Sonderdruck aus
Arch. Math., Vol. 48, 217–222 (1987)

0003-889X/87/4803-0217 $ 2.70/0
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# Overrings of right chain domains

By

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For numerous interesting classes $C$ of commutative rings assertions of the following type hold: Let $R$ be a domain in $C$, $T$ an overring with $T \subseteqq Q(R)$, then $T$ is a quotient ring of $R$ ($QR$-property; Gilmer [6]) resp. an intersection of quotient rings of $R$ ($QQR$-property; Gilmer [6]) and $T \in C$. Generalizations to classes of non-commutative domains are also known, e.g. for left/right principal ideal domains (Brungs [4]) resp. for Bezout domains (Beauregard [3]). However these rings are left/right symmetric. In general, it can not be expected that similar results hold for rings with only a one-sided structure as the counter example of a right chain domain in Brungs [4] demonstrates.

The aim of this paper is the analysis of overrings in exactly this class of rings with a one-sided structure.

In Section 1 we give examples to show that in contrast to the situation for chain domains overrrings $T$ of a right chain domain need not be right chain domains. Furthermore these rings need not lie in a chain. Even if the ring $R$ is of rank 1, overrings which are assumed to be right chain domains may have arbitrarily large rank (Section 2). In Section 3 we analyze overrings which contain "bad" elements (Theorem 7, Theorem 8).

Finally we prove a structure theorem for right invariant overrings $T$ of $R$ in terms of subrings of $R$ (Section 4).

Rings in this paper are not necessarily commutative, but they are domains and possess an identity 1. The Jacobson radical of the ring $R$ is denoted by $J(R)$ whereas $U(R)$ stands for the group of units. Throughout, overrings of $R$ are always assumed to be contained in $Q(R) = K$.

**1. Preliminaries and a counter example.** A ring $R$ without zero-divisors is called a *right chain domain* (sometimes right valuation domain) if the lattice of right ideals is linearly ordered by inclusion. If $R$ is in addition a left chain domain we speak of a *chain domain*. An element $a \in R$ satisfying $Ra \subseteqq aR$ is called *right invariant* resp. *left invariant* in case of $aR \subseteqq Ra$ resp. *invariant* or *duo* in case of $Ra = aR$. Analogously *rings* are called *right invariant* (r.i. for short), *left invariant*, *invariant* or *duo* if all elements are right invariant, left invariant or invariant.

Chain domains, esp. duo chain domains are the obvious generalizations of classical valuation rings. For chain domains the answer of the overring-problem is the same as in the commutative case:

**Theorem 1.** (Brungs/Törner [5]). *Let $R$ be a chain domain, $K = Q(R)$ its skew field of quotients. Then there is a one-to-one correspondence between the set of rings $T$ between $R$ and $K$ and the set of completely prime ideals of $R$ given by $P \to R_S$ where $S = R \backslash P$ is an Ore-system, $T \to P = R \backslash S$ with $S = \{s \in R \mid s \in U(T)\}$. $R_S$ is the ring of quotients of $R$ with respect to the Ore-system $S$.*

As an easy consequence of the result above overrings of a chain domain are again chain domains and they lie in a chain. That overrings are localizations is not true for right chain domains (Brungs [4]). There are even overrings which are not right chain domains:

Example 1. Let $L = \mathbb{Q}(x_i \mid i \in \mathbb{N}_0)$ be the function field in a countable number of indeterminates. By $x_i^\sigma = x_{i+1}$ we define a monomorphism on $L$, which is not an automorphism. Then we consider the skew power series ring $R = L[[t, \sigma]]$ which is determined by the multiplication rule $a\,t = t\,a^\sigma$, for all $a \in L$. The ring $R$ is a r.i. right chain domain, which is not a left chain domain. In $K = Q(R)$ we have the rings $t^j R\, t^{-j}$, which we may identify with $L_j[[t, \sigma]]$, where $L_j = \mathbb{Q}(x_i \mid i \in \mathbb{Z},\ i \geqq -j)$ and $\sigma$ is extended naturally. Set $T = \{f \in L_1[[t, \sigma]] \mid f(0) \in L\}$; this is clearly a subring of $K$ which contains $R$, but it is neither a left nor a right chain ring. (Try to compare $t$ and $t\,a$, $a \in L_1 \backslash L$!)

