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RESEARCH ARTICLE

A STRUCTURE THEOREM FOR RIGHT INVARIANT RIGHT HOLOIDS

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Introduction

A totally ordered semigroup H with identity e which is left cancellative and right naturally ordered is called a right invariant right holoid, r.i.r. holoid for short. The extended positive cone $G^+ = \{g \in G \mid g \geq e\}$ of an ordered group G or the ordinal numbers less than a power of ω, the order type of the natural numbers, are examples of r.i.r. holoids. These semigroups have been discussed by KLEIN-BARMEN [7] in the commutative case, additional results were obtained by CLIFFORD [4] and CONRAD [5], and related structures were defined by SCHEIN [9] as left holoids. R.i.r. holoids of rank 1 are subsemigroups of the additive semigroup of the non-negative real numbers. The structure of rank two r.i.r. holoids is described in [1] and the right noetherian r.i.r. holoids are exactly the above mentioned semigroups $0_I = \{\alpha \mid \alpha < \omega^I\}$ of ordinals less than a power of ω, see [2]. The set $H(R) = \{aR \mid 0 \neq a \epsilon R\}$ of non-zero principal right ideals of a right chain domain R forms a semigroup with ideal multiplication as operation if and only if R is right invariant (i.e. all right ideals are two-sided). In this case, $H(R)$ is a r.i.r. holoid and these semigroups play for right invariant right chain rings the role, that ordered groups play for invariant chain rings and commutative ordered groups for commutative valuation domains.

Here, we prove a structure theorem (Theorem 3.1) for certain r.i.r. holoids of

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finite rank using the methods developed in [3]. All rank 2 r.i.r. holoids and all noetherian r.i.r. holoids of finite rank fall into this class. Essential is the existence of semigroups of representatives for certain factor structures of the given semigroup (Theorem 2.2).

1 Definitions and preliminary results

We recall some definitions and results from [3], for the standard notations and definitions see [8].

DEFINITION 1.1. *A totally ordered semigroup H with identity e is called a r.i.r. holoid if the following two conditions hold:*

(i) $hh_1 = hh_2$ *implies* $h_1 = h_2$ *for* $h, h_1, h_2 \in H$*; i.e. H is left cancellative.*

(ii) $h \geq h'$ *for* $h, h' \in H$ *if and only if there exists an element* $h'' \in H$ *with* $h = h'h''$*; i.e. H is right naturally ordered.*

It follows that a r.i.r. holoid H is *positively ordered*, since $h = eh$, i.e. $h \geq e$, for every $h \in H$. If a, b are elements in H then $ba \geq a$ and $ba = ab'$ for some $b' \in H$,. hence H is *right invariant*, i.e. every right ideal in H is two-sided. We observe that a *right cancellative r.i.r. holoid H* can be embedded into the extended positive cone of an ordered group G.

Throughout the paper we assume that *convex subsemigroups* contain the identity e. The ideal I will be called *prime* if $A \cdot B \subseteq I$, A, B ideals, implies $A \subseteq I$ or $B \subseteq I$. Again, a prime ideal P in a r.i.r. holoid H will be always *completely prime*, i. e. $a, b \notin P$ implies $a \cdot b \notin P$ for $a, b \in H$. The complement $S = H \backslash P$ of a (completely) prime ideal $P \neq H$ in H is a *convex subsemigroup.* We remark that in r.i.r. holoids all prime ideals are completely prime.

We say the r.i.r. holoid H has *rank n* if there exist exactly n prime ideals $\neq H$ in H, i.e. we have the following chain of distinct prime ideals:

$$H = P_0 \supset P_1 \supset P_2 \supset ... \supset P_n$$

and a corresponding chain of convex subsemigroups:

$$\{e\} = S_0 \subset S_1 \subset S_2 \subset ... \subset S_n = H$$

where $P_i = H \backslash S_{i-1}, i = 1, ..., n$.

Since only the left cancellation law is assumed for a r.i.r. holoid H, it can happen that $E(a) = \{x \epsilon H \mid xa = a\} \neq \{e\}$ for some element $a \in H$. This set $E(a)$ of elements absorbed by a is a convex subsemigroup of H. If $E(a) \neq \{e\}$ the set $E(a)$ is strictly contained in each convex subsemigroup containing a. The set $A(H) = \{a \in H \mid E(a) \neq \{e\}\}$ is a prime ideal of H or $\emptyset$, called the *absorber radical* of H. Exactly in the case $A(H) = \emptyset$ is H embeddable into a group.

