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ON THE IDEAL STRUCTURE OF RIGHT DISTRIBUTIVE RINGS

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Introduction

A right distributive ring, right D-ring for short, is a ring whose lattice of right ideals is distributive. It is well-known that the class of commutative D-domains coincides with the class of Prüfer domains. Noncommutative right D-rings were investigated in a paper of Stephenson [13]. Brungs [5] proved that right D-domains are locally right chain rings. So the class of right chain rings is an interesting class of examples (see [3] and the literature quoted therein). Recently two papers by Mazurek and Puczyłowski respectively Mazurek showed that some features for right chain rings can be carried over to right D-rings ([10], [9]).

The purpose of this paper is to study the structure of right ideals in right D-rings. In §1 we first recall some results from [9]. We introduce the relevant condition (MP). A right D-ring is said to satisfy (MP) if it has a completely prime ideal contained in the Jacobson radical. Some examples of right D-rings satisfying (MP) are given.

In §2 we first introduce some notions such as right multiplicative ideals, waists and the right associated completely prime ideal $P_r(I)$ of a right multi-

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plicative ideal I (see [3]). A waist is a right ideal I such that for every right ideal K we have either $K \subseteq I$ or $I \subseteq K$. A good part of this paper is devoted to the study of waists. The discussion on right multiplicative waists is started in §2 and then, as an application we prove that if P is a completely prime right multiplicative ideal contained in the Jacobson radical, then P is a completely prime two-sided ideal. Next we consider the question of whether $P_r(I)$ is always a right ideal. It follows that this condition is equivalent to $\dim_R R/I = 1$, where $dim_R B$ means the Goldie dimension of the right R-module B. As a consequence we prove that if I is a right ideal of R such that $I = \bigcap_{a \notin I} aP$, for some completely prime ideal P contained in the Jacobson radical J of R, then $\dim_R R/I = 1$ and $P_r(I)$ is an ideal contained in J.

In §3 we discuss right ideals that are waists. In particular it follows that under some assumptions a right ideal I is a waist if and only if $P_r(I)$ is contained in J and this is also equivalent to I being an intersection of ideals of the type aP, P a completely prime ideal contained in J.

The following question raised in [11] was open for a long time: *Do there exist prime ideals in right chain rings which are not completely prime?* (see [3], [6], [12]). Recently Dubrovin presented affirmative examples by construction (private communication). Hence, at the moment there is less known about the similar question in the class of right D-rings. So it is natural to ask for additional conditions which imply that every prime ideal is completely prime.

In §4 we consider right D-rings which are algebraic over their center. We prove that in this case if R is prime, then it is a domain. Also, if R is a right D-ring which is (almost) integral over its center, then every one-sided prime ideal is two-sided and completely prime. Finally, in §5 we prove that under some assumption on the group of units, the prime radical of a right D-ring is completely prime.

Throughout this paper R is always a ring which has an identity element. By $J = J(R)$ we denote the *Jacobson radical*, $L = L(R)$ the *prime (lower nil) radical* and $A = A(R)$ the *generalized nil radical* of R. The set of *nilpotent elements* is denoted by N, the *group of units* by U and the *center* by C. An element $a \in R$ is said to be a right (resp. left) *zero divisor* if $l(a) \neq 0$ (resp. $r(a) \neq 0$), where $l(a)$ (resp. $r(a)$) denotes the left (resp. right) annihilator of a. The set of all right (resp. left) zero divisors of R will be denoted by $N_r(R)$ (resp. $N_l(R)$). The notations $\supset$ and $\subset$ will mean strict inclusions. Ideals are assumed to be two-sided unless otherwise stated.

1 Right D-rings with (MP)

DEFINITION 1.1 *A ring R is called a right distributive ring, right D-ring for short, provided its lattice of right ideals is distributive.*

Mazurek pointed out in [9] that the following condition, first stated in [10], is of interest for a right D-ring R:

(MP) There exists a completely prime ideal of R contained in the Jacobson radical J.

Condition (MP) is satisfied in right D-rings if and only if the generalized nil radical A of R is completely prime ([10], p. 469). Obviously the condition is automatically true provided R is a right chain ring. This condition was already used by the authors in [7] (stated there as Condition (C)).

We begin recalling the following results (see [13], Proposition 2.1 and [9], §3).