For this reason we will *always restrict* our attention in this paper *to right chain overrings* $T \subseteqq Q(R) = K$. Further we will often assume that $R$ is of rank 1 in order to keep aside effects by localizations. Here we call a *right chain domain $R$ of rank* 1 if $J(R)$ is the only non-trivial prime ideal.

As can be seen from the ring $R$ in Example 1 we may have an infinite ascending chain of overrings $R^{t^i}$ which are not localizations. But not even the sublattice of those overrrings which are themselves chain domains is a chain.

Example 2. We consider again the right chain ring $R$ constructed as Example 1. The overrings $T_1 = L(x_{-1}^2)[[t, \sigma]]$ and $T_2 = L(x_{-1}^3)[[t, \sigma]]$ are right chain domains which are not comparable.

**2. Some easy properties for overrings.** As we will often assume that $R$ is of rank 1 the following observation is useful.

**Lemma 2.** *Let $R$ be a right chain domain of rank* 1, *$T$ an overring in $K$. Then $y^{-1} \in T$ with $y \in J(R)$ implies $T = K$.*

Proof. If $J(R)$ is not simple then $R$ is right invariant (use Prop. 3.1 in Bessenrodt et al. [2]); for chain rings see Brungs/Törner [5]). With Corollary 3.2 (Bessenrodt et al. [2]) we get for any $x \in J(R)$ $n \in \mathbb{N}$ and $r \in R$ satisfying $x\,r = y^n$, hence $x \in U(T)$. If $J(R)$ is simple, then the two-sided ideal $\bigcap_{u \in U(R)} u\,y\,R$ equals the zero-ideal and similarly as above $x \in U(T)$ follows; hence $T = K$.

Elements of $T$ are of the form $a\,b^{-1}$ with $a, b \in R$, $b \neq 0$. As $R$ is a right chain domain we have the possibilities $a\,r = b$ or $a = b\,s$ which lead to $a\,b^{-1} = a\,r^{-1}\,a^{-1}$ or $a\,b^{-1} = b\,s\,b^{-1}$.

**Theorem 3.** *Let $R$ be a r.i. right chain domain and $R^a = aRa^{-1}$ with $a \in R\backslash\{0\}$. Then the following assertions are valid:*

a) *$R \subseteqq R^a$ and $R \cong R^a$ for all $a \in R\backslash\{0\}$.*
b) *The inclusion $aR \supseteqq bR$ with $a, b \in R\backslash\{0\}$ implies $R^a \subseteqq R^b$, and equality holds if $aR = bR$.*
c) *Let $a, b \in R\backslash\{0\}$ with $aR \supseteqq bR$. If $x = ar^{-1}a^{-1} \in Q(R)$ with $r \in R$, then there exists $s \in R$ with $x = bs^{-1}b^{-1}$.*
d) *$\tilde{R} = \bigcup_{a \in J\backslash\{0\}} R^a$ is an invariant chain domain.*

Proof. a) The right invariance of $R$ implies that $R$ is a subring of $R^a$. Obviously $R$ and $R^a$ are isomorphic via $r \to ara^{-1}$.

b) Let $b = as$ with $s \in R\backslash\{0\}$. Then $bRb^{-1} = a(sRs^{-1})a^{-1} \supseteqq aRa^{-1}$ by a). The equality is obvious if $s \in U(R)$.

c) With similar arguments as in b).

d) As the rings $R^a$ form a chain, $\tilde{R}$ is a r.i. right chain domain. In order to prove $R$ to be a left chain domain consider elements $x = ara^{-1}$ and $y = asa^{-1}$ (using b)). If $s = rt$ for some $t \in R$ then $y = asa^{-1} = (asr^{-1}a^{-1})(ara^{-1}) = art(ar)^{-1}x$. To show that $\tilde{R}$ is left invariant use $xy = ara^{-1}asa^{-1} = arsa^{-1} = ars(ar)^{-1}ara^{-1} = y'x$, where $y' = ars(ar)^{-1}$.