DEFINITION 1.2. *Let H be a r.i.r. holoid and S a convex subsemigroup with $S' \supset S$ its immediate upper neighbour in the lattice of convex subsemigroups of H. We say that there is a jump at S if $E(a) \supset E(s)$ for some $a \in S' \backslash S$ and all $s \in S$. A jump at S is complete if $E(b) = S$ for all $b \in S' \backslash S$, i.e. $sb = b$ for all $s \in S, b \in S' \backslash S$.*

2 Semigroups of representatives and denominator sets

We recall from [3, Theorem 4.3] the definition of a special type of factor structure which is central in the following. Let S be a convex semigroup of a r.i.r. holoid H that satisfies $Sa \subseteq aS$ for all $a \in P = H \backslash S$. The set of classes $[a] = \{x \in H \mid \exists s \in S : xs = a \text{ or } \exists s \in S : x = as\}$ with $[a][b] = [ab]$ as operation is a r.i.r. holoid, denoted by H/S (Theorem 4.3 in [3]). The additional assumption $Sa \subseteq aS$ for all $a \in H \backslash S$ holds automatically in the finite-rank-case ([3, Theorem 3.7]).

The following lemma gives information on the factor structures originating in complete jumps.

LEMMA 2.1. *Let H be a r.i.r. holoid of finite rank. Let H have jumps at S_1 as well as at $S_1 \subset S_2$ or $S_2 = H$, but no jumps at S for all convex subsemigroups S with $S_1 \subset S \subset S_2$. Then S_2/S_1 exists and is right cancellative.*

PROOF: The existence of S_2/S_1 follows from Theorem 4.3 in [3], S_2/S_1 is a r.i.r. holoid and thus left cancellative. Furthermore S_2/S_1 is right cancellative: Assume that $[t_2][s_2] = [s_2]$ for elements $s_2, t_2 \in S_2 \backslash S_1$. This implies $t_2 s_2 = s_2 s_1$ or $t_2 s_2 s_1 = s_2$ for some element $s_1 \in S_1$. The first of these equations leads to $t_2 \in E(s_2^m)$ for large enough $m \in \mathbb{N}$ (Theorem 3.8 in [3]), and the second equation implies $s_2 = t_2 s_2 s_1 = s_2 t_2' s_1$, with $t_2 s_2 = s_2 t_2'$, for some $t_2' \in S_2$, hence $t_2' s_1 = e, t_2' = e = s_1$, and $t_2 \in E(s_2)$. As there is no jump between $S_1 \subset S_2$ we have $E(s_2) = E(s_2^m)$, hence $E(s_2) = E(s_2^m) \subseteq S_1$ and thus $[t_2] = [e]$. This observation is sufficient to prove that S_2/S_1 is right cancellative. Let be $[h_1][h] = [h_2][h]$; w.l.o.g. assume $[h_1][r] = [h_2]$, hence $[h_1][h] = [h_1][r][h]$. As the r.i.r. holoid S_2/S_1 is left cancellative we obtain $[h] = [r][h]$ and by the previous arguments $[r] = e$ follows, which implies $r \in S_1$ and therefore $[h_1] = [h_2]$. Therefore S_2/S_1 is right cancellative and embeddable into an ordered group. ∎

Let S_2, S_1 as in Lemma 2.1. We then call $D = S_2/S_1$ a *factor* of H and denote by $Q(D) = G = \{ab^{-1} \mid a, b \in D\}$ the ordered group of quotients of D. Then D is not necessarily equal to $G^+ = \{ab^{-1} \mid a \geq b, a, b \in D\}$.

