditions

Before we present further concepts we introduce some invariance terminology.

DEFINITION 7.11 *Let R be a ring.*

(i) *An element $a \in R$ is called* right invariant, *if $Ra \subseteq aR$, that is: the right ideal generated by a is two-sided. If all elements $a \in R$ are right invariant, R is called* right invariant.

(ii) *An element $a \in R$ is called* duo *or* invariant *if $Ra = aR$. The ring R is said to be* duo *or* invariant *if all elements are duo.*

(iii) *Let P be a completely prime ideal of R, then R is called* right P-semiinvariant *if $Pa \subseteq aR$ for all $a \in R$.*

(iv) *The ring R is said to be* right semiinvariant *if R is right J-semiinvariant.*

Similarly, left versions of (i), (iii) and (iv) are defined.

We continue with some easy observations relating our invariance conditions.

LEMMA 7.12 *Let R be a ring and $a \in R$.*

(i) *If R is a right semiinvariant right chain ring and $Ra \not\subseteq aR$. Then $Ja \subseteq aJ$.*

(ii) *Left invariant right chain rings are chain rings.*

(iii) *If R is a chain ring, then for $b \neq 0$ we have $Ra \subseteq bR$ if and only if $aJ \subseteq Jb$.*

(iv) *If R is a chain ring, then $Ra \subseteq aR$ if and only if $aJ \subseteq Ja$.*

(v) *For a chain ring R the following are equivalent:*

(a) *R is right semiinvariant.*

(b) *R is left semiinvariant.*

(c) *For any $a \in R$ we have $aR \subseteq Ra$ or $Ra \subseteq aR$.*

Hence, we call right semiinvariant chain rings semiinvariant.

PROOF: (i) Assume $na \in aU$ with $n \in J$, say $na = au$. On the other hand $va \notin aR$ for some $v \in U$. Hence $vna = vau \notin aR$, a contradiction.

(ii) Let $a, b \in R$ with $aR \subseteq bR$. Then also $Ra = RaR \subseteq RbR = Rb$.

(iii) Suppose $Ra \subseteq bR$. If we assume $aJ \not\subseteq Jb$, we have $b = ras = br's$ for some $s \in J$, $r', r \in R$ and $ra = br'$. Hence $b = 0$, a contradiction. On the other hand consider the case $aJ \subseteq Jb$. If there exists $ra \in Ra \backslash bR$, then $b \in raJ \subseteq Jb$, a contradiction.

(iv) is a special case of (ii) if $a \neq 0$, but obviously also true for $a = 0$.

(v) We show that (a) and (c) are equivalent.

(a) $\Rightarrow$ (c) If $Ra \not\subseteq aR$ and $aR \not\subseteq Ra$, then $a \in RaJ$ and $a \in JaR$, so $a \in JaJ \subseteq aJ$, leading to $a = 0$.

(c) $\Rightarrow$ (a) If $Ra \subseteq aR$ then trivially $Ja \subseteq aR$. By (iii), $aR \subseteq Ra$ implies even $Ja \subseteq aJ$. ■

Part (iii) of the Lemma shows that the A-valuations and V-valuations introduced by Mathiak [82] are equivalent.

We observe the following result for P-semiinvariant right chain rings.

LEMMA 7.13 *Let R be a right P-semiinvariant right chain ring and $Q \subseteq P$ a prime ideal. Then Q is completely prime.*

PROOF: By definition, if $Q = P$, then Q is completely prime. If we assume that $Q \subset P$ is not completely prime we find a completely prime ideal $P' \subseteq P$ with $[P', Q[$ simple. Take $a \in [P', Q[$, then we have $P'aR \subseteq aR$, hence there is a two-sided ideal between P' and Q contradicting the fact that $[P', Q[$ is simple. ■

Among the P-semiinvariant conditions the right J-semiinvariance (i.e. the semiinvariance) can be characterized in various ways. For chain domains R semiinvariance implies a linear order on the set of conjugate rings $aRa^{-1}, a \in Q(R)\backslash\{0\}$. Mathiak [77] calls these rings *subinvariant* where we prefer the term *semiinvariant.* Hence *the structure of the set of conjugate rings* of a right chain domain R can be viewed as a further key in studying different invariance properties.

Although right invariance is already a large step towards the commutative situation we observe phenomena which are far away from commutativity. For example, in case of right invariance the set $\{x \in R \mid xa \in aU\}$ is multiplicatively closed, so the maximal two-sided ideal $P \neq (0)$ with $Pa \subseteq aJ$ is a completely prime ideal. There is not much known about the ideal generated by X where $Pa = aX$. If we denote by P^π the minimal prime ideal Q with $Pa \subseteq aQ$ (see Proposition 4.12), the function π, is not continuous as we will see in a later chapter. If $P \supset P'$ are neighbours in the lattice of prime ideals $P^\pi \supset P'^\pi$ are in general not neighbours; however in the situation of chain rings (see Proposition 4.12) the situation is clear. In particular π need not be surjective so that large gaps can occur.

These observations suggest the following useful, however very restrictive notion for right chain rings:

DEFINITION 7.14 *A right invariant right chain ring R is called* strongly invariant *if for any $a \in R$ and (completely) prime ideal P we have $P^\pi(a) = P$.*

Note that in a strongly invariant right chain ring we have $Pa \subseteq aP$ and for each prime ideal Q with $Pa \subseteq aQ$ we get $P \subseteq Q$. If R is a right invariant right chain ring with exactly one prime ideal $\neq (0)$, R is strongly invariant: Assume $ra = au$ for some $a \neq 0$, $r \in J$, $u \in U$, so $r^n au^{-n} = a$ with $r^n R \subseteq aR$ for a sufficient large n, leading to a contradiction. For further examples see a forthcoming part. Strongly right invariance is a rather restrictive condition, e.g. no prime ideal $\neq J$ and (0) can be finitely generated:

LEMMA 7.15 *Let R be a right chain ring. Suppose $Ja \subseteq aJ$ for all $a \in R$. Then no prime ideal $P \neq (0)$, $P \neq J$, is finitely generated.*

PROOF: Suppose $P = pR$ and $P \neq J$. Take $x \in J \backslash P$, then there exists $y \in J$ with $p = xy$, so $y \in P$, as P is completely prime by Lemma 7.13. Hence $p \in JpR \subseteq pJ$, so $p = 0$. ■

PROPOSITION 7.16 *If R is a strongly right invariant duo chain ring (strongly duo chain ring, for short), then $Pa = aP$ for any $a \in R$ and completely prime ideal P.*

PROOF: follows from Proposition 4.12. ■

THEOREM 7.17 *Let R be a strongly right invariant right chain ring.*

(i) *If $Q \subset P$ are neighbour prime ideals then $(P\backslash Q)a \subseteq a(P\backslash Q)$ for all $a \in R$.*

(ii) *For any completely prime ideal P we have $(R\backslash P)a \subseteq a(R\backslash P)$ for all $a \in R$.*

(iii) *For any two-sided ideal I we have $P_l(I) = P_r(I)$.*

PROOF: (i) Let $p_1 \in P\backslash Q$, so $p_1 a = ap_2$ with $p_2 \in P$. We want to show that we can choose $p_2 \in P\backslash Q$. Since R is strongly right invariant there exist $q_1, q_2 \in P\backslash Q$ with $q_1 a = aq_2$. Then for some $n \in \mathbf{N} : q_1^n = p_1 r$ with $r \in R$, so $p_1 ra = aq_2^n$. If $p_2 \in Q$ and $p_1 a \neq 0$ we get a contradiction. Thus we now assume $p_1 a = 0$. But then $0 = p_1 ra = q_1^n a = aq_2^n$, so $0 \in a(P\backslash Q)$.

(ii) Let $sa \in (R\backslash P)a$ for some $s \in J$. Then there exist prime ideals P_1, P_2 which are neighbours with $P \subseteq P_2 \subset sR \subseteq P_1$. By (i) we have $sa \in a(P_1\backslash P_2) \subseteq a(R\backslash P)$. It remains to show that $ua \in a(R\backslash P)$. Note that $ua = av$ implies $v \in U$, since otherwise $a = u^{-1}ua = u^{-1}av = av'v$ for some $v' \in R$ leads to $a = 0$.

(iii) Let $s_1 \in P_l(I)$, hence there exists $t \notin I$ with $s_1t \in I$. Together with (i) we obtain $s_1t = ts_2$, thus $s_2 \in P_l(I)$ which proves $S_l(I) \supseteq S_r(I)$. Let $s_1 \in S_l(I)$ and $ts_1 \in I$. If $s_1 \in U$ then, of course, $t \in I$, so $s_1 \in S_r(I)$. Thus we can assume that there are neighbour prime ideals $Q_2 \subset Q_1$ with $s \in Q_1\backslash Q_2$. Clearly, $P_l(I) \subseteq Q_2$. Let $p_1 \in Q_1\backslash Q_2$, then $p_1t = tp_2$ with $p_2 \in Q_1\backslash Q_2$ by (i). Without loss of generality $s_1R \supseteq p_2R$ (otherwise replace p_1 by a suitable power). Then also $p_1t = tp_2 \in I$. As $p_1 \notin Q_2$ we have $p_1 \in S_l(I)$, hence $t \in I$. This proves $s_1 \in S_r(I)$, so $S_r(I) = S_l(I)$. ∎

7.4 Invariance properties and localization

In Section 5.2 we discussed the general question whether rings of fractions exist for right chain rings. Now we want to present result on the preservation of invariance properties under localization. It turns out that the classes of locally archimedean rings, semiinvariant rings and strongly right invariant rings are closed with respect to localizations.

THEOREM 7.18 *Let R be a locally archimedean right chain ring and $S = R\backslash P$, where P is a prime ideal. Then $R[S^{-1}]$ is locally archimedean.*

PROOF: We want to use the characterization of locally archimedean rings given in Theorem 7.1(d). Let $[P_1[S^{-1}], P_2[S^{-1}][$ be any prime segment different from the radical segment in $R[S^{-1}]$. Then, $[P_1, P_2[$ contains a two-sided ideal I, for example take $I = aP_1$, $a \in P_1\backslash P_2$. Then $sx = a$ for some $s \in S$ implies $sx = xt$ (Case 1) or $sxt = x$ (Case 2), $s \notin P_1$ in both cases using Corollary 1.22. We obtain $S^{-1}(aP_1[S^{-1}]) \subseteq aP_1[S^{-1}]$. Hence $[P_1[S^{-1}], P_2[S^{-1}][$ is not simple. ∎

PROPOSITION 7.19 *Let R be a right P-semiinvariant right chain ring, P a prime ideal and $S = R\backslash P$. Then $R[S^{-1}]$ is a right semiinvariant right chain ring.*

PROOF: Let $0 \neq as^{-1} \in R[S^{-1}]$. We have to show: $P[S^{-1}]as^{-1} \subseteq aR[S^{-1}]$. Let $p \in P$ and $a = sb$, so $ps^{-1}a = pb$. If $pb \in aR[S^{-1}]$ we are done. Otherwise we have to consider the case $pbq = a$ with $p \in P$. Let $sp' = p$ with $p' \in P$. Then we conclude: $s(p'bq - b) = 0$ which implies $sb = 0$, hence $a = 0$ in $R[S^{-1}]$. ∎

COROLLARY 7.20 *Localizations of right semiinvariant right chain rings are again right semiinvariant rings. In particular, localizations of right invariant right chain rings are semiinvariant rings.*

It follows from the corollary that localizations of right invariant chain rings are semiinvariant. The next result shows that the converse holds for semiinvariant right chain domains.

