Forum Math. 2 (1990), 307–322

Forum Mathematicum

On Two-Sided Right Ideals in Chain Domains

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(Communicated by Rüdiger Göbel)

Abstract. A chain ring R is a (local) ring whose lattices of right ideals resp. left ideals are chains. With each two-sided ideal I we associate two completely prime ideals $P_l(I)$ resp. $P_r(I)$. Equality of these two prime ideals implies interesting conditions on the two-sided ideal I which is then called P-symmetric. It is shown that the fact that I is two-sided as well as that I is P-symmetric transfers to all right ideals with the same associated right prime ideal in a neighbourhood of I. In particular, specific two-sided ideals which are associated with a fixed completely prime ideal $P_l(I) = P_r(I) = P$ define a linearly ordered semigroup $\Gamma^+(P)$. The connection between the semigroups $\Gamma^+(P)$ for neighbouring prime ideals is analysed. A class of examples is described.

1980 Mathematics Subject Classification (1985 Revision): 16A10; 16A15.

In analogy to the commutative case, Schilling [7] defined a valuation ring as a (not necessarily commutative) ring without zero-divisors whose one-sided ideals are two-sided and linearly ordered by inclusion. More generally, for the notion of chain domains one only assumes that the lattices of right resp. left ideals are chains. Until recently, algebraic structure theorems for the ideal lattice of such a ring were only discussed in situations which were close to commutativity, for example for right invariant right chain domains; this analysis lead to the study of holoids (see [3]).

In this paper we start an investigation on the algebraic structure of suitable subsets of the lattice of two-sided ideals for an arbitrary chain ring R, that is, without imposing any commutativity conditions. Given a right ideal I and a completely prime ideal $P = R \setminus S$ of R, we consider the right ideals IP and IS^{-1}. These right ideals and their relationship are studied in the first section. Then we proceed with a discussion of some results on their associated prime ideals. Section 2 is devoted to an analysis of left-right phenomena for two-sided ideals with respect to their generating type. Both of these

* Supported by the Deutsche Forschungsgemeinschaft

concepts are then used to prove that in a suitably defined neighbourhood of a two-sided ideal all right ideals with the same associated prime ideal are also two-sided (Theorem 3.4). Also the fact that a two-sided ideal I is P-symmetric transfers to all two-sided ideals in a neighbourhood of I (Theorem 3.8). Of particular importance are *standard P-associated right ideals*; provided these ideals are two-sided we also have precise information on the left associated prime ideal for a neighbourhood. Generalising from the situation of invariant chain domains, in Section 4 we introduce certain subsets of P-symmetric ideals of R which are proved to be positively and linearly ordered semigroups (Theorem 4.3). Finally, we give an example to show how the semigroups of two neighbouring prime ideals are related.

1. Some arithmetic for right ideals and associated prime ideals

First we want to fix some notations, and we will also recall some definitions. For further unexplained terminology we refer the reader to [1] or [2].

All rings have an identity, and they are not necessarily commutative. The *Jacobson radical* of the ring R is denoted by $J = J(R)$, and $U = U(R)$ stands for the group of *units*. Rings may have zero-divisors, otherwise they are called *domains*. *Prime ideals P* are two-sided ideals $\neq R$ defined by the property that for right ideals I, L the inclusion $IL \subseteq P$ always implies $I \subseteq P$ or $L \subseteq P$. A prime ideal P is called *completely prime* if for all $x, y \in R$ with $xy \in P$ we have $x \in P$ or $y \in P$.

Definition 1.1. A ring R is called *right* (*left*) *chain ring* if for any $a, b \in R$ we have $aR \subseteq bR$ or $bR \subseteq aR$ ($Ra \subseteq Rb$ or $Rb \subseteq Ra$). If R is both a right and left chain ring, it is called a *chain ring*.

We will now define some sets and ideals associated with a given right ideal. In the following, R will always be a right chain ring and I a right ideal of R.

For $s \in R$ we set

$$Is^{-1} = \{x \in R \mid xs \in I\}.$$

This set contains I and is closed under addition, but in general it is not a right ideal. Obviously, if I is a two-sided ideal, then Is^{-1} is a left ideal. By the following construction we do get a right ideal associated with I:

Lemma 1.2. *Let R be a right chain ring, I a right ideal of R, P a left ideal of R and $S = R \setminus P$. Then $I \subseteq IS^{-1} = \bigcup_{s \in S} Is^{-1}$ is a right ideal of R.*

Proof. Take $x \in IS^{-1}$, so $xs \in I$ for some $s \in S$. Let $r \in R$. If $rs \in sR$, then $xrs \in xsR \subseteq I$, so $xr \in IS^{-1}$. In the other case, $s = rst$ for some $t \in R$, and then $st \in S$ because P is a left ideal. Thus $xs = xr(st) \in I$ implies $xr \in IS^{-1}$. □

We make a few easy observations on the right ideal IS^{-1} above.

Remark 1.3. (a) *Clearly, if I is a two-sided ideal then so is IS^{-1}.*

(b) *If $I \cap S \neq \emptyset$, then $1 \in IS^{-1}$ and $IS^{-1} = R$. Therefore we will usually assume that I is contained in $P = R \setminus S$.*

(c) *Let $P = R \setminus S$ be a completely prime ideal. Then IS^{-1} is closed under S-quotients, i.e. $(IS^{-1})S^{-1} = IS^{-1}$.*

(d) *Furthermore, one can easily check that for any right ideals I_1, I_2 of R with $I_1 \subseteq I_2 S^{-1}$ we have $I_1 S^{-1} \subseteq I_2 S^{-1}$.*

For the following investigations we will always take for P a completely prime ideal of R, and we set again $S = R \setminus P$.

Of course, given a right ideal I and a completely prime ideal $P \supseteq I$ as above we can also consider the associated right ideal IP. To clarify the relationship between IS^{-1} and IP we introduce one further right ideal associated with I and P:

$$\mathfrak{T}(I) = \bigcap_{x \in P} (IP)x^{-1} = \{r \in R \mid rP \subseteq IP\}$$

In other words, if I is a two-sided ideal, $\mathfrak{T}(I)$ is the annihilator of the left R-module P/IP.