It was proved in [3, Theorem 5.2] that there exists a semigroup of representatives in the r.i.r. holoid H for H/S provided there is a complete jump at the maximal convex subsemigroup $S \subset H$. These semigroups of representatives are not uniquely determined, on the contrary for any element h with $[h] \neq e$ there exists at least one system containing h. In that case, H/S is a rank 1 r.i.r. holoid and as such isomorphic to a (commutative) subsemigroup of $(\mathbb{R}^+, +)$, the additive semigroup of non-negative real numbers. This result will now be used to show that there exists a subsemigroup of representatives in a r.i.r. holoid of finite rank for *every* factor S_2/S_1 of H, provided the jump at S_1 is complete.

THEOREM 2.2. *Let H be a r.i.r. holoid with a convex subsemigroup T such that*

(i) *$sp = p$ for all $s \in T$ and $p \in H \backslash T$.*

(ii) *H/T is a r.i.r. holoid of finite rank which is right cancellative.*

Then there exists in H a subsemigroup R of representatives for H/T.

We remark that a convex subsemigroup T satisfying (i) and (ii) defines a jump which is complete by definition.

PROOF: We denote by $[h] = \{k \in H \mid kt = h \text{ or } k = ht \text{ for some } t \in T\}$ the elements in H/T where $h \in H$. Let T' be the convex subsemigroup in H minimal above T and $D' = T'/T$ exists and is isomorphic to a subsemigroup of $(\mathbb{R}^+, +)$. By the above quoted result (Theorem 5.2 in [3]) there exists in T', and hence in H, a semigroup R_1 of representatives of T'/T. In order to find an extended set of representatives let $h \in H \backslash T'$ be arbitrarily and choose $e \neq p \in R_1 \cap T'$. We have $h = ph_1$ for some $h_1 \in H \backslash T'$ and $h = ph_1 = h_1 p'$ for some $p' \in H$ follows. By Theorem 3.8 in [3] the elements p' lies in the same prime segment as p. However,

$[h] = [p][h_1] = [h_1][p'] \in H/T$, which is a right cancellative r.i.r. holoid of finite rank. Hence, $[p] \neq [e]$ implies $[p'] \neq [e]$ and $[p'] \in D'$ using Corollary 3.6 and Theorem 3.8 in [3]. Then there exists an element $e \neq r \in R_1$ with $[p'] = [r]$ and $[h_1 r] = [h]$ follows.

Let $h \in H \backslash T'$ and assume that elements $h_1 r_1, h_2 r_2$ exist in $[h]$ with $r_1, r_2 \in R_1$. We can assume that $r_1 \geq r_2$ and $r_1 = v r_2 = r_2 v$ follows for some $v \in R_1$, as R_1 is commutative. Hence, $[h_1 r_1] = [h_1 v r_2] = [h_2 r_2]$ and $[h_1 v] = [h_2]$ since H/T is assumed to be right cancellative. Therefore, $h_1 v$ and h_2 differ by a factor $s \in T$. W.l.o.g. assume $h_1 v s = h_2$, hence $h_2 r_2 = h_1 v s r_2 = h_1 v r_2 = h_1 r_1$ since the jump at T is complete. We conclude that there is exactly one element in $[h]$ of the described form.

We define $h_1 r$ as the representative of $h \in H \backslash T'$ if $h_1 r \in [h]$ and $r \neq e \in R_1$; the elements in R_1 are the representatives of the elements in T'. We proved that to every element in H a unique representative has been assigned and it is easy to check that the representative of a product is the product of the representatives of the factor. We obtain $R = R_1 \cup \{e \neq hr \mid r \in R_1, h \in H \backslash T'\}$ as a system of representatives of H/T in H. ∎

In the situation of Theorem 2.2 it is not enough to know T and the semigroup of representatives R of H/T in order to describe H. An element $h \in H$ is of the form $h = rs$ or $hs = r$ for $r \in R$, $s \in T$. The elements r and s are uniquely determined by h in each case, however s and r do not determine h uniquely in the second case. We recall the following definition from [3, Definition 6.1]:

DEFINITION 2.3. *Let P be a prime ideal in the r.i.r. holoid $H, S = H \backslash P$, $p \in P$. We write $N(p, S) = \{s \in S \mid$ there exists $x \in H$ with $xs = p\}$ and say $N(p, S)$ is the set of denominators s of p with respect to S. The set of solutions $L(p, s) = \{x \in H \mid xs = p\}$ is called a quotient set.*

We are interested in the set of denominators $N(p, T)$ where $p \in R$ in the situation of Theorem 2.2 and prove a generalization of Theorem 6.2 in [3].