THEOREM 7.21 *Let R be a right semiinvariant right chain domain. Then there exists a right invariant right chain ring A and a prime ideal $P = A\backslash S$ in A such that $R = A[S^{-1}]$.*

PROOF: Set $A = \{r \in R \mid ra \in aR$ for all $a \in R\}$. Then A is a ring and $J(R) \subseteq A$. Let $u \in U(R)\backslash A$. Claim: $u^{-1} \in A$. Let $a = uar$ with $r \in J$. If $b = u^{-1}bs$ for some $b \in R$, $s \in J$, we consider the cases $aq_1 = b$ and $a = bq_2$. By symmetry, it suffices to treat the case $aq_1 = aq = b$. Then $uarq = aq = uaqr'$ with $r' \in R$. Thus $br' = aqr' = u^{-1}b$, so $br's = u^{-1}bs = b$, a contradiction. Obviously, this implies that A is a right chain ring. As R is a domain, $ra = ar'$ with $r \in A$ implies $r' \in A$, hence A is right invariant. Localizing at $P = J(R) \subseteq A$ gives R, that is $R = A[S^{-1}]$. ∎

An example of a semiinvariant chain domain which is not invariant is given in Section 7.5. The next result gives conditions under which the localization of a right invariant right chain ring is again right invariant.

PROPOSITION 7.22 *Let R be a right invariant right chain ring, P a prime ideal and $S = R\backslash P$. Then the following are equivalent:*

(a) *$R[S^{-1}]$ is right invariant.*

(b) *Let I be the kernel of the canonical homomorphism $R \to R[S^{-1}]$. Then for all $a \in R\backslash I$: $Sa \subseteq aS$.*

If (a) *or* (b) *holds, then $N_r(S) \subseteq N_l(S)$, hence $I = N_l(S)$.*

PROOF: (a) $\Rightarrow$ (b) Let $R[S^{-1}]$ be right invariant, so $s^{-1}a \in aR[S^{-1}]$ for all $s \in S$. This implies $s^{-1}a = apt^{-1}$, respectively $sap = at$ in $R[S^{-1}]$. If $sa \in aP$ (in R), then $at \in I$, so $a \in I$ which implies $a = 0$ in $R[S^{-1}]$.

(b) $\Rightarrow$ (a) Let $a \in P\backslash I, s \in S$, so $a = sb$ for some $b \in R$. If $b = ar$, then $s^{-1}a \in aR[S^{-1}]$ - done. Otherwise $a = br$, and by (b) $r \in S$. Therefore $s^{-1}a = b = ar^{-1}$.

The additional assertion follows by (b). ∎

The result above and the next proposition give another justification for the introduction of strongly right invariant right chain rings.

THEOREM 7.23 *Let R be a strongly right invariant right chain ring, P a prime ideal and $S = R\backslash P$. Then $R[S^{-1}]$ is also a strongly right invariant right chain ring. Moreover, if the kernel $I \neq (0)$, then $J(R[S^{-1}]) = N_l(R[S^{-1}]) = N_r(R[S^{-1}])$, that is, $R[S^{-1}]$ is an affine Hjelmslev ring.*

PROOF: That $R[S^{-1}]$ is right invariant follows directly from Proposition 7.22(b) together with Theorem 7.17. In the same way the strong right invariance can be checked. That $N_l(S) = N_r(S)$ follows by 7.17(i) using the arguments of the proof of Theorem 7.17(iii). ∎

7.5 An example of a semiinvariant chain domain

We consider an example of a semiinvariant, but not invariant chain domain. This construction was essentially given by Radó [70] and provided the first examples of such rings.

We begin with $F = \mathbf{R}(t)$ the field of rational functions in one indeterminate t over the real numbers and define an order on F by

$$0 < \frac{a_0 t^n + \cdots + a_0}{b_0 t^m + \cdots + b_m} \Longleftrightarrow 0 < a_0 b_0.$$

This order is nonarchimedean, in particular $t > a$ for any $a \in \mathbf{R}$. Next, consider the group

$$G = \{(a,b) \mid a, b \in F, a > 0\}$$

with

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1)$$

as operation and

$$(a_1, b_1) \le (a_2, b_2) \Longleftrightarrow a_1 < a_2 \text{ or } a_1 = a_2 \text{ and } b_1 \le b_2$$

defines an order on G with

$$G^+ = \{(a,b) \mid a = 1 \text{ and } b \ge 0 \text{ or } a > 1 \text{ and } b \in F\}$$

as the (extended) positive cone.

The Malcev-Neumann generalized power series ring $D = k[[G]]$ over a field k contains a chain domain $R = k[[G^+]]$ which is invariant (Proposition 1.24).

We will define an overring R_1 of R which is not invariant, but semiinvariant by Theorem 7.21. Denote

$$\Delta = \{x \mid x \in G,\ \exists r \in \mathbf{R}:\ x \ge (1, r)\}.$$

If $x_1, x_2 \in \Delta$, $x_1 x_2 \ge (1, r_1) \cdot (1, r_2) = (1, r_1 + r_2)$, hence $\Delta \cdot \Delta \subseteq \Delta$. On the other hand Δ is not an invariant set in G. For, if $(a,b) \in G$, then $(a,b)^{-1} = (a^{-1}, -a^{-1}b)$ and

$$(a,b)(1,-1)(a,b)^{-1} = (1, -a).$$

Selecting an element $a \in F$ such that $r < a$ for all $r \in \mathbf{R}$ (in the case of $F = \mathbf{R}(t)$ we may take $a = t$), we obtain a transform of the element $(1,-1) \in \Delta$ which does not belong to Δ. We consider now the following subset of $k[[G]]$

$$R_1 = \{a = \sum g a_g \in k[[G]] \mid supp(a) \subseteq \Delta\}.$$

Obviously $R_1 + R_1 \subseteq R_1$ and $a \in R_1$ implies $-a \in R_1$. Since $\Delta \cdot \Delta \subseteq \Delta$, we get $R_1 \cdot R_1 \subseteq R_1$. Hence R_1 is a subring in $k[[G]]$ containing $k[[G^+]]$. By Proposition 5.3 R_1 is a chain domain and hence a localization at a prime ideal P. It is easily checked that P consists of all elements in $k[[G]]$ whose smallest element of the support lies in

$$\Pi = \{x = (a,b) \in G \mid \forall r \in \mathbf{R}:\ (1,r) \le (a,b)\}.$$

R_1 is not invariant as the following calculation shows. Take $t = a$ and let $x = (a,0) \cdot 1$, $y = (1,-1) \cdot 1$ be one-element series in $k[[G]]$. With the same arguments as above we notice

$$x \cdot y \cdot x^{-1} = (1,-a) \cdot 1 \notin R_1,$$

hence the subring R_1 is not invariant, however semiinvariant as proved in Corollary 7.20.

7.6 Examples of locally invariant chain domains

We consider the following situation: Let D be a division algebra, finite dimensional over its center K. If R is a total subring of D, i.e. $x \in D \backslash R$ implies $x^{-1} \in R$, then $V = R \cap K$ is a valuation ring of K and we say R is an extension of V in D. Cohn [81] showed that the extension R of V is invariant if $J(V)$ and (0) are the only prime ideals of V, i.e. V has rank 1. We will consider the general situation in Part 2, but discuss here an example of an extension R of a rank 2 valuation ring V which is not invariant in D.

LEMMA 7.24 *Let R be a chain domain with $Q(R) = D$ and $[D : K]$ finite for K, the center of R. Then R is a locally invariant chain domain.*

PROOF: We show that (d) in Theorem 7.1 holds for R. Let $xP \in [P,Q[$ where $[P,Q[$ is a prime segment. There exists an $n > 1$ such that the elements $\{x, x^2, \ldots, x^n\}$ are linearly dependent over K. Hence, there exist $c_1, \ldots, c_n \in K$ with $c_1x + \ldots c_nx^n = 0$, not all $c_i = 0$. Therefore, indices $i < j$ exist with $c_ix^iR = c_jx^jR$ and $x^{j-i}R = c_j^{-1}c_iR = zR$ for $0 \neq z = c_j^{-1}c_i \in K$ and hence $x^mR = zR = Rz$ for some m and $z \in K$. Note that in each case some power of x defines a two-sided ideal, thus by Proposition 6.7 we have $x^m \notin Q$. Therefore using Theorem 7.1 a two-sided ideal $\neq P$ exists in $[P,Q[$. ∎

Consider $L = \mathbb{Q}(i)$ and the two extensions V_1, V_2 of $\mathbb{Z}_5 = \{a/b \in \mathbb{Q} \mid 5 \nmid b\}$ the 5-adic valuation ring in L. Hence $V_1 = \mathbb{Z}[i]_{(2-i)}$, $V_2 = \mathbb{Z}[i]_{(2+i)}$ and let $D = L[[t,\sigma]]$ be the skew Laurent series field with elements $\sum_{i>-m}^{\infty} t^i a_i$ and $at = ta^\sigma$ defining the multiplication. The center of D is $\mathbb{Q}[[t^2]] = K$ and $[D : \mathbb{Q}[[t^2]]\,] = 4$. The subring

$$V = \{\sum_{i=0}^{\infty} (t^2)^i a_i \mid a_i \in \mathbb{Q},\ a_0 \in \mathbb{Z}_5\}$$

has rank 2 and $5V = J(V)$, $P = (t, t^{\frac{1}{5}}, t^{\frac{1}{25}}, \ldots)$, (0) are its prime ideals.

There are two extensions of V in D:

$$R_1 = \{\sum_{i=0}^{\infty} t^i a_i \mid a_i \in L,\ a_0 \in V_1\}$$

and

$$R_2 = \{\sum_{i=0}^{\infty} t^i a_i \mid a_i \in L,\ a_0 \in V_2\}$$

and $t^{-1}R_1t = R_2$. However, the ideal tR_1 is not a left ideal, since $\frac{1}{2+i}t = t\frac{1}{2-i}$ and $t = \frac{1}{2+i}t(2-i)$ with $2 - i \in J(R_1)$.