Lemma 1.4. *Let R be a right chain ring, $P = R \setminus S$ a completely prime ideal and I a right ideal.*

(i) *$(IS^{-1})P = IP$.*

(ii) *If $IP \subset I$, then $\mathfrak{T}(I) = IS^{-1}$.*

(iii) *IP is a two-sided ideal if only if IS^{-1} is a two-sided ideal.*

Proof. (i) Let $x \in IS^{-1}$, say $xs \in I$ for $s \in S$. Hence $xP = xsP \subseteq IP$.

(ii) Suppose $x \in \mathfrak{T}(I)$, so $xP \subseteq IP$. Since $IP \subset I$, this immediately implies that $xs \in I$ for some $s \in S$, i.e. $x \in IS^{-1}$. The inclusion $IS^{-1} \subseteq \mathfrak{T}(I)$ follows from (i).

(iii) If IS^{-1} is two-sided then so is $IP = (IS^{-1})P$. If IP is a two-sided ideal, then obviously so is $\mathfrak{T}(I)$. Hence IS^{-1} is also a two-sided ideal. □

Corollary 1.5. *Let R be a right chain ring, $0 \neq a \in R$, $P = R \setminus S$ a completely prime ideal. Then the following assertions are equivalent:*

(a) *$(aR)S^{-1}$ is a two-sided ideal of R.*

(b) *aP is a two-sided ideal of R.*

(c) *$a \notin RaP$.*

Proof. Lemma 1.4(i) implies (a) ⇒ (b), (b) ⇒ (c) is obvious. We only have to show that (c) implies (a). Let $x \in (aR)S^{-1}$, $r \in R$. Take $s \in S$ with $xs \in aR$, say $xs = ab$. If $rxs \notin aR$, then for some $v \in R$: $a = rxsv = rabv$, and hence $bv \in S$ by (c). Thus $v \in S$ and so $sv \in S$, which proves that $rx \in (aR)S^{-1}$. □

In the corollary above we have described the relationship between two rather special right ideals associated with a principal right ideal aR. As we will see in the following these ideals play an important role. Also then the reason for the term P-associated will become clear. We will call $(aR)S^{-1}$ and aP *standard P-associated* right ideals. In the case where the completely prime ideal P is not idempotent, the two types are the same. Note that P equals $(pR)S^{-1}$ for each $p \in P \setminus P^2$.
The following observation is easy to check:

Corollary 1.6. *Suppose $0 \neq aR \subseteq bR$. Then the following assertions are equivalent:*

(a) *$a = bs$ for some $s \in S$.*
(b) *$(aR)S^{-1} = (bR)S^{-1}$.*
(c) *$aP = bP$.*

Elements $a, b \in R^*$ satisfying the conditions above are called *S-right-associated*, and we abbreviate this as $a \sim_S b$. Similarly, we define the left hand version. If I is a right ideal with $IP \subset I$, then all elements in $I \setminus IP$ are S-right-associated. In particular, $IS^{-1} = (aR)S^{-1}$ for all $a \in I \setminus IP$.

Now we introduce a second important concept, the associated prime ideals for a two-sided ideal I of a right chain ring R. The left resp. right associated prime ideals are defined by:

$$P_l(I) = \{p \in R \mid \exists t \notin I \text{ with } pt \in I\}$$
$$P_r(I) = \{p \in R \mid \exists t \notin I \text{ with } tp \in I\}$$

Using the same terminology as above we can also describe $P_l(I)$ (and symmetrically $P_r(I)$) as follows:

$$P_l(I) = \{x \in R \mid I \subset x^{-1}I\} = \bigcup_{t \notin I} It^{-1} = I(R \setminus I)^{-1}$$

These notions were first introduced in [8], independently they were used in the domain case in [6].

It is obvious that I is contained in $P_l(I)$ and $P_r(I)$. To prove that $P_r(I)$ is a completely prime ideal we only need I to be a right ideal (see [1]). In general, the two associated prime ideals for I are different; examples can be found in [8]. But we note that for any completely prime ideal P we clearly have $P_l(P) = P = P_r(P)$.

For various types of right ideals we can describe their associated prime ideals. Some of the results below can be found in [1]. As the proofs are straightforward we omit them:

Lemma 1.7. *Let R be a right chain ring and P a completely prime ideal with $P = R \setminus S$. Then the following holds:*

(i) *For all $a \in R$ with $aP \neq 0 : P_r(aP) = P$.*
(ii) *For all $0 \neq a \in J : J = P_r(aJ)$.*

(iii) *If R is a chain ring, then for all $0 \neq a \in P$: $P_r((aR)S^{-1}) = P$, in particular $P_r(aR) = J$.*

The following lemma provides a link between the associated ideals:

Lemma 1.8. *Let R be a right chain ring, $P = R \setminus S$ a completely prime ideal and I a right ideal of R. Then we have:*

(i) *$P_r(I) \subseteq P$ if and only if $I = IS^{-1}$.*
(ii) *$P_r(I) \supset P$ if and only if $I \subset IS^{-1}$. In this case we have $IS^{-1} = (aR)S^{-1}$ for some $a \notin I$.*
(iii) *Let $0 \neq IP \subset I$. Then for any $a \in I \setminus IP$ we have $IP = aP$ and $P_r(IP) = P$ follows.*
(iv) *Let $IP \neq 0$. Then $P_r(IP) \subseteq P$.*

Note that the condition $IP \subset I$ is satisfied for any non-zero principal right ideal $I = aR$.

Proof. (i) Suppose $P_r(I) \subseteq P$ and take $x \in IS^{-1}$, so $xs \in I$ for some $s \in S$. As $s \notin P_r(I)$, this implies $x \in I$. Conversely, assume $I = IS^{-1}$. If $x \in P_r(I)$, then $tx \in I$ for some $t \notin I$. Since $I = IS^{-1}$, we must have $x \in P$.

The first part of (ii) follows from (i). In this case take $a \in IS^{-1} \setminus I$. Then $I \subseteq aR \subseteq (aR)S^{-1} \subseteq IS^{-1}$ and by Remark 1.3 $IS^{-1} \subseteq (aR)S^{-1} \subseteq IS^{-1}$ follows.