THEOREM 2.4. *Let H and T be as in Theorem 2.2 and R a semigroup of representatives in H for H/T as constructed in Theorem 2.2. Then $N(p, T) = N(q, T)$ for all $p \neq e \neq q$ in R.*

PROOF: If $p \neq e \neq q$ are elements in R_1 the statement follows from the quoted result: Theorem 6.2 in [3]. It is therefore enough to show that $N(p, T) = N(q, T)$

for $e \neq p \in R_1$ and $q \in R \backslash R_1$. We know that $q = h_1 r$ for some $e \neq r \in R_1, h_1 \in H$. We will show $N(q,T) = N(r,T)$, which will prove the theorem. Let $s \in N(r,T)$, i. e. $xs = r$ for some $x \in H$, hence $h_1 xs = h_1 r = q$ and $N(r,T) \subseteq N(q,T)$ follows.

If conversely $ys = q = h_1 r$ for some $y \in H, s \in N(q,T)$ then either $h_1 = yt$ or $h_1 t = y$ for some $t \in H$.

We have $ys = h_1 r = ytr, s = tr \notin T$ a contradiction in the first case. The other possibility $h_1 t = y$ implies $h_1 ts = ys = h_1 r$ and $ts = r$, i. e. $s \in N(r,T)$ and $N(q,T) = N(h_1 r, T) = N(r,T)$ follows and proves the theorem. ∎

The semigroup R of representatives of H/T in Theorem 2.2 is by no means unique. One of the key arguments in the proof of the main theorem is the changing of the semigroup of representatives in order to change the common set of denominators. We list here an easy result which will be used later several times.

PROPOSITION 2.5. *Let R be the semigroup of representatives of H/T in Theorem 2.2, d an element in T. Then $\tilde{R} = \{rd \mid e < r \in R\} \cup \{e\}$ is again a semigroup of representatives in which all elements $\neq e$ have the same set of denominators.*

PROOF: Since the jump defined by T is complete it follows immediately that $\tilde{R}$ is again a semigroup of representatives for H/T. If we replace R_1 in the proof of Theorem 2.2 by $\tilde{R}_1 = \{r_1 d \mid e < r_1 \in R_1\} \cup \{e\}$ then $\tilde{R}_1$ is a semigroup of representatives of T'/T and $N(\tilde{r}_1, T) = N(\tilde{r}_2, T)$ for any $\tilde{r}_1 \neq e \neq \tilde{r}_2$ in $\tilde{R}_1$. However, $\tilde{R}$ is the semigroup of representatives of H/T obtained by the proof of Theorem 2.2 if we begin with $\tilde{R}_1$ in place of R_1. Theorem 2.4 then shows that the elements $\neq e$ in $\tilde{R}_1$ all have the same set of denominators with respect to T. ∎

We will also need the following process for changing R, the semigroup of representatives of H/T. Let s be a common denominator for the elements $r \neq e$ in R (by Theorem 2.4 without loss of generality). Choose one $r \neq e$ in R_1 arbitrary and one $x \in H$ with $xs = r$; s is of course an element in T. We write $\chi(r) = x$. Let r' be another element $\neq e$ in R_1. If $r' = kr$ for $k \in R_1$ - we use the fact that R_1 is commutative - then $\chi(r') = kx$. If on the other hand $r = kr' = xs$, $s \in T$, we must have $x > k$ and $x = kx'$ for some uniquely determined $x' \in H$. The equation $kr' = kx's$ implies $r' = x's$ and we put $\chi(r') = x'$. The set $\{\chi(r) \mid e \neq r \in R_1\} \cup \{e\} = R_1''$ is a semigroup of representatives for

T'/T, i.e. it can replace R_1. We apply the proces of Theorem 2.2 to obtain a semigroup of representatives, but start with R_1'' instead of R_1. Let R'' be the resulting semigroup of representatives. It follows that if $h_1 r \in [h]$ was the representative of $[h]$ in R then $r = \chi(r) \cdot s$ and $h_1\chi(r)$ is the representative of $[h]$ in R'' where $r \neq e$ and hence $\chi(r) \neq e$. It follows from Theorem 2.4 that all elements $\neq e$ in R'' have the same set of denominators with respect to T. We summarize these results.