8 Chain conditions on prime ideals

It will be shown that chain conditions on prime ideals imply the symmetry of certain right-left conditions. For example, $P_l(I) \subseteq P_r(I)$ for any two-sided ideal I in a locally archimedean right chain ring with DCC for prime ideals (Theorem 3.4). This implies that a semiinvariant right chain domain is right invariant if DCC holds for prime ideals (Theorem 3.6). An example of a chain ring R without DCC or ACC for prime ideals given and $P_l(I) \neq P_r(I)$ is shown for a certain two-sided ideal I of R.

8.1 Chain conditions - preliminary remarks

If an ordered group G satisfies the minimum or maximum condition on convex subgroups then every convex subgroup is normal in G (see Fuchs [66], p. 82). This result carries directly over to the ring case:

LEMMA 8.1 *Let R be an invariant chain domain with minimum or maximum condition on prime ideals. Then the localization R_P at any prime ideal P is again an invariant chain domain.*

PROOF: Note that by assumption each principal left ideal is a principal right ideal and vice versa, hence $Ra = aR$ for any $a \in R$. These ideals form the positive cone of an ordered group G under ideal multiplication. As described in Schilling [50] the elements in the complements of prime ideals exactly generate the convex subgroups in G. Thus by the previously cited result we have $aPa^{-1} \subseteq P$ for all $0 \neq a \in R$ and P an arbitrary prime ideal and the statement follows from Proposition 7.22. ∎

We will prove related results for chain rings and right chain rings. We begin with some general observations on chain conditions on prime ideals in a right chain ring and recall that in such a ring the union as well as the intersection of prime ideals is again a prime ideal.

PROPOSITION 8.2 *Let R be a right chain ring. Then the following conditions are equivalent.*

(a) *The set of prime ideals satisfies the minimum conditions.*

(b) *Every descending chain of prime ideals becomes stationary (DCC).*

(c) *Each prime ideal $\neq J$ has an upper neighbour in the lattice of prime ideals.*

(d) *The set of completely prime ideals satisfies the minimum condition.*

PROOF: The equivalence of (a) and (b) is obvious.

(a) $\Rightarrow$ (c) Let $P_0 \neq J$ be a prime ideal. By assumption the set of prime ideals strictly containing P_0 has a minimal element which is an upper neighbour.

(c) $\Rightarrow$ (a) Let $\mathcal{S}$ be a set of prime ideals. Obviously the intersection on all prime ideals $P \in \mathcal{S}$ is a prime ideal, say Q_0. As Q_0 has an upper neighbour, say Q_1, the prime ideal Q_0 must be an element of $\mathcal{S}$.

(a) $\Rightarrow$ (d) is obvious.

(d) $\Rightarrow$ (c) Let Q be an arbitrary prime ideal $\neq J$. Let Q_1 be the minimal element in the set of all completely prime ideals $\supset Q$. If there is no prime ideal between Q_1 and Q, we are done. Otherwise there exists an exceptional prime ideal Q_2 and by Theorem 6.2 the intersection of the powers of Q_2 equals Q. Thus Q_2 is the upper neighbour of Q and Q_1 the upper neighbour of Q_2 as prime ideals. ∎

We obtain a dual version for the maximum condition:

PROPOSITION 8.3 *Let R be a right chain ring. Then the following conditions are equivalent.*

(a) *The set of prime ideals satisfies the maximum conditions.*

(b) *Every ascending chain of prime ideals becomes stationary (ACC).*

(c) *Each prime ideal which is not the prime radical has a lower neighbour in the set of prime ideals.*

(d) *The set of completely prime ideals satisfies the maximum condition.*

The following example shows that the two chain conditions are independent. Let G be the group $G = \sum_{i\in\mathbb{N}} C_i$ with C_i a copy of $(\mathbb{Z}, +)$ for every $i \in \mathbb{N}$. A commutative valuation ring associated with G and the lexicographical order does not satisfy the minimum condition for prime ideals, but the maximum condition, a valuation ring associated with the same G but inverse lexicographical order satisfies DCC for prime ideals but not ACC.

8.2 Minimum conditions for prime ideals in right chain rings

We remind the reader that by $Rad(R) = \bigcap P$, P prime in R, we denote the prime radical of the ring R.

THEOREM 8.4 *Let R be a locally archimedean right chain ring whose prime ideals satisfy the minimum condition and I a two-sided ideal which is not in the prime radical $Rad(R)$ if R is not a domain. Then we have $P_l(I) \subseteq P_r(I)$.*

Note that only in the non-domain case there is a restriction on I.

PROOF: Let I be a two-sided ideal. If I is prime, then I is completely prime by Corollary 7.3(i) and $P_l(I) = I = P_r(I)$ follows by Proposition 4.10(i). So I can be assumed not to be prime, in particular $I \neq (0)$ by assumption. The ideal I defines a prime segment, say $[P, Q[$. We will prove the assertion of the theorem segment by segment. Hence, assume $[P, Q[$ contains a counterexample L. If $P_r(L) = P$, Lemma 4.6 can be applied and $P_l(L) = P$ follows. Thus for a counterexample L we have the following situation:

$$P_l(L) = P_1 \supset P_r(L) = P_2 \supset P \supset Q$$

Since the minimum conditions holds for prime ideals there exists an upper neighbour of P_2, say P_1', in the lattice of prime ideals.

We choose L minimal with respect to P_2. Note that $P_2 = P_r(L) \supset P$ holds and any two-sided ideal $L' \in [P, Q[$ (under the general assumption) with $P_r(L') \subset P_2$ satisfies $P_l(L') \subseteq P_r(L')$. Choose any $s \in P_1' \backslash P_2$, so $\bigcap_{n \in \mathbb{N}} s^n R = P_2$. Hence there exists $x \notin L$ with $sx \in L$. As P is completely prime, we have $xR \in [P, Q[$. Since $x \notin L$, we have $sxR \subseteq xR$, thus $sx = xt$ for some $t \in R$ and $t \in P_2 = P_r(L)$ follows. In the case when R is not a domain, the element t can never lie in the prime radical $Rad(R)$, since $sx \notin Rad(R)$. So the powers of t define a completey prime ideal, say $P_3 = \bigcap t^n R$. Note that t never lies in P. Assume otherwise $t \in P$, then $s^n x = xt^n = x^4 r$ for a sufficient large n and $r \in R$. We conclude $s^n v = x$, so $s^n x = s^n v x^2 \cdot xr = s^n x v' x r$ for some $v' \in R$ using Theorem 7.1. This leads to $s^n x = 0$ which is impossible by $s^n, x \notin Q$ and Q completely prime.

Hence the situation is the following:

$$P_l(L) = P_1 \supseteq P_1' \supset P_2 = P_r(L) = \bigcap_{n \in \mathbb{N}} s^n R \supseteq tR \supset P_3 = \bigcap_{n \in \mathbb{N}} t^n R \supseteq P \supseteq L \supset Q.$$

From $P_2 \backslash P_3$ we choose an element p under the following restrictions with x as specified above.

Case 1: There exists $p \in P_2 \backslash P_3$ with $pxq = x$ for some $q \in R$.

Case 2: $(P_2 \backslash P_3)x \subseteq xR$.

Case 1: We assume $pxq = x$ for some $p \in P_2 \backslash P_3$, $q \in R$. Since $p^n x q^n = x$ follows for each n, the element q can never lie in the prime radical nor the smallest prime ideal $\neq (0)$ in the domain case. Each q generates a prime segment $[P_q, Q_q[\ni qR$. Choose p under these conditions such that $P_q = P'$ is minimal. Such a prime ideal P' is never the union of prime ideals strictly contained in P' since $q \in P'$. Again we have $s^n q_n = p$ for all $n \in \mathbb{N}$ and $q_n \in P_2 \backslash P_3$ follows.

If for some n we have $q_n x R \subseteq xR$, say $q_n x = x q_n'$, we obtain

$$x = pxq = s^n q_n x q = s^{n-1} s x q_n' q \in L,$$

since $sx \in L$, a contradiction. Thus we may assume $q_n x R \supset xR$ for all n, say $q_n x q_n' = x$. By assumption on q we have $q_n' R \supset q^k R$ for some k, since $q_n' \notin Rad(R)$ and each prime segment is not simple (use Theorem 1.21). We set $q_n' z = q^k$ and obtain

$$x = p^k x q^k = p^{k-1} p x q^k = p^{k-1} s^n q_n x q^k = p^{k-1} s^n q_n x q_n' z = p^{k-1} s^n x z = p^{k-1} s^{n-1} (sx) z,$$

thus $x = p^{k-1} s^{n-1} (sx) z \in L$, a contradiction.

Case 2: We assume $(P_2 \backslash P_3)x \subseteq xR$. Since $L \supset Q$ and $P_2 Q \supset Q$ follows. Then choose $p \in P_2 \backslash P_3$ arbitrarily. For each $n \in \mathbb{N}$ we find $q_n \in P_2 \backslash P_3$ with $s^n q_n = p$. Then we have $px = s^n q_n x = s^n x q_n'$ for some $q_n' \in R$ and any n by assumption. Thus $px = s^n x q_n' = xt \cdot t^{n-1} q_n'$, hence $px \in xt \bigcap_{n \in \mathbb{N}} t^n R \subseteq (xt) P_3 \subseteq LP_3$. Note that even if R is not a domain, $LP_3 \neq (0)$, since $px \neq 0$ for some $p \in P_2$. Since $P_3 \subset P_2$ holds, $P_r(L) = P_2$ implies $P_r(LP_3) = P_3 \subset P_2$ by Proposition 4.10(vi). Again $LP_3 \supset Q$ holds, so $P_l(LP_3) \subseteq P_3$ by induction. However, $p \notin P_3$, $px \in LP_3$ implies $x \in LP_3 \subseteq L$, a contradiction.

Hence, there cannot exist a counterexample. ∎

We now draw some conclusions from Theorem 8.4.

In Section 7.2 for a prime ideal P with $Pa \subseteq aJ$ we had defined $P^\pi = P^\pi(a)$ as the minimal completely prime ideal Q with $Pa \subseteq aQ$. It was proved in Lemma 7.9 that in a locally archimedean right chain ring we have $P = P^\pi(a)$ where P is the *minimal prime ideal containing* aR. This result turns out as a special case of a more general statement.

THEOREM 8.5 *Let R be a locally archimedean right chain ring whose prime ideals satisfy the minimum condition. Let P be any prime ideal and $a \neq 0$ an element of R which is not in the prime radical provided R is not a domain. Assume further $Pa \subseteq aJ$. Then we have $P \subseteq P^\pi(a)$.*

PROOF: If P is the prime radical the assertion is obvious. Thus assume $P \supset Rad(R)$.