From Theorem 3 b) we deduce:

**Corollary 4.** *Let $R$ be a r.i. right chain domain. If $0 \neq a \in R$ is a duo element and $aR \subseteqq bR$ then $b$ is duo. In particular: If $a$ lies in the minimal prime ideal $\neq (0)$ then $R$ is an invariant chain domain.*

Proof. By Theorem 3 b) $R \subseteqq R^b \subseteqq R^a = R$ as the element $a$ is duo, hence $R = R^b$ which implies that $b$ is duo. For the second assertion note that $\bigcap_n a^n R = (0)$.

The following example shows that we can not expect the overrings to be of rank 1, in fact the rank is not even bounded.

Example 3. Replace the monomorphism in Example 1 by $\sigma: x_i \to x_{i+n} (n \in \mathbb{N})$. Take the free abelian group $G$ generated by $\{x_{-n}, \ldots, x_{-1}\}$ and canonically linearly ordered with $1 < x_{-n} < \cdots < x_{-1}$. The quotient ring of the group ring $L[G]$ contains a valuation ring $N \supseteqq L[x_{-n}, \ldots x_{-1}]$ which can be obtained by the induced order valuation. The ring $T = N[[t, \sigma]]$ is a right invariant right chain domain of rank $n+1$ satisfying $R \subset T \subset Q(R)$.

**3. Overrings under restricted conditions.** It is natural to ask how the lattices of right ideals of $T$ and $R$ are related. Here we deduce conditions on the structure of $J(T)$ induced by $J(R)$.

**Lemma 5.** *Let $R$ be a right chain domain and $T$ an overring $\neq K$. Then we have: $\bigcap_{b \in J(R)\backslash\{0\}} bT = (0)$.*

Proof. Let $x \in \cap\, b\,T$. If $x = a\,r\,a^{-1}$ we have $(a\,r\,a^{-1})\,a = a\,r \in \cap\, b\,T$, hence $a\,r = 0$. If $x = a\,r^{-1}\,a^{-1}$ we get $(a\,r^{-1}\,a^{-1})\,a\,r = a \in \cap\, b\,T$, hence $a = 0$, a contradiction.

**Lemma 6.** *Let $R$ be a right chain domain of rank 1 and $T$ an overring. For any completely prime ideal $P \neq (0)$ in $T$ we have $J(R)\,T \subseteqq P$.*

Proof. Suppose $P \subset a\,T$ for some $a \in J(R)$. In case of $R$ right invariant we have $a^n \in P$ for some $n$. Contradiction. Otherwise $J(R)$ is simple (see Proof of Lemma 2) and $\bigcap_{u \in U(R)} u\,a\,R = (0)$. By Lemma 5 we obtain $u\,a \in P$ for a suitable $u \in U(R)$. Contradiction.

In the general case there might appear "bad" elements. In condition (A) we define one type of these elements.

(A) *There exist elements $a\,v^{-1}\,a^{-1} \in T$ with $v \in J(R)$, $a \in R$.*

If the ring $T$ is right invariant, elements of type (A) do not occur as can be seen from the equation $(a\,v^{-1}\,a^{-1})\,a\,v = a$.