PROPOSITION 2.6. *Let R be a semigroup of representatives for H/T constructed as in Theorem 2.2. Assume $s \in T$ is a common denominator for all elements $r \neq e$ in R. Then one can choose for each $r \neq e$ in R an element $\chi(r) \in H$ with $\chi(r)s = r$ such that $R'' = \{\chi(r) \mid e \neq r \in R\} \cup \{e\}$ is again a semigroup of representatives of H/T in H such that all elements $\neq e$ in R'' have the same set of denominators with respect to T.*

Even though R'' is not uniquely determined by R and s we occasionally use the notation Rs^{-1} instead of R''. We recall and reprove the following result from [3]:

PROPOSITION 2.7. *The set $N(r,T)$ of denominators of r in T is left naturally ordered in H.*

PROOF: Let $x_1 s_1 = r = x_2 s_2$. Then $x_1 = x_2 y$ or $x_1 y = x_2$ for $y \in H$. This implies $ys_1 = s_2$ or$s_1 = ys_2$. ■

3 The structure of r.i.r. holoids of finite rank with complete jumps only

We assume in this section that H is a r.i.r. holoid of finite rank in which every jump is complete. Let

$$H = T_0 \supset T_1 \supset \dots \supset T_{n-1} \supset T_n$$

be the chain of convex subsemigroups defining the jumps and let $D_i = T_i/T_{i-1}, i = 1, \dots, n$, be the factors of H.

As it was observed earlier we can embed D_i into the extended positive cone $G_i^+ = \{ab^{-1} \mid a, b \in D_i, a \geq b\}$ of the ordered group $Q(D_i) = G_i = \{ab^{-1} \mid a, b \in D_i\}$. We define a semigroup $\hat{H} = \{(e, \dots, e, d_i, g_{i-1}, \dots, g_1) \mid d_i \in G_i^+, g_j \in G_j, j < i, d_i > e \text{ for } i > 1, i = 1, \dots, n\}$ with the operation

$$\begin{aligned}(e, ..., e, d_i, g_{i-1}, ..., g_1)&(e, ..., e, d'_j, g'_{j-1}, ..., g'_1) \\ &= (e, ..., e, d'_j, g'_{j-1}, ..., g'_1) && \text{for } j > i; \\ &= (e, ..., e, d_i d'_j, g'_{j-1}, ..., g'_1) && \text{for } j = i; \\ &= (e, ..., d_i, g_{i-1}, ..., g_{j+1}, g_j d'_j, g'_{j-1}, ..., g'_1) && \text{for } j < i.\end{aligned}$$

This operation is best described as *lexicographically absorbing.*

The semigroup $\hat{H}$ has a subsemigroup $H_0 = \{(d_n, ..., d_1) \in \hat{H} \mid d_i \in D_i, i = 1, ..., n\}$. Checking the defining axioms one proves the following result:

PROPOSITION 3.1. *H_0 and $\hat{H}$ are r.i.r. holoids of the same rank as H in which all jumps are complete and the factors of H_0 are isomorphic to the factors of H, and the factors of $\hat{H}$ are isomorphic to $G_i^+, (i = 1, ..., m)$.*

The main result states that H is isomorphic to a subsemigroup H' of $\hat{H}$ with $H_0 \subseteq H' \subseteq \hat{H}$.

THEOREM 3.2. *Let H be a r.i.r. holoid of finite rank in which all jumps are complete. Then there exists a semigroup monomorphism ϕ from H into $\hat{H}$ such that every element of H_0 has a preimage in H.*

PROOF: We use the notation as defined at the beginning of this section and induction on the number of complete jumps n. The statement of the theorem is true for $n = 1$ and we assume that it is true for $T_{n-1} = T$ in place of H.