Let $Rad(R) \subset Pa \subseteq aP^\pi(a) = aP^\pi$ and consider the two-sided ideal aP^π (use Proposition 7.10. P^π is not the prime radical when R is not a domain and so $aP^\pi \supset Rad(R)$. We apply Theorem 8.4 to the two-sided ideal aP^π which satisfies the prerequisites of the assertion. The rest follows from Proposition 7.10. ∎

THEOREM 8.6 *Let R be a right semiinvariant right chain ring with the minimum condition for prime ideals and $0 \neq a \in R$ satisfying $a \notin Rad(R)$ provided R is not a domain. Then $Ra \subseteq aR$.*

PROOF: If $Ra \not\subseteq aR$, then $a = uas$ for some $u \in U$, $s \in J$ and hence $Ja \subseteq aJ$ as R is right semiinvariant (use Lemma 7.12(i)). Thus $J^\pi \subseteq J$ and by Theorem 8.5 we obtain $J^\pi = J$.

Consider the two-sided ideal RaR. If $J = RaR$ then $P_l(J) = P_r(J) = J$, otherwise choose $j \in J \backslash RaR$ and b exists in R with $jb = a$. If $b = r_1ar_2 \in RaR$, then $a = jb = jr_1ar_2 \in aJ$, using $Ja \subseteq aJ$ we obtain a contradiction and $P_l(RaR) = J$ follows. Since a semiinvariant ring R is locally archimedean we obtain $P_r(RaR) = J$ by Theorem 8.4.

We turn back to the chosen elements a, u, s with $a = uas, s \in J$. We may assume that in case $J^2 \subset J$ the element s is not a generator of J, otherwise consider $a = u^n as^n$. Hence, there exists $t \in J$ with $tR \supset sR$, say $tv = s$ for some $v \in J$. By $P_r(RaR) = J$ we find $b \notin RaR$ with $bt = r_1ar_2 \in RaR$. If r_1 is a non-unit, we have $r_1a = ar_1'$ for some $r_1' \in R$, that is $bt = ar_1'r_2$. If r_1 is a unit rename $r_1^{-1}b$ as b and we have $bt = ar$ for some $r \in R$. Obviously $bw = ua$ for some $w \in J$ and u as chosen above. Thus we conclude $a = uas = bwtv = btw'v = arv$ using $Jt \subseteq tJ$ and so $a = 0$ follows which finally proves the theorem. ∎

COROLLARY 8.7 *Let R be a right semiinvariant right chain domain whose prime ideals satisfy the minimum condition. Then R is right invariant.*

PROPOSITION 8.8 *Let R be a right invariant right chain domain whose prime ideals satisfy the minimum condition, further let $S = R \setminus P$. Then $R[S^{-1}]$ is right invariant.*

PROOF: If $P = J$, we are done. So assume $P \subset J$ and $sa = ap$ for some $s \in S$, $p \in P$. Let $[P_1, P_2[$ be the prime segment generated by sR. Note that $P_1 \supset P$ holds. By Theorem 8.5 we have $P_1 \subseteq P_1^{\pi}(a)$, thus $s_1 a = ap_1$ for some s_1 with $P_1 \supseteq s_1 R \supset P_2$ and $p_1 \notin P$. W.l.o.g. we may assume $s_1 R \subset sR$, say $st = s_1$ which leads to $ap_1 = s_1 a = sta = sat' = apt'$ for some $t' \in R$. Since pt' lies in P, however $p_1 \notin P$ we obtain a contradiction. ■

8.3 Minimum and maximum conditions for prime ideals in chain rings

As it can be expected one obtains more specific results in case R is a chain ring.

THEOREM 8.9 *Let R be a chain ring and I a two-sided ideal of R. Assume that the minimum condition holds for the prime ideals of R that contain I. Then $P_l(I) = P_r(I)$.*

In view of Proposition 5.6(ii) and Lemma 4.5 it is sufficient to prove the following theorem:

THEOREM 8.10 *Let R be a chain ring with minimum condition for prime ideals. Then $N_l(R) = N_r(R)$.*

PROOF: Without loss of generality it can be assumed that $N_r(R) \subseteq N_l(R)$, and after localizing at $S = R \setminus N_l(R)$ we may further assume that $N_r(R[S^{-1}]) \subseteq N_l(R[S^{-1}]) = J(R[S^{-1}])$ (Proposition 5.6). Therefore it suffices to consider the case $N_l(R) = J(R)$, which implies in particular $(Ra)^{rl} = Ra$ for all $a \in R$ by Theorem 2.9(ii) . Therefore we may restrict ourselves to the following situation: We set $N_r(R) = Q$, $N_l(R) = J$ and *assume* $Q \subset J$. For $b \in J \setminus Q$ we have $Q \subseteq Rb$ and thus $Q^r \supseteq (Rb)^r \neq (0)$ follows. By Proposition 4.14(ii) the ideal Q^{rr} is a completely prime ideal.

Independent of any chain conditions for prime ideals $Q = Q^{rr}$ is impossible. To prove this assume:

Step 1: $Q = Q^{rr}$. Thus $Q^l = Q^{rrl}$ and by Proposition 2.10(ii) $Q^r = Q^{rrl}$ since $Q^r \neq (0)$, hence $Q^l = Q^r$ and therefore $Q^l \neq (0)$. Take any $a \neq 0, a \in Q^l$. We conclude $Q \subseteq (Ra)^r$ and as $Q = N_r$, we thus have $Q = (Ra)^r$ and finally $Q^r = Q^l = (Ra)^{rl} = Ra$ (Theorem 2.9(ii)). As the element $a \in Q^l$ was chosen arbitrarily, we have $Ja = (0)$. Hence $J \subseteq (Ra)^l = (Q^l)^l = Q^{rl} = Q$. Contradiction!

Step 2: Since $N_r(R) = Q$ we have $Q^{rr} \subseteq N_r(R) = Q$. We now assume $Q^{rr} \subset Q$ and hence $Q^{rrr} \neq (0)$. The process can be iterated. We write $Q_0 = Q$, $Q_{i+1} = Q_i^{rr}$, and we have $Q_1 \subset Q_0$ by assumption. By the minimum condition for prime ideals there must exist a smallest index k with $Q_k = Q_{k-1}^{rr} = Q_k^{rr}$, hence $Q_{k-1}^{rrll} = Q_k^{rrll} = Q_k$ by Lemma 2.10(iii), a contradiction. ■

COROLLARY 8.11 *Let R be a chain ring with minimum condition for prime ideals, P a completely prime ideal in R. Then $P^l = P^r$.*

PROOF: If both annihilators of P are zero, we are done. Thus we can assume that $P^l \neq (0)$. By the left symmetric version of Proposition 4.14(i) it follows that $P_l(P^l) = P^{ll}$. Applying Lemma 4.13 we obtain $P_r(P^l) = P$. Theorem 8.9 implies $P = P^{ll}$. Now we get by Proposition 2.10 $P^l = P^{llr} = P^r$. ■

We don't know whether an analogous result holds if we assume the maximum condition for prime ideals. However, Mazurek [89] gave an equivalent characterization under this condition:

THEOREM 8.12 *Let R be a chain ring with the maximum condition for prime ideals. Then $N_r(R) = N_l(R)$ if and only if $P^r = P^l$ holds for any completely prime ideal P of R.*

Before we give his proof we need a further lemma which can also be found in Mazurek's paper:

LEMMA 8.13 *Let R be a chain ring with $N_r(R) \subseteq N_l(R)$ and P a completely prime ideal with $P^r \subset P^l$. If we put $A_0 = P$ and $A_{n+1} = A_n^r$ for $n \in \mathbb{N}_0$, then A_{2n} is completely prime and $A_{2n+3} \subset A_{2n+1} \subset A_{2n} \subset A_{2n+2}$ for any $n \in \mathbb{N}_0$.*

PROOF: If $P^r = 0$, then Lemma 2.8 gives $N_l \subset N_r$, a contradiction. Therefore $P^r \neq 0$, and $A_2 = P^{rr}$ is a completely prime ideal in view of Proposition 4.14. If $P^{rr} \subsetneq P$, then Proposition 2.10 and 4.14 imply $P^l \subseteq P^{rrl} = P^r$, a contradiction. Hence, $A_0^r \neq 0, A_2$ is a completely prime ideal and $A_0 \subset A_2$. We shall prove by induction on n that $A_{2n}^r \neq 0$, that A_{2n+2} is a completely prime ideal, and that $A_{2n} \subset A_{2n+2}$ for all $n \in \mathbb{N}_0$. Let us suppose that the result is true for some n. From the induction hypothesis and Proposition 2.10 we get $A_{2n+2}^l = A_{2n}^{rrl} = A_{2n}^r \neq 0$.

So Lemma 2.8 implies $A_{2n+2}^r \neq 0$, and therefore $A_{2n+4} = A_{2n+2}^{rr}$ is a completely prime ideal by Proposition 2.10. If $A_{2n+4} \subsetneq A_{2n+2}$, then Proposition 2.10 gives $A_{2n+2} = A_{2n+2}^{rrll} = A_{2n+4}^{ll} \subseteq A_{2n+2}^{ll} = A_{2n}^{rrll} = A_{2n}$, a contradiction. Thus we have $A_{2n+2} \subset A_{2n+4}$.

It follows from the above and from Proposition 2.10 that $A_{2n+3} = A_{2n+2}^r \subset A_{2n}^r = A_{2n+1}$. ■

Now we are able to prove Theorem 8.12.

PROOF: If $N_r = N_l$, then Lemma 8.13 and the symmetric version of Lemma 8.13 imply $P^r = P^l$ for any completely prime ideal P.

Now assume that $N_r \neq N_l$. We consider only the case when $N_r \subset N_l$. Then $N_r^r \neq 0$, since otherwise $N_l \subseteq N_r$ by Lemma 2.3. If $N_r^l = N_r^r$, then $N_l^r = N_r^r$ by Proposition 2.5, so Proposition 2.10 implies $N_l = N_l^{rl} = N_r^{rl} = N_r$, a contradiction. Hence we have $P^r \neq P^l$ for a completely prime ideal. ■

For (projective) Hjelmslev rings R we have $N_r(R) = N_l(R) = J$ and Lemma 8.13 can be applied to obtain a result by Törner ([74], Satz 5.28).

THEOREM 8.14 *Let R be a projective Hjelmslev ring whose prime ideals satisfy the maximum or minimum condition. Then for each completely prime ideal P we have $P^l = P^r$.*

A detailed analysis shows that Radó's ring [70], see the next section, provides an example of a chain ring which neither satisfies DCC nor ACC for prime ideals. Furthermore, in this ring there exist two-sided ideals whose left and right associated prime ideals do not coincide.

We close this section with a short proof of a consequence on semiinvariant rings:

THEOREM 8.15 *Let R be a semiinvariant chain ring with minimum condition on prime ideals. Then R is a duo ring.*

PROOF: We may assume without loss of generality that $aR \subseteq Ra$ for an element $0 \neq a \in R$.

Case 1: $Ja \neq (0)$. Since $aR \subseteq Ra$ implies $Ja \subseteq aJ$ (see 7.12) we can apply Lemma 4.12 and obtain a completely prime ideal Q with $Ja = aQ$. From Theorem 8.9 and Proposition 4.12 we get $P_l(Ja) = J = Q = P_r(aQ)$, so $J = Q$, that is $Ja = aJ$ and by 7.12(iv) $Ra = aR$ follows.