(iii) follows from Corollary 1.6 and the preceding remark, and Lemma 1.7.

(iv) By (iii) we may assume $I = IP$. Take $s \in R \setminus P$ and assume $xs \in IP$, so $xs = yp$ for some $y \in I, p \in P$. If $x \notin I$, then $y \in xR$ and hence $xs \in xP$. This implies $xs = 0$ and hence $IP \subseteq xP = xsP = (0)$, a contradiction. So $x \in I$, and thus $P_r(IP) \subseteq P$ is proved. □

The lemma above emphasizes the importance of the standard P-associated right ideals. This is also brought out by the next proposition.

If I has P as its associated prime ideal, then it can be approximated by standard P-associated right ideal ideals:

Proposition 1.9. *Let R be a right chain ring, $P = R \setminus S$ a completely prime ideal and I a right ideal with $P_r(I) = P$. Then the following holds:*

(i) *I is the union of right ideals of the type $(aR)S^{-1}$, $a \in I$, that is $I = \bigcup_{a \in I} (aR)S^{-1}$.*
(ii) *I is the intersection of right ideals of the type aP for all $aP \supseteq I$, that is $I = \bigcap_{a \notin I} aP$.*

Proof. (i) follows directly from $P_r(I) = P$.

(ii) By definition of $P_r(I)$, $I \subseteq aP$ for all $a \notin I$. On the other hand, if $x \notin I$ then $\bigcap_{a \notin I} aP \subseteq xP \subset xR$, so equality holds.

Remark 1.10. *If I is a two-sided ideal with $P_r(I) = P$ and $Q = R \setminus T$ is an arbitrary completely prime ideal with $I \subseteq Q \subseteq P$, then we deduce immediately from Lemma 1.7 (i) and Lemma 1.8 (ii) that IT^{-1} is a two-sided Q-associated ideal. So in the prime segment containing I there are also two-sided Q-associated ideals for any such Q.*

From Lemma 1.8 above we conclude:

Corollary 1.11. *Let R be a chain ring, $P = R \setminus S \supseteq Q = R \setminus T$ completely prime ideals and $P_r(I) = P$.*

(i) *Suppose $I \subseteq Q$. Then IT^{-1} is the smallest right ideal containing I with Q as its right associated prime ideal.*
(ii) *Suppose $IQ \neq (0)$. Then IQ is the largest right ideal contained in I with Q as its right associated prime ideal.*
(iii) *There exists $a \in I$ with $IQ = aQ \subset I \subset (aR)T^{-1} = IT^{-1}$.*
(iv) *If L is a right ideal with $aQ \subset L \subset (aR)T^{-1}$, then $Q \subset P_r(L)$.*

2. Standard P-associated right ideals as left ideals

If R is a chain ring and IP is a two-sided ideal, then it is an ideal of the same type from the left. More precisely:

Lemma 2.1. *Let R be a right chain ring, P a completely prime ideal of R, and I a right ideal of R such that $0 \neq IP \subset I$ is a two-sided ideal of R. Take $z \in I \setminus IP$ with $zP \neq 0$ and set $P_1 = (IP)z^{-1}$. Then we have:*

(i) *P_1 is a completely prime ideal of R.*
(ii) *If R is a chain ring, then $P_1 z = P_1 I = IP = zP$.*

Proof. (i) Clearly P_1 is a left ideal of R. First we want to prove that P_1 is also right ideal of R. Take $r \in R$ and $x \in P_1$, so $xz \in IP$. If $rz \in zR$, then $xrz \in xzR \subseteq IP$, and so $xr \in P_1$. Otherwise $z = rzv$ with $v \in J \setminus P$. If $xrz \notin zR$, then $z = xrzy$ with $y \in J \setminus P$. Now $y \in vR$ leads to $z \in xrzvR = xzR \subseteq IP$, a contradiction. Thus $v = yw$ for some $w \in S$, and then $zw = xrzv = xz \in IP$, which leads again to a contradiction. Hence, we must have $xrz \in zR$. If $xrz = zs$ with $s \in S$, then $xz = xrzv = zsv \notin IP$ – a contradiction. So $xrz \in zP \subseteq IP$, and thus $xr \in P_1$. It remains to show that P_1 is completely prime. If $x \notin P_1$, then $xz = zs$ for some $s \in S$, or $z = xzt$ for some $t \in S$. In the first case, $x^2 z = zs^2 \notin IP$, so $x^2 \notin P_1$. In the other case $z = x^2 zt^2$, so again $x^2 z \notin IP$ and thus $x^2 \notin P_1$. By [1] this implies that P_1 is completely prime.

(ii) The inclusion $P_1 I \subset IP$ is obvious. Now take $x \in IP$; since R is also a left chain ring, $x = rz$ for some $r \in R$. By definition, this implies $r \in P_1$, so $x \in P_1 z \subseteq P_1 I$. □

From the result above we easily deduce that the same relationship holds also on the next level downwards:

Corollary 2.2. *Let R be a chain ring and Q be the lower neighbour of P as a prime ideal and assume $IQ \neq 0$. Then we have $IQ = Q_1 I$, where $Q_1 = (IQ)z^{-1}$ is the lower neighbour (as a prime ideal) of P_1.*

Similar results can also be obtained for the other associated ideals:

Lemma 2.3. *Let R be a right chain ring, P a completely prime ideal and I a two-sided ideal with $0 \neq IP \subset I \subseteq P$. Take $z \in I \backslash IP$ and set $S_1 = \{t \in R \mid z \in t(IS^{-1})\}$. Then the following holds:*

(i) *$P_1 = R \backslash S_1$ is a completely prime ideal.*
(ii) *If R is a chain ring, then $S_1^{-1}(Rz) = IS^{-1}$.*

Proof. (i) First we prove that S_1 is multiplicatively closed. So let $v, w \in S_1$, say $z = vx = wy$ with $x, y \in IS^{-1} \subseteq P$. If $w \in xR \subseteq IS^{-1}$, then $ws \in I$ for some $s \in S$. Suppose $z = wst$ for some $t \in S$, then $z = wy = wst \in wP$ implies $z = 0$, a contradiction. But $ws \in zR \subseteq wP$ is also impossible. Hence $x = wr$ for some $r \in J$. Now $z = vx = vwr$, so it suffices to show that $r \in IS^{-1}$. If not, then $y \in rP$ and $z = wy \in wrP = xP \subseteq IP$, contradiction.