Even more can be proved:

**Theorem 7.** *Let $R$ be a right chain domain of rank 1 and $T$ an overring satisfying* (A). *Then the section $P \supseteqq (0)$ is simple where $P$ is the minimal completely prime ideal over $J(R)\,T$.*

For a right ideal $I$ we denote by $\overline{I}$ resp. $\underline{I}$ the smallest (resp. largest) two-sided ideal containing $I$ (lying in $I$). A straightforward proof shows $\overline{I} = \bigcup_{u \in U(R)} u\,I$ resp. $\underline{I} = \bigcap_{u \in U(R)} u\,I$ (Bessenrodt et al. [2])

Proof. We set $\tilde{v} = a\,v^{-1}\,a^{-1}$ and obtain $\tilde{v}^n\,a\,v^n = a$ for arbitrary $n \in \mathbb{N}$. Suppose $\underline{a\,T} \neq (0)$. If $R$ is right invariant then $v^n \in \underline{a\,T}$ for some $n$. Otherwise $\bigcap_{u \in U(R)} u\,v\,R = (0)$, hence $u\,v \in \underline{a\,T}$ for some $u \in U(R)$ and thus $v \in \underline{a\,T}$. In both cases we obtain $\tilde{v}^n\,a\,v^n = a \in \underline{a\,T}$ which implies $Ta \subseteqq \underline{a\,T} \subseteqq a\,T$ contradicting $\tilde{v}\,a\,v = a$. Thus $\underline{a\,T} = (0)$. Since $R$ is a domain and $\overline{a\,T}/\underline{a\,T}$ is simple the two-sided ideal $\overline{a\,T}$ must be idempotent, hence $\overline{a\,T}$ is completely prime by Bessenrodt et al. [1, Theor. 2.4]. Applying Lemma 6 we know $J(R)\,T \subseteqq \overline{a\,T}$.

As locally archimedian right chain rings (Bessenrodt et al. [2]) are characterized by the fact that between any two prime ideals there is a further two-sided ideal, condition (A) is conflicting with $T$ being locally archimedian.

We do not have examples deciding whether $\overline{a\,T} = J(T)$ might hold or not. However, in case of $\overline{a\,T} \subset J(T)$, we may localize at $\overline{a\,T}$ which will lead to an overring $T' \neq K$ with exactly two two-sided ideals, namely (0) and $J(T')$.

Another pathological situation is described by the following condition:

(B) *There exist elements $a\,u\,a^{-1} \in J(T)$ with $u \in U(R)$, $a \in R$.*

As the semigroup of principal right ideals of a r.i. right chain domain of rank 1 is commutative a ring $T$ with (B) cannot be of rank 1 *and* right invariant.

What can be said about the consequences of condition (B)? If, in addition, condition (A) holds the prime segment $P \supseteqq J(R)T \supseteqq (0)$ (see Theorem 7) is simple. Let us now assume that there are no elements of type (A), further let $\tilde{u} = a u a^{-1} \in J(T)$ with $u \in U(R)$.

If $P = \overline{J(R)T}$, the assumption $\tilde{u} \in P$ implies $\tilde{u} = v x r$ with $v \in U(T)$, $x \in J(R)$ and $r \in T$. Without loss of generality we may assume $v = a v' a^{-1}$, $x = a x' a^{-1}$ and $r = a r' a^{-1}$ with $v' \in U(R)$, $x' \in J(R)$ (Theorem 3 b) keeping in mind that there are no elements of type (A). The equation $\tilde{u} = v x r$ implies $u = v' x' r'$, contradiction.

If $P \supset \overline{J(R)T}$ then $J\overline{(R)T}$ is not completely prime by definition of $P$. We are done if we can exclude $\overline{J(R)T}$ being prime using the same arguments as above. If we assume $\tilde{u} \in P \backslash \overline{J(R)T}$ and $\overline{J(R)T}$ prime there exists $v \in U(T)$ with $\tilde{u} v \tilde{u} \in \overline{J(R)T}$ (Bessenrodt et al. [1]). As we can assume $v = a v' a^{-1}$ with $v' \in U(R)$ (!) we obtain $a u v' u a^{-1} \in J(R)T$. Contradiction.