Case 2: Now suppose $Ja = (0)$ which implies $J \subseteq (aR)^l$, hence $J = (aR)^l \supseteq (Ra)^l$. However, take any $j \in J$ and $ra \in Ra$, thus $jra = 0$, which shows $J = (Ra)^l$. Obviously we have $(Ra)^{lr} = J^r$ and by Corollary 8.11 $Ra \subseteq (Ra)^{lr} \subseteq J^r = J^l$ follows. Thus $aJ = 0$, which leads to $Ra = aR$. ∎

THEOREM 8.16 *Let R be a chain ring with maximum or minimum condition on prime ideals. Assume $aR = Ra$ for some $a \in R$, i.e. a is a duo element. Then $Pa = aP$ for any completely prime ideal P, that is, a is strongly duo.*

PROOF: (a) Suppose the maximum condition on prime ideals holds. Set $\mathcal{S} = \{P \mid P \text{ completely prime ideal and } Pa \neq aP\}$. Assume $\mathcal{S} \neq \emptyset$. As $Ja = aJ$, the set $\mathcal{S}$ has a maximal element $Q \neq J$. As $Qa \neq aQ$, say $Qa \subset aQ$, in particular $aQ \neq (0)$. By Proposition 4.12 $aQ = Q_1a$ for some completely prime ideal Q_1. If $Q_1 \subseteq Q$, $aQ = Q_1a \subseteq Qa \subset aQ$ follows, a contradiction. Hence, we have $Q \subset Q_1$, so $aQ = Q_1a = aQ_1$ since Q was maximal under the restriction. But this implies $aQ = (0)$, a contradiction to $aQ \neq (0)$.

(b) Suppose the minimum condition on prime ideals holds. From Theorem 8.8 we know $P_l(I) = P_r(I)$ for any ideal I. With the arguments from above and by Proposition 4.12 we get $Pa = aP$. ∎

COROLLARY 8.17 *Let R be an invariant (or duo) chain ring with maximum or minimum condition on prime ideals. Then $P_l(I) = P_r(I)$ for all two-sided ideals I. In particular R is strongly duo.*

PROOF: Let I be a two-sided ideal of R and $p \in P_l(I)$. To be more precise, let $[P_1, P_2[$ the prime segment generated by p. There exists $x \notin I$ with $px \in I$. Hence, by Theorem 8.16 we have $P_1x = xP_1$ and $P_2x = xP_2$, so there exists $p' \notin P_1 \backslash P_2$ with $px = xp'$. Such an element p' however lies in $P_r(I)$ showing $P_l(I) \subseteq P_r(I)$. With symmetric arguments the proof is finished. ∎

8.4 A counterexample

We return to the example of Section 7.5 to make the following observations about invariant chain ring $k[[G^+]] = R$ constructed there:

(a) R does not satisfy DCC or ACC for prime ideals.

(b) There exist in R two-sided ideals I with $P_l(I) \neq P_r(I)$.

To recall the definition of R let $F = \mathbf{R}(t)$ be the field of rational functions in one indeterminate t over the real numbers with

$$(a_nt^n + \cdots + a_1t + a_0)(b_mt^m + \cdots + b_1t + b_0)^{-1} > 0 \text{ if and only } a_nb_m > 0$$

defining an order on F. Then $G = \{(a, b) \mid a, b \in \mathbf{R}(t), a > 0\}$ with

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, a_2b_2 + b_1)$$

as operation is an ordered group and we show that (a) holds by defining sequences of convex subsemigroups of G^+.

Consider the sets

$$\begin{aligned} \Pi_i &= \{x \in G^+ \mid \forall r \in \mathbf{R}^+ : (1, rt^i) < x\} \text{ with } i = 1, 2, \ldots \\ \Omega_i &= \{x \in G^+ \mid \exists r \in \mathbf{R}^+ : (1 + rt^{-i}, 0) < x\} \text{ with } i = 1, 2, \ldots \end{aligned}$$

The complement of Π_i is multiplicatively closed and convex in G and we obtain a strictly decreasing sequence of prime ideals in R:

$$P_1 \supset P_2 \supset \cdots \supset P_n \supset \cdots,$$

where P_i equals the set of elements in R with support in Pi_i for $i = 1, 2, \ldots$

If $a_1, a_2 \in F$ with $a_1, a_2 < 1 + st^{-i}$ for all $s \in \mathbf{R}^+$ then

$$a_1a_2 < (1 + st^{-i})^2 = 1 + 2st^{-i} + (st^{-i})^2 \leq 1 + 3st^{-i}$$

since $st^{-i} < 1$ in F which shows that the complement of Ω_i is a convex subsemigroup of G. Hence, we obtain a strictly increasing sequence of prime ideals in R:

$$Q_1 \subset Q_2 \subset \cdots \subset Q_n \subset \cdots,$$

where Q_i equals the set of elements in R with support in Ω_i.

(b) Next we construct a two-sided ideal I in R which is not symmetric, that is, $P_l(I) \neq P_r(I)$. We define I by those elements of R whose support lies in

$$\Theta = \{x \in G \mid \exists r \in \mathbf{R}^+ : (t^2, rt) < x\}$$

Θ is an upper set, that is, $x \in I$ and $x < y$ for any $y \in G$ implies $y \in \Theta$. Again we do the calculation in G instead of in R.

We claim: $P_l(I) \neq P_r(I)$. Take the element $(1, t^{-1} + 1) \in G$. To multiply this element in Θ, we use $(t^2, 0)$ and obtain

$$(t^2, 0)(1, t^{-1} + 1) = (t^2, (t^{-1} + 1)t^2) = (t^2, t + t^2) \in \Theta.$$

However $(t^2, 0) \notin \Theta$.

On the other side let

$$\Theta \ni (1, t^{-1} + 1)(t^2, a) = (t^2, a + t^{-1} + 1)$$

leading to $rt \leq a + 1 + t^{-1}$ for some $r \in \mathbf{R}^+$. By $t^{-1} < 1$ we obtain as a consequence the necessary condition $rt < a + 2$. Since $2 < (r/2)t$ this implies $(r/2)t < a$ showing $(t^2, a) \in \Theta$. These observations carry over to the ring case by choosing element in R with a corresponding support and show $P_l(I) \neq P_r(I)$.

9 Overrings of right chain domains

Some examples show that in the lattice of overrings of a right chain domain R much more complicated phenomena occur than in the case of a chain ring. Overrings T are no longer localizations of the ground ring in general. In particular in the case of R a right invariant chain domain some construction principles are discussed (Proposition 9.3 and Theorem 9.11). This suggests to study right chain overrings with additional properties for which we obtain some structure theorems. An expansion theorem is presented describing overrings in terms of unions of conjugated subrings (Theorem 9.13). The chapter closes with some examples.

9.1 Introductory remarks and some easy properties of overrings

In commutative ring theory a domain R with the property that each overring of R is a quotient ring of R is said to have the *QR-property* (see Gilmer [72], p. 334). It is known that a domain with the QR-property is Prüfer. The converse fails, but under the additional hypothesis of integral closure for R it can be proved (see Gilmer [72], p. 324) that any overring of a Prüfer domain is an intersection of quotient rings. Rings R of that type are said to have the *QQR-property* (see Gilmer [72], p. 339).

Generalizations to classes of *noncommutative domains* are also known, e.g. for left/right principal ideal domains (Brungs [71]) resp. for Bezout domains (Beauregard [73]). However these rings are left/right symmetric. In general, it can not be expected that similar results hold for rings with only a one-sided structure as the right chain domains considered in Brungs [71] demonstrate.

The aim of this chapter is an analysis of overrings of a *right chain domain* R in its quotient ring $Q(R) = D$. For chain rings the situation is similar to the commutative case as pointed out in Section 5.1. We will show that in the lattice of right chain overrings there is no chain domain between any two right chain domains unless the larger one is already a chain domain.

As by Example 5.7 an overring of a right chain domain need not to be a right chain ring again, we will mostly restrict our attention in this chapter to overrings $T \subseteq Q(R) = D$ which are again right chain domains.

Obviously the trivial condition $U(R) \subseteq U(T)$ and so $J(T) \cap R \subseteq J(R)$ hold for any overring T. On the other side non-units of R may turn into units of T. We set $P = \{x \in R \mid x^{-1} \notin T\} \subseteq J(R)$. It is evident that $R \backslash P$ is multiplicatively closed. If $x \in P$, $r \in R$ and $xr \in R\backslash P$, then also $x \notin P$, a contradiction. So P is a right ideal, hence a two-sided ideal by Lemma 1.14 and P is completely prime. Hence, we obtain $R \subseteq R_P \subseteq T \subseteq Q(R)$. Therefore, in order to keep aside effects caused by localizations we may often assume that no non-unit in R becomes a unit in T, that is R is *maximally localized* in T.

Any element of T is of the form ab^{-1} with $a, b \in R$, $b \neq 0$. Let $ab^{-1} \in T$ and R a right chain domain. Since $a = bx$ or $ay = b$ holds for some $x \in R$, $y \in J$ each element of T is of one of the types

$$ab^{-1} = axa^{-1} \text{ (Type 1) or } ab^{-1} = by^{-1}b^{-1} \text{ (Type 2).}$$

This terminology does not a priori provide a disjoint classification. Since any overring T is a sophisticated union of elements of Type 1 as well as Type 2, the structure of T heavily depends on these elements.

Note that a right chain domain R has *rank* 1 provided R has exactly one completely prime ideal $\neq (0)$.

We summarize a few observations:

LEMMA 9.1 *Let R be a right chain domain and $T \subseteq Q(R)$ an overring of R. Then we have:*

(i) *The set P of elements of R which are non-units in T is a completely prime ideal in R and $R \subseteq R_P \subseteq T$.*

(ii) *Assume R to be of rank 1, i.e. $J(R)$ is the only nonzero completely prime ideal. Then $y^{-1} \in T$ for some $y \in J(R)$ implies $T = Q(R)$.*

(iii) *Assume $T \subset Q(R)$. Then we have: $\bigcap_{b \in J(R)\setminus\{0\}} bT = (0)$.*

PROOF: (i) was already proved.

(ii) By (i) we have $P \subset J(R)$, hence $P = (0)$ and so $R_P = Q(R)$.

(iii) Let $x \in \bigcap_{b \in J(R)\setminus\{0\}} bT$. If x is of the form ara^{-1} we have $(ara^{-1})a = ar \in \bigcap_{b \in J(R)\setminus\{0\}} bT$. Since $ar \notin ar^2T$ it follows $x = 0$. On the other hand, if $x = ar^{-1}a^{-1}$ we get $(ar^{-1}a^{-1})ar = a \in \bigcap_{b \in J(R)\setminus\{0\}} bT$, hence $a = 0$, a contradiction. ■

Whereas the lattice of right ideals in R is cofinal with that of T, the maximal ideal $J(T)$ even when T is local, is in general not T-generated by $J(R)$ (see again Example 9.14). Example 9.15 will show that we can not expect the overrings to be of rank 1, even if R is of rank 1. In fact the rank is not even bounded.