By [1] it remains to show that P_1 is a right ideal. Let $t \in P_1$, $r \in R$. If $t = zx$ for some $x \in J$, then $tr = zxr \in zJ$, so $tr \notin S_1$. In the other case, $z = tx$ for some $x \notin IS^{-1}$. If $tr \in S_1$, then $z = try$ for some $x \in IS^{-1}$, implying $t \in T$, because IS^{-1} is an ideal by Lemma 1.4. This shows that we must have $tr \in P_1$.

(ii) Take $x \in IS^{-1}$. If $z = tx$, then by definition $t \in S_1$, so $x \in S_1^{-1}(Rz)$. Otherwise $x \in Rz \subseteq S_1^{-1}(Rz)$. Conversely, take $y \in S_1^{-1}(Rz)$, so $ty \in Rz$ for some $t \in S_1$. If $y \in Rz$, we are done because IS^{-1} is an ideal. So $z \in Ry$, and thus we have $z = sy$ for some $s \in S_1$. By definition, there exists $x \in IS^{-1}$ such that $z = sx$, hence we must have $y \in IS^{-1}$, because $z \neq 0$. □

Corollary 2.4. *In the situation of 2.3 (ii), let $Q = R \backslash T$ be the lower neighbour of P as a prime ideal, $Q_1 = R \backslash T_1$ the lower neighbour of P_1 as a prime ideal. Then $T_1^{-1}(Rz) = IT^{-1}$.*

For principal right ideals we have the following close connection between the associated ideals:

Lemma 2.5. *Let R be a chain ring, $P = R \backslash S$ and $P_1 = R \backslash S_1$ completely prime ideals of R. Let $a \in P$ be such that aP is a non-zero two-sided ideal. Then we have*

$$P_1 a = aP \text{ if and only if } S_1^{-1}(Ra) = (aR)S^{-1}.$$

Proof. Suppose $aP = P_1 a$ and take $y \in S_1^{-1}(Ra) \backslash Ra$, so $a = ty$ for some $t \in S_1$. If $aP \subset yP$, then $as \in yP$ for some $s \in S$. Thus $tas \in tyP = aP = P_1 a = tP_1 a = taP$, which implies $0 = taP = aP$ – a contradiction. So $yP \subseteq aP$ and hence $y \in (aR)S^{-1}$ by Lemma 1.4(ii). By symmetry, we have $S_1^{-1}(Ra) = (aR)S^{-1}$.

For the converse, we note first that $P_1 a = aP_1'$, where $P_1' = a^{-1}(P_1 a)$ is a completely prime ideal (by Lemma 2.1). If $P_1 a \nsubseteq aP$, then $qa = ap$ for some $q \in P_1$, $p \in S$. As $a \in P$, $a = xp$ for some $x \in (aR)S^{-1}$. So $ap = qxp$, which implies $Ra = Rqx$, since otherwise $ap = 0$ and thus $aP = 0$. But as $x \in S_1^{-1}(Ra)$, there exists $t \in S_1$ such that $tx \in Ra \subseteq P_1 x$, leading to $P_1 x = 0$, a contradiction. Hence we must have $P_1 a \subseteq aP$, and then by symmetry $P_1 a = aP$. □

If the right ideals $(aR)S^{-1}$ resp. aP are two-sided we can also consider their left associated prime ideals. The following lemma serves as a helpful observation.

Lemma 2.6. *Let R be a chain ring, P a completely prime ideal and $a \in R$ such that $0 \neq aP$ is a two-sided ideal. Then $P_l(aP) = \{x \in R \mid xa \in aP\}$.*

Proof. If $x \in P_l(aP)$, then $xas \in aP$ for some $s \in S = R \setminus P$. Suppose $a = xar$ for some $r \in S$. If $r \in sR$, then $a \in xasR \subseteq aP$. On the other hand $s = rt$ implies $xas = xart = at \in aP$, which is again a contradiction since $t \in S$. So we have $xa = ar$ for some $r \in R$. Now from $xas = ars \in aP$ we deduce $r \in P$, thus $xa \in aP$. The converse inclusion is obvious. □

The following lemma gives some informations on the connections between the associated prime ideals:

Lemma 2.7. *Let R be a chain ring, $P = R \setminus S$ a completely prime ideal and $a \in P$. Assume $(aR)S^{-1}$ is a two-sided ideal (or equivalently, that aP is two-sided). Then the following assertions are equivalent:*

(a) $P_l((aR)S^{-1}) = P_1 = R \setminus S_1$.

(b) $(aR)S^{-1} = S_1^{-1}(Ra)$.

If $aP \neq (0)$, then these are also equivalent to:

(c) $P_l(aP) = P_1 = R \setminus S_1$.

(d) $aP = P_1 a$.

Proof. The equivalence of (a) and (b) follows from Lemma 2.3, Lemma 1.7 and their symmetric versions. For the last equivalences use Lemma 2.5 and Lemma 2.1. □

In the lemma above, the situation where $P = P_1$ is particularly nice. Let I be a two-sided ideal; we call it *P-symmetric* if $P_r(I) = P_l(I) = P$ holds for some completely prime ideal P. Consequently, an element $a \in R$ is P-symmetric if $Pa = aP$ is satisfied. Schilling's assumption of invariance for valuation rings R can be rewritten as the fact that all elements of R are J-symmetric.

Recall that an element $a \in R$ is *right invariant* if the right ideal aR is two-sided. For such elements we state explicitly the following consequence:

Corollary 2.8. *Let R be a chain ring and $a \in R$ right invariant with $aJ \neq 0$. Then the following conditions are equivalent:*

(a) $aR = Ra$.

(b) $aJ = Ja$.