We therefore have a monomorphism ψ from T into $\hat{T}$ and $D_1, ..., D_{n-1}$ or $G_1^+, ...G_{n-1}^+$ respectively are the factors of T and $\hat{T}$. Using identification we can assume that the $D_i's, i = 1, ..., n-1$ are subsemigroups of T and hence of H. Of course, $H/T \cong D_n$ and by Theorem 2.2 there exists a semigroup R' of representatives of H/T in H. We choose an element $d_{n-1} \neq e$ in D_{n-1} and replace R' by $R = \{r'd_{n-1} \mid e \neq r' \in R'\} \cup \{e\}$. By Proposition 2.7, R is again a semigroup of representatives of H/T in H and every element $r \neq e$ in R has the same set of denominators with respect to T.

Let h be any element in H. As observed earlier, there exist $r \in R$, $s \in T$ with $h = rs$ or $hs = r$ and s and r are uniquely determined. We are mainly concerned with the second possibility. The equation $hs = r$ means that $s \in N(r, T)$. By construction we have $d_{n-1} \in N(r, T)$ and by Proposition 2.7 we know that the set $N(r, T)$ is left naturally ordered in T. The element s is therefore equal to yd_{n-1} or ys equals d_{n-1} for some $y \in T$. By induction, the statement of the theorem holds for T which implies that $N(r, T) \subseteq \bigcup_{i=1}^{n-1} D_i$.

Assume $N(r,T)$ contains a largest element, say d, in D_{n-1}. Using Proposition 2.6 we form $Rd^{-1} = R_{n-2}$ where the initial $r \neq e$ in R is chosen arbitrarily, and x with $xd = r$ is equal to $x = x'd_{n-2}$ for an arbitrary $x' \in H$ with $x'd = r$ and $d_{n-2} \neq e$ in D_{n-2}.

It follows that R_{n-2} is a semigroup of representatives for H/T whose elements $\neq e$ have common denominator sets contained in $\bigcup_{i=1}^{n-2} D_i$.

We repeat this argument and end up with one of the following two possibilities:

(A) H/T has a semigroup R_0 of representatives in H whose denominator sets consist of $\{e\}$.

(B) H/S has a semigroup R_m of representatives in H with $N(r,T) \subseteq \bigcup D_i, i = 1, ..., m$ such that $\emptyset \neq N(r,T) \cap D_m$ does not contain a largest element for $r \neq e$.

The proof of Theorem 3.1 is easily finished in the situation (A). For every $h \in H$ there exists an $s \in T$ and an $r \in R_0$ with $h = rs$. We define $\phi(r) = (r, e, ..., e) \in \hat{H}$ and $\phi(s) = (e, \psi(s))$. Finally $\phi(h) = \phi(r)\phi(s) = (r, \psi(s))$ in $\hat{H}$ defines a mapping from H to $\hat{H}$ that satisfies the statements of the theorem.

We are left with the situation (B) where we write R instead of R_m. By construction, $N(r,T) \subseteq \bigcup_{i=1}^{m} D_i$ for $e \neq r$ in R and $N(r,T) \cap D_m$ is not empty and does not contain a largest element. Let $h \in H$ with $hs = r, r \in R, s \in T$. Then there exists an element $d \in N(r,T) \cap D_m$ with $d > s$; assume $xd = r$ for $x \in H$. We have $x < h$ (the other possibility leads to a contradiction) and $h = xs'$ for some $s' \in T$ follows.

We will now show that for every $d \in N(r,T) \cap D_m$ we can choose $x \in H$ with $xd = r$ such that the mapping ϕ defined by $\phi(x) = (r, e, ..., e, d^{-1}, e, ..., e) \in \hat{H}, d^{-1}$ appears in the m^{th} component counted from the right, $\phi(s) = (e, \psi(s)), s \in T$, and $\phi(h) = \phi(x)\phi(s')$ for $h = xs', s' \in T$, satisfies the statements of the theorem.

To choose the element x with image $(r, e, ..., e, d^{-1}, e, ..., e)$ in the solution set $L(r,d)$ let $x'd = r$ for some $x' \in H$ and we pick an element $d' > d$ in $N(r,T) \cap D_m$. There is an element $y \in H$ with $yd' = r$. Since $x'd = r$, there is an element $t \in T$ with $x' = yt$ and $td = d'$ follows. We have $\psi(t) = (e, e, ..., e, k, g_{m-1}, ..., g_1)$ in $\hat{T}$ with $k \neq e$ in D_m and $kd = d'$ follows. We now define $\phi(yk) = (r, e, ..., e, d^{-1}, e, ..., e)$, i. e. $x = yk$, with $xd = ykd = r$, is the chosen element in $L(r,d)$.