We recall from Chapter 7 that in particular in a strongly right invariant right chain ring we have $Ja \subseteq aJ$ for any $a \in R$. Under additional assumptions for R we obtain the following results.

LEMMA 9.2 *Let R be a right semiinvariant right chain domain and $T \subset Q(R)$ an overring of R. Then we have:*

(i) *Assume $axa^{-1} = by^{-1}b^{-1} \in T$. Then $x \in U(R)$ follows. If R is in addition strongly invariant, each element is either of Type 1 or of Type 2.*

(ii) *Assume R is of rank 1. Then each element of T is either of Type 1 or of Type 2.*

(iii) *Let T be right invariant and R maximally localized in T, then each element in T is of Type 1.*

PROOF: (i) We consider the cases $br_1 = a$ and $b = ar_2$. In the first case we have $r_1xr_1^{-1} = y^{-1}$ and so $yr_1x = r_1$ follows. As R is semiinvariant and $y \in J$ we obtain $r_1y'x = r_1$ for some $y' \in R$ showing $x \in U(R)$. If $b = ar_2$ holds we get $x = r_2y^{-1}r_2^{-1}$, thus $xr_2y = r_2$ leading again to $x \in U(R)$. If R is assumed to be strongly invariant, then in the first case $y' \in J(R)$ follows leading to $r_1 = 0$, a contradiction. In the second case we get $r_2 = xr_2y = r_2x'y$ for some $x' \in R$, hence by $y \in J$ again a contradiction.

(ii) follows directly from (i).

(iii) By $(av^{-1}a^{-1})av = a$ and the right invariance of T we obtain $av'v = a$ for some $v' \in T$ which shows that v is a unit in T, a contradiction. ■

9.2 Conjugated rings as overrings in the right invariant case

Next we are considering for some obvious overrings of R. Let $b \neq 0$ be a *right invariant element* of R, i.e. $Rb \subseteq bR$ and so $R \subseteq bRb^{-1}$ follows. Hence $bRb^{-1} = R^b$ is an overring of R, the *b-conjugated ring* of R, which is again a right chain domain $\simeq R$. These overrings will play a central role. We may interpret the situation as follows: in the overring R^b a step forward is made in order to let b become a duo element. Since $(brb^{-1})b = br$ holds for any $r \in R$, each element of R occurs in the overring R^b as a right hand factor of b caused by a shifting process from left to right. This procedure can be iterated: Consider R^{b^2} as an overring of R^b. Then each element of R^b originates in shifting over b some element of R from left to right. We remark that this process will never become stationary if b was not duo to begin with. Thus a right invariant element b defines an overring $T = \bigcup_{n \in \mathbb{N}} R^{b^n} \subseteq Q(R)$ in which the element b is duo!

The next proposition summarizes and extends some of these elementary facts.

PROPOSITION 9.3 *Let R be a right chain domain and $a, b \in R^*$. Then the following assertions are valid:*

(i) *R^a is an overring of R if and only if only if the element a is right invariant, i.e. aR is a two-sided ideal. $R^{a^{-1}}$ is an overring of R if and only if a is left invariant.*

(ii) *$R^a \simeq R$ for any $a \in R^*$.*

(iii) *Let T be an overring of R and $a \in T$ right resp. left invariant in T, however not duo. Then $\{T^{a^n}\}$ resp. $\{T^{a^{-n}}\}$ are infinite ascending chains of overrings of T and hence right chain domains and overrings of R in $Q(R) = D$.*

(iv) *Let a, $s \neq 0$ be right invariant elements in R and $b = as$. Then $R^a \subseteq R^b$, and equality holds if $aR = bR$. In particular, if R is right invariant, the conjugated overrings R^a, $a \in R^*$ form a chain.*

(v) *Let $a \in R$ be a right invariant element. Then $T = \bigcup_{n \in \mathbb{N}} R^{a^n}$ is a right chain overring of R in which a is duo.*

(vi) *Let $a, b \in R$ with $aR \supseteq bR$ be right invariant elements where the element b is in addition assumed to be duo. Then a is also a duo element.*

(vii) *Let R be a right invariant right chain ring and assume that the minimal prime ideal $\neq (0)$ contains a duo element. Then R is an invariant chain domain.*

PROOF: (i) Obviously $R \subseteq R^a = aRa^{-1}$ if and only if $Ra \subseteq aR$.

(ii) R and R^a are isomorphic via $r \to ara^{-1}$.

(iii) follows from (i).

(iv) Let $b = as$ with $s \in R^*$. Then $bRb^{-1} = a(sRs^{-1})a^{-1} \supseteq aRa^{-1}$ by (i). The equality is obvious if $s \in U(R)$.

(v) see the remarks before Proposition 9.3

(vi) By (iv) we have $R \subseteq R^a \subseteq R^b = R$ as the element b is assumed duo, hence $R = R^a$ which implies that a is duo.

(vii) Note that with a duo any power of a is again duo, thus for any $b \neq 0$ we find a duo element a with $aR \subseteq bR$. Thus by (vi) any b is duo and since R is a domain, R is also a left chain ring. ∎

Next we turn to the question posed at the beginning of this chapter and provide a partial answer.

PROPOSITION 9.4 *Let R be a right invariant right chain domain with the minimum condition for prime ideals. If R satisfies the QR-property, that is, each overring is a localization of R, then R is a chain domain.*

PROOF: We assume that R is not a chain domain. Then there exists some $a \in R$ which is not a duo element. Hence $Ra \subset aR$ and so $R \subset R^a \subset R^{a^2} \subset \cdots \subset R^{a^n} \subset$ is a strictly increasing set of overrings by Proposition 9.3(iii). Since each overring is required to be a localization, there must be a decreasing sequence of prime ideals which does not become stationary. Contradiction. ∎

Nevertheless overrings T obtained by conjugation may possess elements of R which have become units in T without T being a localization as the following example shows.

EXAMPLE 9.5 *Take R as a right noetherian right chain domain with exactly two prime ideals $xR \supset yR \neq (0)$ (see Example 3.8). In particular we have the commutation rule $xy = y\varepsilon$ for some $\varepsilon \in U(R)$. Then yRy^{-1} is an overring of R in which $x = y\varepsilon y^{-1}$ is a unit, however yxy^{-1} is a nonunit. We will show that yRy^{-1} is never a localization. Assume otherwise $R_{(yR)} = yRy^{-1}$ holds. Then each yry^{-1} with arbitrary $r \in R$, $r \neq 0$ can be written as $y^m x^k u x^{-l}$, $u \in U(R)$, $m, l, k \in \mathbb{N}_0$. Set $yry^{-1} = y^m x^k u x^{-l}$, hence $ry^{-1}x^l = y^{m-1}x^k u$. Since $xy = y\varepsilon$ and so $y\varepsilon^{-1} = xy$ holds for some $\varepsilon \in U(R)$, we obtain $ry^{-1}x^l = r\varepsilon^l y^{-1} = y^{m-1}x^k u$. It follows*

$$\begin{aligned} r\varepsilon^l &= y^{m-1}x^k uy \\ &= y^{m-1}x^k yu' \text{ for some } u' \in U(R) \\ &= y^m \varepsilon^k u' \end{aligned}$$

We obtain $r = y^m \varepsilon^k u' \varepsilon^{-l}$ which shows that the r's are restricted to a subset of R, a contradiction.

The conjugated rings R^a discussed so far are again contained in some specific overring, namely the ring constituted by all the elements of Type 1. This ring is a chain domain as the next proposition states.

PROPOSITION 9.6 *Let R be a right invariant chain domain and $\tilde{R} = \bigcup_{a \in R^*} R^a$. Then $\tilde{R}$ is an invariant chain domain all of whose overrings in $Q(R)$ are again chain domains. If R is of rank 1, then so is $\tilde{R}$.*

PROOF: As the rings R^a, $a \in R^*$ form an increasing chain as far the corresponding sequence of right ideals aR is decreasing (Proposition 9.3), $\tilde{R}$ is a right invariant right chain domain. In order to prove $\tilde{R}$ to be a left chain domain consider elements $x = ara^{-1}$ and $y = asa^{-1}$, again using the fact that the conjugated rings are linearly ordered by inclusion. Let $s = rt$ for some $t \in R$ then $y = asa^{-1} = (asr^{-1}a^{-1})(ara^{-1}) = as(ar)^{-1}(ara^{-1}) = (ar)t(ar)^{-1}x$. To show that $\bigcup_{a \in R^*} R^a$ is left invariant use $xy = ara^{-1}asa^{-1} = arsa^{-1} = ars(ar)^{-1}ara^{-1} = y'x$ where $y' = ars(ar)^{-1}$.

By Proposition 5.3 each overring of a chain domain in its quotient field is a localization at a prime ideal.

Now assume that R is of rank 1. Let $ava^{-1} \in \tilde{R}$, hence ava^{-1} lies in $J(\tilde{R})$ if and only if $v \in J(R)$. We remark that the semigroup of principal right ideals in case where R is of rank 1, is commutative (a result which will be proved independently in a forthcoming part). Thus $av = vau$ for some $u \in U(R)$ and so $ava^{-1} = v(aua^{-1})$ with $aua^{-1} \in U(\tilde{R})$, hence $ava^{-1}\tilde{R} = v\tilde{R}$. This shows that each principal right ideal of $\tilde{R}$, say ava^{-1}, is contained in some right ideal which is generated by an element of R, namely $v \in R$. Hence $J(\tilde{R}) = \bigcup_{b \in J(R)} b\tilde{R}$. Since also $\bigcap_{bR \subseteq R} b\tilde{R} = (0)$ holds by Lemma 9.1(iii), $\tilde{R}$ is of rank 1. ■

Overrings lying in $\tilde{R}$ can be described more closely. In this context the following condition is of interest defining an extraordinary situation for elements of Type 1.

($\star$) A conjugate of a unit in R may be a non-unit in T, that is, there may exist elements $u \in U(R)$, $x \in R$ with $xux^{-1} \in J(T)$.

PROPOSITION 9.7 *Let R be a right invariant right chain domain of rank 1 and T an overring of R lying in $\tilde{R}$. Assume further that T is again a right chain domain. Then we have:*

(i) *Let $xax^{-1} = yby^{-1} \in T$. Then $aR = bR$.*

(ii) *Let xbx^{-1} be in T with $b \in J(R)$. Then we have $aT \supseteq xbx^{-1}T \supseteq cT$ provided $aR \supset bR \supset cR$.*

(iii) *The set of elements $\{z = xax^{-1} \in T \mid a \in J(R)\}$ defines a completely prime ideal P_0 which is also the minimal prime ideal $\neq (0)$ in T. Further we have $\bigcap_{n \in \mathbb{N}} z^n T = (0)$ for any $z \in P_0$.*

(iv) *The overring T is of rank 1 if and only if there exists no elements satisfying Condition ($\star$). In this case we have: $xax^{-1} \in U(T)$ if and only if $a \in U(R)$.*

PROOF: (i) Let $xax^{-1} = yby^{-1}$. Without loss of generality we assume $xR \supseteq yR$, thus $xr = y$ for some $r \in R$. We obtain $yr^{-1}ary^{-1} = yby^{-1}$ and so $r^{-1}ar = b$ leading to $aRrR = rRbR$ since R is right invariant. The semigroup of principal right ideals in a right invariant right chain ring of rank 1 is commutative (see a forthcoming Part), hence $aR = bR$ follows.