(c) $P_l(aR) = J$, *i.e.* aR *is* J*-symmetric.*

(d) $P_l(aJ) = J$, *i.e.* aJ *is* J*-symmetric.*

Now it is natural to ask: under which conditions do we have globally $P_l((aR)S^{-1}) = P$? An answer in this direction is given in ([9], Theorem 2.1): If the prime ideals of the chain ring R satisfy the minimum condition, then $P_l(I) = P_r(I)$ for all two-sided ideals I of R. As mentioned earlier, there are examples of chain ring with $P_l(I) \neq P_r(I)$ for suitable two-sided ideals I (see [8]).

3. The neighbourhood of two-sided ideals

In this section we want to study ideals with the same associated prime ideal. Starting from a given two-sided ideal I we look at ideals in a "neighbourhood" of I, a notion that will be made precise below. If $P = R\setminus S$ is a completely prime ideal and $0 \neq I = aR \subseteq P$ a principal right ideal, then we know from Lemma 1.7 and Lemma 1.8 that $P_r(IS^{-1}) = P = P_r(IP)$, but that there is no ideal between IP and IS^{-1} with P as its right associated prime ideal. So the neighbourhood should be larger than this segment. If we want to have similar boundaries, but with another prime ideal Q, we know by Lemma 1.8 that we should take $Q \subset P$. In fact, we will take $Q = R\setminus T$ to be the lower neighbour of P as a prime ideal.

In this Section 3 we generally denote by $Q = R\setminus T$ the lower neighbour of P as a prime ideal. Furthermore, we will assume that there is a further two-sided ideal between Q and P, i.e. P/Q is not simple.

We quote the following result on this setup from [1]:

Lemma 3.1. *Let R be a right chain ring and $Q \subset P$ neighbouring prime ideals such that P/Q is not simple. Then the following holds:*

(i) *Q is completely prime.*

(ii) *For all $a \in P\setminus Q$: $\bigcap_{n\in\mathbb{N}} a^n R = Q$.*

Remark 3.2. *In the situation of Lemma 3.1, we have the following useful fact: Let $p \in P\setminus Q$, $r \in R$. Then $p \notin rpP$. Furthermore, if $r \notin P$, then $rp \notin pP$.*

Now we want to give the precise definition of neighbourhood that will be used in our investigations. As above, we set $T = R\setminus Q$.

Definition 3.3. Let R be a right chain ring and I be a right ideal of R with $P_r(I) = P$. Then the *Q-neighbourhood* $U_Q(I)$ is defined by

$$U_Q(I) = \{X \mid X \text{ right ideal of } R,\ IQ \subseteq X \subseteq IT^{-1}\}.$$

Note that $P_r(I) = P$ implies $I \subseteq P$ and that in the case $Q \subset I \subseteq P$ we obtain $U_Q(I) = \{X \mid X$ right ideal of R, $Q \subseteq X \subseteq R\}$. Furthermore we observe that the boundaries of $U_Q(I)$ are of the form aQ and $(aR)T^{-1}$, where $a \in I \setminus IQ$ (see Corollary 1.11).

What is the range of all right associated prime ideals of right ideals in a neighbourhood $U_Q(I)$? With the notations above, let R be a chain ring and Q a completely prime ideal. If $Q \subset I$, then obviously we have $Q \subset P_r(X)$ for all right ideals X with $IQ \subset X \subset IT^{-1}$. If $I \subseteq Q$ and if $IQ \neq 0$ use Corollary 1.11 (i) and (ii) to obtain the same result. Furthermore, $X \neq IQ, IT^{-1}$ is in this neighbourhood if and only if $XT^{-1} = IT^{-1}$ resp. $XQ = IQ$, and it is easy to check that in this case $U_Q(X) = U_Q(I)$ holds. In particular, for any principal right ideals $aR, bR \in U_Q(I)$ we have $a \sim_T b$. Using Lemma 1.7, we know that the union and the intersection of all P-associated ideals $X \in U_Q(I)$ gives the Q-associated ideals IT^{-1} and IQ; this should be kept in mind for this section. Furthermore, these ideals share some properties with I. For example, all these right ideals are two-sided if I is:

Theorem 3.4. *Let R be a chain ring. Let I be a two-sided ideal with $P_r(I) = P$. Then all right ideals $X \in U_Q(I)$ with $P_r(X) = P$ are also two-sided.*

Proof. If X is not two-sided, then $ua \notin X$ for some $u \in R$, $a \in X$. So $a = uar$ for some $r \in R$, and by definition $r \in P_r(X) = P$.

Case 1: $I \subset X \subset IT^{-1}$.
Since $ua \in IT^{-1}$, we have $uat \in I$ for some $t \notin Q$. Hence, by Lemma 3.1, $r^n \in tR$ for sufficiently large $n \in \mathbb{N}$. Now $a = u^n ar^n \in u^n atR \subseteq I$, since I is two-sided – a contradiction.
Case 2: $IQ \subset X \subset I$.
Take $b \in IT^{-1} \setminus I$, so $a = bq$ for some $q \in P \setminus Q$. Now let L denote a two-sided ideal with $Q \subset L \subset P$. By Lemma 3.1, $r^m \in L$ for some $m \in \mathbb{N}$, and $L^k \subset qR$ for some $k \in \mathbb{N}$, so for sufficiently large n we have $Rr^n \subseteq qJ$. Furthermore, since $I \subset bP \subset IT^{-1}$ and $P_r(bP) = P$, bP is a two-sided ideal by Case 1. Thus we deduce: $a = u^n ar^n = u^n bqr^n \in bPr^n \subseteq bqJ = aJ$, so $a = 0$, a contradiction. □

The following is an interesting consequence of Theorem 3.4 and a generalisation of a well-known fact: In the situation where R is a right chain domain of rank 1, the existence of at least one two-sided ideal $I \neq J$, (0) implies R to be right invariant.