LEMMA 1.2 *Let R be a right D-ring and Q a completely prime ideal of R contained in J.*

(i) *For any right ideal I of R we have either $I \subseteq Q$ or $Q \subseteq I$.*

(ii) *For any $a \in R \backslash Q$ we have $Q = aQ$.*

(iii) *For any $a, b \in R$ one of the following holds: $aR \subseteq bR$, $bR \subseteq aR$ or $aQ = bQ$.*

(iv) *For any ideal I of R with $I \subseteq Q$ either I is nilpotent or $\bigcap_{n \in \mathbb{N}} I^n$ is a completely prime ideal.*

LEMMA 1.3 *Let R be a right D-ring which satisfies (MP). Then the following holds:*

(i) *The prime radical L of R is prime.*

(ii) *There is no two-sided ideal I of R with $L \subset I \subset A$. Moreover we have $A = L$ or $A/L = N_r(R/L)$ is the unique minimal non-zero ideal of R/L.*

As remarked earlier Dubrovin recently showed that there exist prime ideals in right chain rings which are not completely prime. So he gave an affirmative answer to an open question first raised in [11] (see also [3], [6], [12]). In right chain rings these prime ideals are called *exceptional.* Here we extend this terminology to right D-rings.

Let P_1, P_2 be two neighbouring prime ideals of R with $P_1 \supset P_2$. We call the set of right ideals X with $P_1 \supseteq X \supset P_2$ a *prime segment*, $[P_1, P_2[$ for short. A prime segment $[P_1, P_2[$ is said to be *simple* if there is no two-sided ideal I with $P_1 \supset I \supset P_2$. As an easy consequence of Lemma 1.2 and 1.3 we obtain the following generalization of a result which is well-known for right chain rings (see [3]).

COROLLARY 1.4 *Let R be a right D-ring and P a prime ideal of R such that there exists a completely prime ideal Q with $P \subsetneq Q \subsetneq J$. Then we have: If P is exceptional, then there exists a completely prime ideal Q_1 of R such that the prime segment $[Q_1, P[$ is simple. Further, either P is nilpotent or there exists a completely prime ideal $Q_2 \subset P$ and the prime segment $[P, Q_2[$ contains no other prime ideal.*

An example of a right D-ring which satisfies condition (MP) and is not a chain ring was given in ([9], Example 3.6). We include here two additional examples.

Let A be a (commutative) Prüfer domain, σ an automorphism of A and F the field of fractions of A. We denote by σ again the extension of σ to F. By $F[[t;\sigma]]$ (resp. $F[t;\sigma]$) we mean the skew power series ring (resp. skew polynomial ring) defined by $ta = \sigma(a)t$ for all $a \in F$.

EXAMPLE 1.5 We put $R = A \oplus tF[[t;\sigma]]$. It is not hard to show that given a right ideal I of R, there exists $s \geq 0$ such that $(t^{s+1}) \subseteq I \subset (t^s)$, where (t^s) denotes the right ideal of R generated by t^s. Also the lattice of all right ideals of R contained in (t^s) and containing in (t^{s+1}) is isomorphic to the lattice of A-submodules of F which is distributive ([13], Proposition 3.3 (ii)). So R is a right (and left) D-ring. Finally, R is a domain and the Jacobson radical of R is $J(R) = J(A) + (t)$.

EXAMPLE 1.6 We put $T = A \oplus tF[t;\sigma]$ and we denote by R the ring T/K, where $K = (t^n)$ is the ideal of T generated by t^n. As above we easily deduce that if I is a right ideal of R, then there exists $s \geq 0$ such that $(t^{s+1} + K) \subseteq I \subset (t^s + K)$. Also, the lattice of all those right ideals is isomorphic to the lattice of A-submodules of F. It follows that R is a right (and left) D-ring. The prime radical of R is the ideal generated by $t + K$ which is completely prime and the Jacobson radical is $J(R) = J(A) + (t + K)$.

2 Right multiplicative ideals and $P_r(I)$

In the next two sections we will discuss on *waists* ([2]) in right D-rings.

DEFINITION 2.1 *Let R be a ring and M a right R-module. A submodule N of M is said to be a* waist *if every submodule of M contains N or is contained in N.*

So a right ideal T of R is a waist provided $I \subseteq T$ or $T \subset I$ for any right ideal I of R. Now we weaken these assumptions and consider right multiplicative ideals of R.

A subset T of R is said to be a *right multiplicative ideal* if for every $a \in T$ and $x \in R$ we have $ax \in T$. A right multiplicative ideal T is said to be *completely prime* if $ab \in T$ implies either $a \in T$ or $b \in T$. We say that a right multiplicative ideal T is a *right multiplicative waist* if for every right multiplicative ideal I of R we have either $I \subseteq T$ or $T \subset I$. We point out that T is a right multiplicative waist (resp. a waist) if and only if for every $a \in R \backslash T$ we have $T \subset aR$.

Every right ideal in a right chain ring is a waist. Also, if R is a right D-ring, every completely prime ideal which is contained in J is a waist (Lemma 1.2 (i)). More generally, every completely prime right multiplicative ideal which is contained in J is a waist. In fact, we note that the following generalization of ([13], Proposition 2.1) holds with virtually the same proof.