We summarize these observations:

**Theorem 8.** *Let $R$ be a r.i. right chain domain of rank* 1 *and $T$ an overring satisfying condition* (B). *Then we have the following possibilities.*

a) *$T$ has at least two non-trivial completely prime ideals.*

b) *The section $P \supseteqq (0)$ is simple where $P$ is the minimal completely prime ideal over $J(R)T$.*

From Theorem 7 and 8 we obtain

**Corollary 9.** *Let $R$ be a r.i. right chain domain and $T$ a r.i. overring of rank* 1. *Then we have*:

*$x \in T$ implies $x = a r a^{-1}$ with $a, r \in R$, $a \neq 0$ and $x \in U(T)$ if and only if $r \in U(R)$.*

**4. A structure theorem for right invariant overrings *T*.** For right invariant overrings $T$ we can prove a structure theorem which covers the examples known so far. As the right-invariance of $T$ contradicts condition (A) we can assume that the elements of $T$ are of the form $a r a^{-1}$ with $a, r \in R$. In other words $T$ is always contained in the chain domain $\tilde{R} = \cup R^a$ (Theorem 3).

We set: $T_a = T \cap R^a = \{x \in T \mid \exists r \in R: x = a r a^{-1}\}$, define $R_a = a^{-1} T_a a$. $T_a$ is a subring of $T$ and $R_a$ a subring of R. We remark:

(1) $$\{r \in R \mid \exists x \in R: x a = a r\} \subseteqq R_a \subseteqq R.$$

If $a u a^{-1} \in U(T)$, then $a u^{-1} a^{-1} \in T_a$ holds too, hence $U(T) \cap T_a = U(T_a)$.

By Theorem 3 we have $T_a \subseteqq T_b$ for $a, b \in R$ with $aR \supseteqq bR$, therefore $a R_a a^{-1} \subseteqq b R_b b^{-1}$ resp.

(2) $$s^{-1} R_a s \subseteqq R_b \quad \text{for } s \text{ with } a s = b.$$

Finally we note: $R_a = R \Leftrightarrow R^a \subseteqq T$ and $R_a = R \Rightarrow R_c = R$ with $cR \supseteqq aR$.

**Lemma 10.** *Let $R$ be a r.i. right chain domain of rank 1 and $T$ a r.i. overring. Then $R_a$ is a r.i. right chain domain.*

Proof. The ring $R_a$ is right invariant, if $T_a$ is right invariant, but $T_a$ is an intersection of the two right invariant domains $T$ and $R^a$. Let $r, s \in R_a$, so $x_r a = a r$, $x_s a = a s$ with $x_r, x_s \in T$. We may assume that $x_r y = x_s$ where $y$ is an element of $T$. As $T$ is right invariant we have $a s = x_s a = x_r a y = x_r a y' = a r y'$ with $y' \in T$, hence $s = r y'$. If for the elements $r, s \in R$ the equation $s v = r$ with $v \in R$ holds we obtain $v y' = 1$ which implies $y' \in R$ by Lemma 2. Because of $y a = a y'$ this proves $y \in T_a$ and $y' \in R_a$.

Now we are able to prove an "expansion" theorem:

**Theorem 11.** *Let $R$ be a r.i. right chain domain of rank 1 and $T$ a r.i. overring. Then for any $0 \neq a \in J(R)$ the rings $R_{a^n}$ form a descending chain of r.i. right chain subrings of $R$ and $T = \bigcup_{n \in \mathbb{N}} a^n R_{a^n} a^{-n}$ holds.*

Proof. First we show: $R_1 = R_{a^k} \supseteqq R_{a^{k+1}} = R_2$. Let $r \in R_2$, hence $x a^{k+1} = a^{k+1} r$ with $x \in T$, so $a x' a^k = a^{k+1} r$ for $x' \in T$ and thus $x' a^k = a^k r$ which means $r \in R_1$. As $R$ is a r.i. ring of rank 1 we have $\cap a^n R = (0)$ and Lemma 5 implies $\cap a^n T = (0)$. By Theorem 3 $\tilde{R} = \cup R^{a^n}$, hence $T = T \cap \tilde{R} = \cup (T \cap R^{a^n}) = \cup a^n R_{a^n} a^{-n}$.

The reader is advised to look again at Example 1, to see how overrings are built up in the way described by Theorem 11.

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Eingegangen am 27. 2. 1986

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