Corollary 3.5. *Let R be a chain ring. Then all right ideals X with $Q \subset X$ and $P_r(X) = P$ are two-sided.*

Proof. For any $a \in P \setminus Q$ the right ideal aP is two-sided by Corollary 1.5(b) and Remark 3.2. Since X is in the Q-neighbourhood of aP, Theorem 3.4 implies the statement. □

If I is a two-sided ideal, then all the standard P-associated right ideals in the Q-neighbourhood of I are two-sided:

Corollary 3.6. *Let R be a chain ring and I a two-sided ideal satisfying $P_r(I) = P$. Then there are standard P-associated right ideals $(aR)S^{-1}$ and bP in the Q-neighbourhood $U_Q(I)$, and all these standard ideals in $U_Q(I)$ are two-sided.*

Proof. The first assertion follows from Proposition 1.9, and the second is an immediate consequence of Corollary 1.11 and Theorem 3.4. □

Thus the neighbourhood of any two-sided P-associated ideal is the neighbourhood of a two-sided standard P-associated ideal $(aR)S^{-1}$ resp. bP. So in the same setup as above, we may choose I to be of the form $(aR)S^{-1}$ or aP. Note that aP and $(aR)S^{-1}$ have the same neighbourhood. Using Corollary 1.6 it is easy to characterise the standard P-associated ideals in the Q-neighbourhood of I:

Lemma 3.7. *Let R be a chain ring, $0 \neq a, x \in R$. Then the following are equivalent:*

(a) $(xR)S^{-1} \in U_Q((aR)S^{-1})$.
(b) $xP \in U_Q(aP)$.
(c) $xQ = aQ$.
(d) $x \sim_T a$.

All the standard P-associated ideals in the Q-neighbourhood of a two-sided standard P-associated ideal have the same *left* associated prime ideal. In particular, the standard P-associated ideals in the Q-neighbourhood of a P-symmetric standard ideal are also P-symmetric.

Theorem 3.8. *Let R be a chain ring. Let $a \in P$ be such that $0 \neq aP$ is a two-sided ideal. If $a \sim_T x$, then xP is a two-sided ideal and $P_l((aR)S^{-1}) = P_l(aP) = P_l(xP) = P_l((xR)S^{-1})$. In particular, if the element a is P-symmetric, so is x.*

Proof. That xP is a two-sided ideal follows from Theorem 3.4 and Lemma 3.7. By Lemma 2.7, it suffices to show that $P_l(aP) = P_l(xP)$. W.l.o.g. we may assume $x = at$ for some $t \in P \backslash Q$. Take $y \in P_l(aP)$, so $ya \in aP$ by Lemma 2.6. If $ya \in xP$, then $y \in P_l(xP)$. So assume $ya \in aP \backslash xP$, say $ya = ap$ for $p \in P$. Now $p^n \in tP$ for some $n \in \mathbb{N}$, and then $y^n a = ap^n \in atP = xP$. Since $a \notin xP$, this implies $y^n \in P_l(xP)$ and hence $y \in P_l(xP)$, because this is a completely prime ideal. So $P_l(aP) \subseteq P_l(xP)$. Conversely, suppose $y \in P_l(xP)$, so $yx \in xP$. We want to show: $ya \in aP$. If $a = yar$ for some $r \in R$, then we must have $r \in S$ and so $t = rv$ for some $v \in P \backslash Q$. Now $yx = yat = av \in xP = atP$ implies $v \in tP = rvP$, contradicting Remark 3.2. On the other hand, $ya = ar$ with $r \in R$ leads to $yx = yat = art \in atP$, so $rt \in tP$ and by Remark 3.2 this implies $r \in P$. Thus $ya \in aP$. □

Corollary 3.9. *Let R be a chain ring and I a two-sided ideal with its right associated prime ideal $P_r(I) = P$ as the smallest prime ideal containing I. Then $P_l(I) = P$, in other words, any such two-sided ideal I is P-symmetric.*

Proof. Clearly we have $P \subseteq P_l(I)$. Conversely, let $s \in S = R \setminus P$ and assume $sx \in I$. If $sx = xt$ for some $t \in S$, then we get immediately $x \in I$. In the case $sx \in xP$, we employ Remark 3.2 to obtain $x \in I$. If $x \in sxR \subseteq I$, we are obviously done. □

We know that we can find standard ideals in the neighbourhood of any ideal. The theorem above leads to the question whether we even have P-symmetric standard ideals in the neighbourhood of a P-symmetric ideal.

Theorem 3.10. *Let R be a chain ring, and let I be a P-symmetric two-sided ideal with $IQ \neq 0$. Then there are P-symmetric standard ideals $(aR)S^{-1} = S^{-1}(aR)$ and $bP = Pb$ in $U_Q(I)$, and all the standard P-associated ideals in $U_Q(I)$ are P-symmetric.*

Proof. By Theorem 3.8 we only have to show that there are P-symmetric standard ideals in $U_Q(I)$. From Proposition 1.9 we have $I = \bigcup_{a \in I}(aR)S^{-1}$. Using Corollary 3.6 and 3.8 we find a subset $I_1 \subseteq I$ such that $I = \bigcup_{a \in I_1}(aR)S^{-1}$ and $(aR)S^{-1}$ is a two-sided ideal for all $a \in I_1$. By Lemma 2.7 and Theorem 3.8, we obtain a prime ideal $P_1 = R \setminus S_1$ with $(aR)S^{-1} = S_1^{-1}(Ra)$ for all $a \in I_1$. Thus $I = \bigcup_{a \in I_1} S_1^{-1}(Ra)$, and this immediately implies $P = P_l(I) \subseteq P_1$. Hence we also have $I = \bigcup_{a \in I_1} S^{-1}(Ra)$.

Again, Corollary 3.6 tells us that there is a subset $I_2 \subseteq I_1$ such that $S^{-1}(Ra)$ is two-sided for all $a \in I_2$ and still $I = \bigcup_{a \in I_2} S^{-1}(Ra)$. By Lemma 2.7, we get a prime ideal $P_2 = R \setminus S_2$ with $S^{-1}(Ra) = (aR)S_2^{-1}$ for all $a \in I_2$, and as before we conclude $P \subseteq P_2$. Hence $(aR)S^{-1} = S_1^{-1}(Ra) \subseteq S^{-1}(Ra) = (aR)S_2^{-1} \subseteq (aR)S^{-1}$, and we deduce $S = S_1 = S_2$. With similar arguments the assertion is proved for the P-standard ideals of type bP; here we need the assumption $IQ \neq 0$. □

4. Semigroups of two-sided ideals

For the convenience of the reader we restrict from now on our considerations to chain domains, although under weak additional assumptions most of the results hold even in arbitrary chain rings.