(ii) We set $ar = b$ and assume contrary $xbx^{-1}T \supset aT$, thus $xb^{-1}x^{-1}a \in T$. By Proposition 9.3 a lies in R^x, hence $a = xa'x^{-1}$ and obviously $aR = a'R$ holds using again the commutativity of the multiplication of ideals in R. By assumption we have $xb^{-1}a'x^{-1} \in T$, so $xb^{-1}a'x^{-1} = ydy^{-1}$ for some $d \in R$ using Lemma 9.2. From (i) we obtain $a'R = bdR$, thus $bR \supseteq a'R = aR$, a contradiction. The rest of the inclusion can be proved by similar arguments.

(iii) Obviously elements $a \in R$ are also elements of R^x for any $x \in R$, say $a = xa'x^{-1}$ and $a \in J$ if and only $a' \in J$ holds. So P_0 is a two-sided ideal whose complements is multiplicatively closed. Let $z = xax^{-1} \in P_0$. Then for any $b \in J(R)$ by (ii) we find $n \in \mathbb{N}$ with $bT \supseteq z^nT = xa^nx^{-1}T$ since R is of rank 1. That $\bigcap_{n\in\mathbb{N}} z^nT = (0)$ follows from Proposition 9.1(iii). By results of Chapter 6 there cannot exist an exceptional prime ideal inside P_0.

If Condition $(\star)$ is satisfied for some xux^{-1}, such an element does not lie in P_0, hence $P_0 \subset J(T)$. On the other side $P_0 = J(T)$ implies that elements xux^{-1}, $u \in U(R)$ are units in T contradicting Condition $(\star)$. The rest is obvious. ∎

9.3 Overrings under restricted conditions

It is natural to ask how the lattices of right ideals of T and R are related. Here we deduce conditions on the structure of $J(T)$ induced by $J(R)$.

The next theorem provides some detailed information on this connection in the case when R is of rank 1 and R does not necessarily consist of elements of Type 1 exclusively. Note that by Corollary 6.3 it cannot be excluded that right chain domains of rank 1 might exist with an exceptional prime ideal $(0) \subset Q \subset J$. In such a case $J(R)$ is idempotent and, in addition, $\bigcap_{n\in\mathbb{N}} Q^n = (0)$ holds.

PROPOSITION 9.8 *Let $R \subseteq T \subset Q(R)$ be right chain domains where R is of rank 1.*

(i) *For any completely prime ideal $P \neq (0)$ in T we have $J(R)T \subseteq P$.*

(ii) *T possesses a minimal completely prime ideal $P_0 \neq$ which is the intersection of all completely prime ideals $\neq (0)$ of T.*

PROOF: (i) Suppose $P \subset aT$ for some $a \in J(R)$. If R is right invariant we have $a^n \in P$ for some $n \in \mathbb{N}$ (use Lemmma 9.1(iii)). Then there exists a unit $u \in U(R)$ such that $(au)^n \in P$ for some $n \in \mathbb{N}$. Contradiction.

(ii) Set $P_0 = \bigcap_{P \supseteq J(R)T} P$. ∎

What can be said about the ideal lattice in T between the two neighbouring completely prime ideals P_0 and (0)? Does there exist an *exceptional prime ideal* in P_0? The next result answers this question negatively.

THEOREM 9.9 *Let $R \subseteq T \subset Q(R)$ be right chain domains where R is of rank 1 and $P_0 \neq (0)$ is the minimal prime of T. Further assume that R does not an exceptional prime ideal.*

(i) *P_0 does not contain an exceptional prime ideal.*

(ii) *$[P_0, (0)[$ is simple provided T contains an element of Type 2 or R is nearly simple.*

Recall that for a right ideal I we denote by $\overline{I}$ resp. $\underline{I}$ the smallest (resp. largest) two-sided ideal containing I (lying in I). A straightforward proof shows $\overline{I} = \bigcup_{u \in U(R)} uI$ resp. $\underline{I} = \bigcap_{u \in U(R)} uI$.

PROOF: (i) If R is right invariant and T lies in $\tilde{R}$, that is, all elements of T are of Type 1, the assertion (i) follows from Proposition 9.7(iii).

Next we assume that R is nearly simple, however T consists only of elements of Type 1. So we have $J(R)^2 = J(R)$. Assume otherwise that there is an exceptional prime ideal $0 \subset Q \subset P_0$. Since $\bigcap_{n \in \mathbb{N}} Q^n = (0)$, however $J(R)$ is idempotent by Corollary 6.3, some elements of $J(R) \cap T$ cannot lie inside Q. On the other side elements in $J(R) \cap Q$ constitute a two-sided ideal in R as it can be checked straightforwardly, a contradiction.

Next we are dealing with the case where T contains an element of Type 2, say $w = av^{-1}a^{-1}$ with $v \in J(R)$, $a \in R$. We will show that the neighbouring prime ideals of the right ideal aT are $\underline{aT}$ and $\overline{aT}$ which are in addition completely prime, hence $\underline{aT} = (0)$ and $\overline{aT} = P_0$, so $[P_0, (0)[$ is simple.

We obtain $wav = w^n av^n = a$ for arbitrary $n \in \mathbb{N}$. Suppose the largest two-sided ideal contained in aT, namely $\underline{aT}$, is not zero. Applying Lemma 9.1(iii) we have $\underline{aT} \cap R \neq (0)$. If R is right invariant we have $v^n \in \underline{aT} \cap R$ for some $n \in \mathbb{N}$. We obtain $a \in \underline{aT}$, hence a is a right invariant element in T and $a = w^n av^n = aw_1^n v^n$ follows, a contradiction. In the other case R is nearly simple, so $\bigcap_{u \in U(R)} uvR = (0)$, hence $uv \in \underline{aT}$ for some $u \in U(R)$ and thus $v \in \underline{aT}$. Again a lies in $\underline{aT}$ and we use the same arguments as before. Thus $\underline{aT} = (0)$.

Since R is a domain and there is no two-sided ideal between $\overline{aT}$ and $\underline{aT} = (0)$, the two-sided ideal $\overline{aT}$ must be idempotent, hence $\overline{aT}$ is a completely prime ideal by Theorem 1.15. As there is no two-sided between $\overline{aT}$ and $\underline{aT}$ the segment $[\overline{aT}, (0)[$ is a simple prime segment. This completes the prove of (i).

(ii) As proved in (i) P_0 does not contain an exceptional prime ideal. By the proof of (i) we may restrict to the case where T contains only elements of Type 1. So we are done if $[P_0, (0)[$ turns out to be simple. If $[P_0, (0)[$ would not be simple, we have $\bigcap_{n \in \mathbb{N}} x^n T = (0)$ for all $x \in P_0$ (use Theorem 1.21). However, by Corollary 6.5 we find elements $0 \neq x, u, v \in R$, $u, v \in U(R)$ with $xT \subset \bigcap_{n \in \mathbb{N}} (uxv)^n T$, again a contradiction. ∎

What can be said about the consequences of the pathological situation described by condition $(\star)$? By Theorem 9.9(ii) we may restrict to the case where all elements of T are of Type 1, so $T \subseteq \tilde{R}$ and by Proposition 9.7 there is no exceptional prime ideal in P_0. Thus $[P_0, (0)[$ defines a prime segment and the following theorem holds.

THEOREM 9.10 *Let $R \subseteq T \subset Q(R)$ be right chain domains where R is in addition right invariant and of rank 1. Further assume that T is an overring satisfying Condition $(\star)$. Then at least one of the following possibilities is true.*

(a) *$P_0 \subset J(T)$, that is, T has at least two non-trivial completely prime ideals.*

$(\star)$ *The prime segment $P_0 \supseteq (0)$ is simple.*

PROOF: As mentioned before we may assume that all elements of T are of Type 1. Further let $v = aua^{-1} \in J(T)$ with $u \in U(R)$. Finally assume that P_0 equals $J(T)$, hence T possesses exactly two prime ideals, namely P_0 and (0). Since $v \cdot au^{-1} \cdot u^{-1} = au^{-1}$ holds with $v \in J(T)$, the element au^{-1} is not right invariant in T, hence by the characterization of right chain domains of rank 1 (see Corollary 6.3) T has to be nearly simple. This shows that the remaining assertion (b) is true. ∎

9.4 A correspondence between overrings and right ideals

Next we describe two elementary, nevertheless important construction methods for overrings which both together lead to a Galois correspondence of two-sided ideals and certain overrings. In Proposition 9.6 a union of a-conjugated rings was taken with a running over the complement of the zero ideal. This idea can be generalized.

Let R be a right chain domain and I a two-sided ideal. *Further assume that any $a \notin I$ is right invariant*, that is $Ra \subseteq aR$ for all $a \notin I$. Each element a with $aR \supset I$ resp. $aR \supseteq I$ defines an a-conjugated ring $R^a \supseteq R$. Next we consider

$$\bigcup_{aR \supset I} R^a = D_1$$

which is an overring of R by Proposition 9.3. Let $I \neq (0)$ be a two-sided ideal, the same applies to

$$\bigcup_{aR \supseteq I} R^a = D_2$$

which is again an overring of R. The question arises when these two constructions do coincide, respectively when different two-sided ideals lead to the same overring. Obviously we have $D_i \supseteq I$. Note that $D_1 \supset I$ implies $D_1 \succ I$, hence I is of the form bJ for some $b \in R$. On the contrary, if I is assumed to be a lower neighbour, we have $I = bJ$ by Lemma 1.2(v). Then $bR \supset bJ$ and $D_1 = bR \succ I$ follows.

Again we have $D_2 \supseteq I$. Assume $D_2 \supset I$. With the same arguments as above we conclude $D_2 \succ I$ leading to $D_2 = bR$ and $I = bJ$ for some $b \in R$. In addition, the maximal ideal J must be not finitely generated and hence J is idempotent otherwise $J = mR$ and $I = bJ = bmR$ follows contradicting $D_2 \supset I$.

The converse is also true, that is, if I is a lower neighbour and J not finitely generated, then $D_2 \succ I$ follows.

Dually since each right ideal $I \neq (0)$ can be represented as a union of principal right ideals contained in I we may consider

$$\bigcap_{(0) \neq aR \subseteq I} R^a$$

provided R is assumed to be right invariant. Since the conjugated rings are linearly ordered under inclusion, these intersections are again right invariant right chain rings.