To prove that x is uniquely determined by r and d we observe first that it does not depend on the choice of $x' \in L(r,d)$. Let x'' be another element in $L(r,d)$, then x' and x'' differ by an element $s \in E(d)$. Hence, if $x' = yt, x'' = yt', t, t' \in T$, then both $\psi(t) = (e, e, ..., e, k, g_{m-1}, ..., g_1)$ and $\psi(t') = (e, ..., e, k, g'_{m-1}, ..., g'_1)$ lead to the same k and the same $x = yk$.

Next, let $x'd = r, yd' = ykd = r, zd'' = ztd = r$ with $d, d', k, d'', t \neq e$ in $D_m, r \neq e$ in $R, x', y, z \in H$. We claim: $yk = zt$.

If $k = t$, then $ykd = zkd, ys = z$ or $zs = y$ for some $s \in E(kd) = E(k) = E(t)$. Therefore, $yk = zt$ in this case.

If $k > t$, then $y < z$ and $z = yt'_1$ for some $t'_1 \in H$. We obtain $r = ykd = ztd = yt'_1td$ and $kd = t'_1td$ by cancelling y from the left in H. As before there exists in D_m an element t_1 with $kd = t'_1td = t_1td$ and, cancelling d from the right in D_m, we obtain $k = t_1t$, which in turn equals t'_1t. Hence $yk = yt_1t = yt'_1t = zt$, which shows that the element x which is mapped to $(r, e, e, ..., e, d^{-1}, e, ..., e)$ is uniquely defined. We call this element rd^{-1}.

To extend the mapping ϕ to all of H we saw that it is sufficient to define ϕ on the elements $(rd^{-1}) \cdot s$ for $s \in T$.

We define: $\phi[(rd^{-1})s] = \phi(rd^{-1})\phi(s)$.

To check that this is well-defined, let $h = (r_1d_1^{-1})s_1 = (r_2d_2^{-1})s_2$ for $r_1, r_2 \neq e \in R, d_1, d_2 \in D_m \cap N(r_1, T), s_1, s_2 \in T$. It follows that $r_1 = r_2 = r$ and if $d_1 = d_2$ then $s_1 = s_2$ and there is nothing to prove. Otherwise we may assume $d_2 > d_1$ and $d_2 = dd_1$ for some $d \neq e$ in D_m follows, since d_1 and d_2 as denominators of r are left naturally ordered.

For $x_1 = rd_1^{-1}, x_2 = rd_2^{-1}$ the equations $x_1d_1 = r = x_2d_2 = x_2dd_1$ imply by definition $x_1 = x_2d$.

Next assume that $\phi(s_1) = (e, \psi(s_1)) = (e, ..., e, d'_i, g'_{i-1}, ..., g'_1)$ and $\phi(s_2) = (e, \psi(s_2)) = (e, ..., e, d''_j, g''_{j-1}, ..., g''_1)$, $i, j \leq n-1$. We compare
$\phi(x_1)\phi(s_1) = (r, e, ..., e, d_1^{-1}, e, ..., e)\ (e, ..., e, d'_i, g'_{i-1}, ..., g'_1)$ with
$\phi(x_2)\phi(s_2) = (r, e, ..., e, d_2^{-1}, e, ..., e)\ (e, ..., e, d''_j, g''_{j-1}, ..., g''_1)$ where d_1^{-1} and d_2^{-1} both appear in the m^{th} position (counted from the right).

We know that $x_1s_1 = x_2ds_1 = x_2s_2$, which implies $ds_1 = s_2$. Hence,
if $i > m, d'_i > e$, then $ds_1 = s_1 = s_2$ and $\phi(x_1)\phi(s_1) = \phi(r)\phi(s_1) = \phi(r)\phi(s_2) =$

$\phi(x_2)\phi(s_2)$.

If $i = m, d'_i > e$, then $dd'_i = d''_j$ with $j = m, g'_t = g''_t$ for $t < m$. In the m^{th} position of $\phi(x_1)\phi(s_1)$ appears $d_1^{-1}d'_i$ which is equal to $d_2^{-1}d''_j = d_1^{-1}d^{-1}dd'_1$, since $d_2 = dd_1$.