In the classical theory of valuations ([7]) the principal right ideals of the invariant valuation ring form a semigroup. In the valuation rings considered by Schilling principal right ideals are two-sided and, using our notation, they are J-symmetric. It is not difficult to check an analogous assertion for arbitrary chain domains.

Lemma 4.1. *Let R be a chain domain, and let $a, c \in R$, $c \neq 0$, be invariant elements, thus $Ra = aR$, $Rc = cR$ and w.l.o.g. $aR \supseteq cR$. Then there exists an invariant element $b \in R$ satisfying $Rc = RaRb = aRbR = cR$.*

These features carry over to our situation and first we obtain

Lemma 4.2. *Let R be a chain domain and A, B non-zero P-symmetric ideals. Then AB is also a P-symmetric ideal.*

Proof. First we prove the inclusion $P_r(AB) \subseteq P$. Let $s \in S = R \setminus P$ with $xs \in AB$, so $xs = ab$ for some $a \in A$, $b \in B$. As $B \subseteq P$, we have $b = rs$ for some $r \in R$, and hence $r \in B$, because $P = P_r(B)$. Thus $xs = ab = ars$, and so $x = ar \in AB$. For the converse inclusion, we may assume $AB \subset AP$ by Lemma 1.7(iii). Then take $x \in AP \setminus AB$ and find $p \in P$ with $xp \in AB$, so $P \subseteq P_r(AB)$. The same arguments show that $P_l(AB) = P$. □

In the set of all P-symmetric ideals we do not have cancellation in general. For example, take a commutative chain domain of rank 1 with an idempotent radical J, then we have $J \cdot Ra = J \cdot Ja$. For the following investigations we will consider a more restricted class of P-symmetric ideals.

Now we define for each completely prime ideal P of a chain domain R three types of semigroups and thus obtain three series of semigroups of two-sided ideals in a chain domain R. Let P be a completely prime ideal and set

$$\Gamma^+(P) = \{(aR)S^{-1} \mid a \in R \setminus \{0\},\ P_l((aR)S^{-1}) = P\}.$$

Thus $\Gamma^+(P)$ consists of the P-symmetric ideals $(aR)S^{-1}$, for which we have by Lemma 2.7: $(aR)S^{-1} = S^{-1}(Ra)$. Hence, $\Gamma^+(J)$ consists of all invariant elements of R^* and the quotient group $Q(\Gamma^+(J)) = \Gamma(J) = \Gamma$ corresponds to the value group in the sense of Schilling.

If P is an idempotent completely prime ideal, we also define

$$\Theta^+(P) = \{aP \mid a \in R \setminus \{0\},\ P_l(aP) = P\}.$$

Note that for a non-idempotent prime ideal P any ideal of type aP is equal to an ideal of the form $(bR)S^{-1}$ (see the remark after the definition of standard right ideals). On the other hand, if P is idempotent, then no ideal of the form aP is also of the form $(bR)S^{-1}$.

Moreover, for idempotent P we define:

$$\hat{\Theta}^+(P) = \{I \mid I \neq (0),\ P_l(I) = P = P_r(I) \text{ and } I \neq (aR)S^{-1} \text{ for all } a \in R\}$$

As mentioned above, the definitions are a generalisation from the classical situation. They are also justified by the following theorem which is an analogue of a corresponding classical result:

Theorem 4.3. *Let R be a chain domain and P a completely prime ideal. Then $\Gamma^+(P)$ is a positively and naturally ordered semigroup with cancellations on both sides. If P is, in addition, idempotent, then the same holds for $\Theta^+(P)$ and $\hat{\Theta}^+(P)$.*

The proof is split into a number of lemmas which contain even more precise informations.

Lemma 4.4. *Let R be a chain domain and P a completely prime ideal. Then $\Gamma^+(P)$ and $\Theta^+(P)$ are semigroups with cancellation on both sides. Furthermore, $\Gamma^+(P)$ and $\Theta^+(P)$ are isomorphic.*

Proof. Let $a, b \in P$ with $(aR)S^{-1}$, $(bR)S^{-1} \in \Gamma^+(P)$. Clearly we have $(abR)S^{-1} \subseteq (aR)S^{-1} \cdot (bR)S^{-1}$. We want to use Lemma 1.4. Now $(aR)S^{-1}(bR)S^{-1} \cdot P = (aR)S^{-1} \cdot bP \subseteq (abR)S^{-1} \cdot P = abP$, because $(aR)S^{-1} \cdot b \subseteq (abR)S^{-1}$ as is easily checked. Thus $(aR)S^{-1} \cdot (bR)S^{-1} \subseteq (abR)S^{-1}$, and hence $\Gamma^+(P)$ is a semigroup. The other assertion can be proved in a straightforward manner. Note that by Corollary 1.6 each P-symmetric standard ideal of type $(aR)S^{-1}$ induces a P-symmetric ideal ot type aP and vice versa. □

Obviously, $\hat{\Theta}^+(P)$ is not trivial, if $\Gamma^+(P) \neq \{1\}$ or $\Theta^+(P) \neq \{1\}$ hold. By Lemma 2.5 and Theorem 3.10 the converse conclusion is also true.

Using the terminology introduced before, we can now show:

Lemma 4.5. *Let R be a chain domain and $P = R \backslash S$ a completely prime ideal, $(0) \neq (aR)S^{-1}, (cR)S^{-1}$ P-symmetric two-sided ideals. Assume further that $aR \supset cR$ holds. Then there exist P-symmetric two-sided ideals $(bR)S^{-1}$, $(dR)S^{-1}$ such that $(aR)S^{-1} \cdot (bR)S^{-1} = (dR)S^{-1} \cdot (aR)S^{-1} = (cR)S^{-1}$ holds. The same holds for elements of $\Theta^+(P)$.*