THEOREM 9.11 *Let R be a right invariant chain domain and $\mathcal{C}$ be the class of overrings which are unions or intersections of a-associated rings of R. Then we have:*

(i) *$\mathcal{C}$ is linearly ordered by inclusion with R as the minimal and $\bigcup_{aR\supset(0)} R^a$ as the maximal element.*

(ii) *Let $0 \neq a$ and $[P,Q[$ the prime segment generated by aR. Then $\bigcup_{n\in\mathbb{N}} R^{a^n} = \bigcup_{bR\supset Q} R^b$.*

(iii) *The mapping $I \longrightarrow \bigcup_{aR\supset I} R^a$ is an order-reversing injective mapping of two-sided ideals which are not lower neighbours to overrings in $\mathcal{C}$.*

(iv) *The mapping $I \longrightarrow \bigcup_{aR\supseteq I} R^a$ is an order-reversing injective mapping of two-sided ideals which are not lower neighbours to overrings in $\mathcal{C}$.*

(v) *The mapping $I \longrightarrow \bigcap_{0\neq aR\subseteq I} R^a$ is an injective order-reversing mapping of two-sided ideals to overrings in $\mathcal{C}$.*

(vi) *Let I be a two-sided ideal which is not a lower neighbour, then $\bigcup_{aR\supseteq I} R^a = \bigcap_{0\neq aR\subseteq I} R^a$.*

PROOF: (i) To prove that $\mathcal{C}$ is linearly ordered use the fact that R is right invariant and apply Proposition 9.3(iv). $\bigcup_{a\in R^\bullet} aRa^{-1} = \bigcup_{aR\supset(0)} aRa^{-1} = \tilde{R}$ is obviously the maximal element of the chain and by Proposition 9.6 a right invariant right chain domain.

(ii) Since R is right invariant, each prime ideal is completely prime, hence $\bigcap_{n\in\mathbb{N}} a^n R$ equals Q by Theorem 1.21. Further $bR \supset cR$ implies $R^b \subseteq R^c$ which proves $\bigcup_{n\in\mathbb{N}} R^{a^n} = \bigcup_{bR\supset Q} R^b$.

(iii), (iv) use the remarks mentioned above.

(v), (vi) can be checked directly applying the considerations above. ∎

9.5 A structure theorem for right invariant overrings

Let R be a right invariant right chain domain of rank 1 and $R \subseteq T \subset Q(R)$ where T is also assumed to be a *right invariant right chain domain* also. We will prove a structure theorem which covers the examples known so far. As R is of rank 1, by Lemma 9.1(ii) no element of $J(R)$ possesses an inverse in T, so we have $J(R) \subseteq J(T)$ and $J(R) = J(T) \cap R$ follows. Lemma 9.2(iii) implies that any element of T is of Type 1. In other words, T is always contained in the chain domain $\tilde{R} = \bigcup_{aR\supset(0)} R^a$ (see Proposition 9.6). In particular this implies that $\tilde{R}$ is the unique invariant chain domain containing the right invariant ring R. Moreover, $\tilde{R}$ is again of rank 1 and with the same arguments used in the proof of Proposition 9.6 the overring T has also rank 1.

As we had realized before, the a-conjugated rings play a major role in the case of overrings of rank 1. These rings will turn out to be the 'building blocks' for any overring T. We set

$$T_a = T \cap R^a = \{x \in T \mid \exists r \in R : x = ara^{-1}\},$$

so T_a is a subring of T. Since T_a is the intersection of two right chain domains contained in $\tilde{R}$, the subring T_a is again a right chain domain. With the same arguments T_a is also right invariant. These subrings T_a generate T; to be precise

$$T = T = \cap\tilde{R} = \bigcup_{a\in R^*} (T \cap R^a) = \bigcup_{a\in R^*} T_a \tag{3}$$

Obviously we have

$$T_a = R^a \text{ if and only } R^a \subseteq T. \tag{4}$$

Since by Proposition 9.3(iv) the conjugated rings R^a are linearly ordered, we have for any $a, b \in R$

$$T_a \subseteq T_b \text{ if } aR \supseteq bR. \tag{5}$$

In particular

$$T_a = T_{au} \text{ for any } u \in U(R) \tag{6}$$

follows, thus the intersections are independent of the generator of the corresponding principal right ideal and form an increasing sequence of overrings provided the 'parameter' right ideals are decreasing. Proposition 9.7(iv) implies

$$ava^{-1} \in U(T) \text{ if and only if } v \in U(R),$$

hence $U(T) \cap T_a = U(T_a)$.

By Equation (4) T_a equals R^a provided $R^a \subseteq T$. Even more is true: for any $a \in R^*$ each T_a is a conjugated ring, however of some subring of R. Thus the external structure of overrings is connected with the internal structure of specific subrings, this will be discussed in the following and will lead to an expansion theorem.

For any $a \in R^*$ we define

$$R_a = \{r \in R \mid \exists x \in T : xa = ar\} = a^{-1}T_a a = R \cap a^{-1}Ta.$$

By definition R_a consists of those elements of R which can be obtained by shifting some $x \in T$ over $a \in R$. We have $(R_a)^a = T_a$ and R_a is a subring of R. Rewriting Equation (4) leads to

$$R_a = R \text{ if and only if } R^a \subseteq T. \tag{7}$$

LEMMA 9.12 *Let R be a right invariant right chain domain of rank 1 and T a right invariant overring. Then $R_a \subseteq R$ is a right invariant right chain domain of rank 1. If $Rb \subseteq Ra$ holds for some $a, b \in R$, we have $R_b \subseteq R_a$.*

PROOF: Since T_a is an intersection of two right invariant right chain domains, so T_a is a right invariant right chain domain and so is R_a.

To prove that R_a is of rank 1 it suffices to show that each $u \in U(R) \cap R_a$ is a unit in R_a. Let $xa = au$, $u \in U(R)$ for some $x \in T$. Since T is of rank 1, x has to be a unit in T, so $x^{-1}a = au^{-1}$ and $u^{-1} \in R_a$ follows.

We have $b = qa$. Hence $T^{a^{-1}} \supseteq T^{b^{-1}}$ and $R_b \supseteq R_a$ follows. In particular we have $R_a = R_b$ provided $b = qa$ for some unit $q \in U(R)$. ∎

It is our aim to exhaust T by conjugated rings of R resp. subrings of R. We have $T = \bigcup_{a \in R^*} T_a$ by Equation (3). Assume that there exists $a \in R$ with $R^a \subseteq T$. Since $T_a = T_{au}$ for any $u \in U(R)$ there exists a right ideal $I_1 \subset R$ with

$$\bigcup_{aR \supseteq I_1} R^a \subseteq T \text{ and } R^a \not\subseteq T \text{ provided } aR \subset I_1 \tag{8}$$

If in (8) the union resp. the associated ring R^b with $I_1 = bR$ equals T, we are done.

In the situation described above we have determined an ideal I_1 and a subring R_1, namely R itself such that conjugated subrings of R_1 exhaust T as far as possible. This process can be extended in the following way.

Now assume that we have already constructed some ideal I_k and some subring R_k such that the a-conjugated subrings of R_k with $a \notin I_k$ approximate T as good as possible. We choose some $b \in I_{k-1} \backslash I_k$ and an element $r \in R$ such that $qb = c \in I_k$ holds. We set $R_{k+1} = R_c$ and $R_{k+1} \subset R_k$ follows. Again we define a right ideal I_{k+1} subject to the condition $R^d_{k+1} \subseteq T$, $d \notin I_{k+1}$. Again we take the union of conjugated rings of R_{k+1} as 'far as possible'.

We summarize: Any overring T as described aboved can be approximated step by step via a union of conjugations of subrings of R. The subrings can be chosen to form a decreasing sequence when the union is taken over segments of ideals of R. To make sure that this process stops after ω steps we make the construction procedure discrete stopping at each power of an arbitrary element $a \in J(R)$. Hence, the fact that the semigroup of principal right ideals is archimedean leads to the following 'expansion' theorem:

THEOREM 9.13 *Let $R \subseteq T \subset Q(R)$ be right invariant right chain domains, R a ring of rank 1. Then for any sequence $(a_n)_{n \in \mathbb{N}}$ of elements of R satisfying $Ra_n \supset Ra_{n+1}$ and $\bigcap_{n \in \mathbb{N}} a_n R = (0)$ the rings R_{a_n} form a descending chain of right invariant right chain subrings of R and*

$$T = \bigcup_{n \in \mathbb{N}} (R_{a_n})^{a_n}$$

holds. In particular for any $0 \neq a \in J(R)$ we have $T = \bigcup_{n \in \mathbb{N}} a^n R_{a^n} a^{-n}$ holds.

PROOF: First we show: $R_1 = R_{a_n} \supseteq R_{a_{n+1}} = R_2$. Let $r \in R_2$, hence $xa_{n+1} = a_{n+1}r$ for some $x \in T$. We set $qa_n = a_{n+1}$, so $xa_{n+1} = xqa_n = qx'a_n$ for some $x' \in T$ and through $xa_{n+1} = a_{n+1}r = qa_nr$ we obtain $a_nr = x'a_n$, hence $r \in R_1$. Since any $a \in R^*$ is contained in some a_n, by Equation (5) $T_a \subseteq T_{a_n}$ follows. Hence $\bigcup_{a \in R^*} T_a = \bigcup_{a_n \in \mathbb{N}} T_{a_n} = (R_{a_n})^{a_n}$ follows.

If in particular a_n is chosen as a^n for some nonunit $a \neq 0$ the assertion follows since $\bigcap_{n \in \mathbb{N}} a^n R = (0)$. ■

9.6 Examples

The reader is advised to look again at Example 5.3 to see how overrings are build up in the way described by Theorem 9.13.

As can be seen from the ring R in Example 5.3 we may have an infinite ascending chain of overrings R^{t^i} which are not localizations. But not even the sublattice of those overrings which are themselves chain domains is a chain.

EXAMPLE 9.14 *We consider again the right chain ring R constructed as Example 5.3. The overrings $T_1 = L(x_{-1}^2)[[t, \sigma]]$ and $T_2 = L(x_{-1}^3)[[t, \sigma]]$ are right chain domains which are not comparable.*

EXAMPLE 9.15 *Replace the monomorphism in Example 5.3 by $\sigma : x_i \to x_{i+n} (n \in \mathbb{N})$. Take the free abelian group G generated by $\{x_{-n}, ..., x_{-1}\}$ and canonically linearly ordered with $1 < x_{-n} < ... < x_{-1}$. The quotient ring of the group ring $L[G]$ contains a valuation ring $N \supseteq L[x_{-n}, ..., x_{-1}]$ which can be obtained by the induced order valuation. The ring $T = N[[t, \sigma]]$ is a right invariant right chain domain of rank $n + 1$ satisfying $R \subset T \subset Q(R)$.*

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