If $i < m$ or $s_1 = e$, then $j = m, d''_j = d$ and $d_1^{-1} = d_2^{-1}d$. This shows that $\phi(x_1)\phi(s_1) = \phi(x_2)\phi(s_2)$ if $h = x_1s_1 = x_2s_2$. It must be checked that the mapping ϕ is one-to-one and multiplicative.

To show that ϕ is one-to-one let $h = xs, x = r_1d_1^{-1}, h' = yt, y = r_2d_2^{-1}, r_1, r_2 \in R$, $d_1, d_2 \in D_m \cap N(r_1, T)$, $s, t \in T$ and assume that $\phi(h) = \phi(h')$. It follows immediately that $r_1 = r_2 = r$. If $d_1 \neq d_2$, say $d_2 > d_1, d_2 = dd_1, e \neq d \in D_m \cap N(r, T)$ (Proposition 2.7), then again $x = yd$. Hence, $\phi(h) = \phi(xs) = \phi(yds) = \phi(y)\phi(ds)$ and $\phi(h') = \phi(y)\phi(t) = \phi(y)\phi(ds)$ implies $\phi(t) = \phi(ds), t = ds$ by induction and $h = h'$.

The multiplicativity of the mapping ϕ will follow from the following two properties:

(*) $\phi(srd^{-1}) = \phi(s)\phi(rd^{-1})$ for $s \in T, e \neq r \in R, d \in D_m \cap N(r, T)$.
(**) $\phi((r_1d_1^{-1})(r_2d_2^{-1})) = \phi(r_1d_1^{-1})\phi(r_2d_2^{-1})$ for $e \neq r_1, r_2 \in R, d_1, d_2 \in D_m \cap N(r, T)$.

Equation (*) follows trivially, since $xd = r$ implies $x \in H\backslash S$ and $sx = x$; both sides in (*) equal $\phi(rd^{-1})$.

To prove (**) let $x_1 = r_1d_1^{-1}, x_2 = r_2d_2^{-1}$ and $x_1x_2d_2 = r_1r_2$ follows. We need to show that $x_1x_2 = (r_1r_2)d_2^{-1}$: Let $d_3 > d_2$ and $d_3 \in D_m \cap N(r_2, T)$ and $yd_3 = r_2$ for some $y \in H$. We have $d_3 = vd_2$ for $e \neq v$ in D_m and $x_2 = yv$ by definition. This leads to $x_1x_2d_2 = x_1yvd_2 = x_1yd_3 = r_1r_2$. It follows that $x_1x_2 = x_1yv = (r_1r_2)d_2^{-1}$ or equivalently $\phi(x_1x_2) = \phi(x_1)\phi(x_2)$.

We are now in a position to show that $\phi(hh') = \phi(h)\phi(h'), h, h' \in H$. The equality follows by induction if both h, h' are in T. If $h' \in H\backslash T$, $h \in T$, then the equality follows from (*). If $h \in H\backslash T$, then $h = xs, x = r_1d_1^{-1}, r_1 \neq e \in R$, $d_1 \in D_m \cap N(r_1, T)$ and $\phi(xst) = \phi(x)\phi(st) = \phi(x)\phi(s)\phi(t) = \phi(xs)\phi(t)$ i. e. $\phi(hh') = \phi(h)\phi(h')$ in case $h' = t \in T$, or $h' \in H\backslash T$, $h' = yt, y = r_2d_2^{-1}, e \neq r_2 \in R$, $d_2 \in D_m \cap N(r_2, T)$, $t \in T$ and $\phi(hh') = \phi(xsyt) = \phi(xyt) = \phi(xy)\phi(t) = \phi(x)\phi(y)\phi(t) = \phi(x)\phi(s)\phi(y)\phi(t) = \phi(h)\phi(h')$, using (**). ∎

It is clear from the above proof that in order to describe the r.i.r. holoids in Theorem 3.1 more precisely, it is necessary to determine the various possibilities for denominator sets at each jump.

It would be interesting to have a construction for right invariant right chain domains R such that $H(R)$ is a given r.i.r. holoid of finite rank with complete jumps only.

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