Proof. To show that $S^{-1}(Rb) = (bR)S^{-1}$, we prove $Pb = bP$. Let $p \in P$ and assume first $b = rbp$ for some $r \in R$. If $a = sar$ for some $s \in R$, then $cp = abp = sarbp = sab = sc$, so $s \in P$ – a contradiction. Thus $ar = sa$ and so $c = ab = arbp = sabp \in RcP$, again a contradiction. Hence we have $bp = rb$. Again, $a = sar$ leads to the contradiction $c = sarb = sabp \in RcP$. So we have $ar = sa$ for some $s \in R$ and then $cp = abp = arb = sab = sc$, which implies $s \in P$ and thus also $r \in P$. This proves $bP \subseteq Pb$. The reverse inclusion holds by similar (and even easier) arguments. The assertion $(aR)S^{-1} \cdot (bR)S^{-1} = (cR)S^{-1}$ follows from the proof of Lemma 4.2. □

The situation of $\hat{\Theta}^+(P)$ needs further careful arguments:

Lemma 4.6. *Let R be a chain domain and P a completely prime ideal which is assumed to be idempotent. Then for all $A, B \in \hat{\Theta}^+(P)$ with $A \supseteq B$ there exist $C, C' \in \hat{\Theta}^+(P)$ satisfying $AC = C'A = B$.*

Proof. Let $A, B \in \hat{\Theta}^+(P)$ with $A \supseteq B$. Because of Lemma 4.5 we may assume that A, B are not both standard ideals. If B is not a standard ideal, then $B = BP = \bigcup_{\lambda \in \Lambda} b_\lambda P$ for a family of P-symmetric ideals $b_\lambda P$. If $A = aP$, we find P-symmetric c_λ with $ac_\lambda = b_\lambda$, for all $\lambda \in \Lambda$. Hence $Pac_\lambda = aPc_\lambda = Pb_\lambda = b_\lambda P$, and thus $b_\lambda P = aPc_\lambda P$. Now for $C = \bigcup c_\lambda P$ we have $AC = B$ and C is also P-symmetric.

Now consider the case where $B = bP$ and A is not a standard ideal. So $A = AP = \bigcap a_\lambda P$ with P-symmetric ideals $a_\lambda P \supseteq B$. Then $b = a_\lambda c_\lambda$ for some P-symmetric $c_\lambda \in R$. Clearly, $C = \bigcup c_\lambda P$ is P-symmetric and $B = AC$.

If both ideals are non-standard, and say $B = \bigcup b_\lambda P$, repeat the second argument for $B_\lambda = b_\lambda P$ to obtain $C_\lambda \in \hat{\Theta}^+(P)$ with $B_\lambda = AC_\lambda$ and then set $C = \bigcup_\lambda C_\lambda$. □

All the lemmas together now provide a proof for Theorem 4.3, the main result of this section.

5. A class of examples

Let $P = R \setminus S \supset Q = R \setminus T$ be prime ideals which are neighbours, and let $\Gamma^+(P)_T$ denote the classes of T-related elements in $\Gamma^+(P)$. Note that the definition of T-related for ring elements can be carried over in a natural way to elements of this semigroup of divisibility. Hence, assigning to each P-symmetric two-sided standard ideal $(aR)S^{-1} \subseteq Q$ the right ideal $(aR)T^{-1}$ gives a well-defined map: $\Gamma^+(P)_T \to \Gamma^+(Q)$, because by Corollary 2.4 the right ideal $(aR)T^{-1}$ is two-sided as well as Q-symmetric. The following construction shows that in this situation we cannot expect a closer relationship between these semigroups; to be more precise, there exist chain domains with $\Gamma^+(P)_T$ discrete and $\Gamma^+(Q)$ not discrete.

Let H be any ordered group and $\phi : H \to H'$ a non-trivial homomorphism into an ordered group H' of rank 1. Furthermore, let K denote the kernel of ϕ. Now take $G = H \times H'$, with lexicographical ordering. The homomorphism ϕ induces a monomorphism σ on G via

$$\sigma(h, h') = (h, \phi(h) + h') \text{ for } h \in H, h' \in H'.$$

Starting from the chain domain R_0 associated with G by the Malcev-Neumann construction, we want to obtain a chain domain R_1 as in [4]. Let σ also denote the extension of σ to R_0. To show that σ is compatible with the Jacobson radical $J(R_0) = J_0$, it suffices to prove the following:

$$\sigma(G^+) \subseteq G^+ \quad \text{and} \quad \sigma^{-1}(G^+) \subseteq G^+.$$

Obviously, $\sigma(0, h') = (0, h')$. If $h > 0$, then $\sigma(h, h') = (h, \phi(h) + h')$ is also positive. So $\sigma(G^+) \subseteq G^+$, and the second assertion is also clear.

Now let $J_1 = J(R_1) = P$, and let $R_1 \setminus T = Q$ be the prime ideal corresponding to the σ-compatible set $\{(h, h') \in G \mid h > 0\}$. Thus Q is the lower neighbour of $P = J_1$ as a prime ideal, since H' was supposed to be of rank 1.

To compute $\Gamma^+(P)_T$ and $\Gamma^+(Q)$ we will use the following two facts:

(a) The two-sided ideals in R_1 correspond to the σ-compatible ideals in R_0.
(b) $\sigma(h, h') = (h, h')$ if and only if $h \in K$.

From this we obtain:

$$\begin{aligned} \Gamma^+(P)_T &= \{aR_1 \mid a \in R_0 \text{ with } aR_1 \text{ two-sided}\}/T\text{-equivalence} \\ &\simeq K \cap H^+ \\ \Gamma^+(Q) &= \{(aR)T^{-1} \mid a \in R_0, P_l(aQ) = Q\} \\ &\simeq H^+. \end{aligned}$$

In particular, all the elements in $(H \setminus K) \times H'$ give rise to principal right ideals which are not two-sided, but the corresponding standard Q-associated right ideals are two-sided and Q-symmetric.

For a more explicit example, take $H = H' = \mathbb{Z}[e] \subset \mathbb{R}$, e the Euler number, with the canonical ordering and define $\phi : \mathbb{Z}[e] \to \mathbb{Z}[e]$ by $\phi(n + me) = me$. Then $\mathbb{Z} = \operatorname{Ker}\phi \simeq \Gamma(P)_T$ is discrete, but $\mathbb{Z}[e] \simeq \Gamma(Q)$ is not discrete.

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Received April 12, 1989

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