LEMMA 2.2 *Let R be a right D-ring. Suppose that P and Q are completely prime right multiplicative ideals of R. Then*

(i) *$P \subseteq Q$, $Q \subseteq P$ or $P + Q = R$.*

(ii) *If $P \subseteq J$, then P is a right multiplicative waist. In particular, P is a waist.*

Let T be a right multiplicative ideal of R. We define the *right associated prime multiplicative ideal* $P_r(T)$ by

$$P_r(T) = \{a \in R : \exists b \in R \backslash T \text{ with } ba \in T\}$$

(see [4] and [3], Chapter 4).

It is easy to check that $P_r(T)$ is a completely prime right multiplicative ideal containing T. Also, if P is a completely prime right multiplicative ideal of R, then $P_r(P) = P$.

If I is an ideal of R, then $P_r(I)$ is the set of all the elements $a \in R$ such that $a + I \in N_r(R/I)$.

LEMMA 2.3 *Let R be a right D-ring, T be a right multiplicative ideal of R and $P = P_r(T)$. Then the following conditions are equivalent:*

(a) *T is a right multiplicative waist.*

(b) *$T = \bigcap_{a \notin T} aP$ or there exists $b \in R \backslash T$ such that $T \subset bR = \bigcap_{a \notin T} aP$ and T is a lower neighbour of bR.*

PROOF: (a) $\Rightarrow$ (b) Take $a \notin T$. Since aP is a right multiplicative ideal we have either $T \subseteq aP$ or $aP \subset T$. Assume there exists $b \in T \backslash aP$. Then $b \in T \subset aR$ and so $b = ax$ for some $x \in R$. Since $b \in T$ we get $x \in P = P_r(T)$, a contradiction. Therefore $T \subseteq \bigcap_{a \notin T} aP$.

If $T = \bigcap_{a \notin T} aP$ we are done. Assume there exists $b \in (\bigcap_{a \notin T} aP) \backslash T$. Hence $T \subset bR \subseteq \bigcap_{a \notin T} aP \subseteq bP \subseteq bR$, so $T \subset bR = \bigcap_{a \notin T} aP$. Suppose H is a right multiplicative ideal with $T \subset H \subseteq bR$ and take any $c \in H \backslash T$. As above we get $cR = \bigcap_{a \notin T} aP$ and consequently $H = bR$.

(b) $\Rightarrow$ (a) Assume $T \subseteq \bigcap_{a \notin T} aP$. If $b \notin T$ we have $T \subseteq bP \subseteq bR$ and so T is a right multiplicative waist. ∎

COROLLARY 2.4 *Let T be a right multiplicative ideal with $P = P_r(T) \subseteq J$. If T is a right multiplicative waist, then $T = \bigcap_{a \notin T} aP = \bigcap_{a \notin T} aJ$.*

PROOF: Suppose there exists $b \notin T$ with $bR = \bigcap_{a \notin T} aP$. Then $bR \subseteq bP$ where $P \subseteq J$. This is a contradiction and so $T = \bigcap_{a \notin T} aP$. Also, if there exists $b \in \bigcap_{a \notin T} aJ \backslash T$, then $b \in bJ$. Hence $\bigcap_{a \notin T} aJ \subseteq T = \bigcap_{a \notin T} aP \subseteq \bigcap_{a \notin T} aJ$. ∎

Mazurek pointed out the following lemma which holds for any ring.

LEMMA 2.5 *Let R be any ring and P a completely prime one-sided ideal of R contained in J. Then P is a two-sided ideal of R.*

PROOF: Assume P is a right ideal and take $x \in P$, $a \in J$. Then $(1 + a)^{-1}(1 + a)x \in P$ and so $(1 + a)x \in P$, since $(1 + a)^{-1} \notin P$. It follows that P is a two-sided ideal of J. Now the Andrunakievich Lemma (if I is an ideal of a ring B, K an ideal of I and K^* is the ideal of B generated by K, then $(K^*)^3 \subseteq K$) immediately implies that P is a two-sided ideal of R. The other case can be treated similarly. ∎

Now we are able to prove one of the main results of this section.

THEOREM 2.6 *Let R be a right D-ring and let P be a completely prime right multiplicative ideal contained in J. Then P is a (two-sided) completely prime ideal of R.*

PROOF: By Lemma 2.2 P is a right multiplicative waist. So $P = P_r(P) = \bigcap_{a \notin P} aJ$, by Corollary 2.4. Then P is a right ideal and Lemma 2.5 completes the proof. ∎

Theorem 2.6 immediately implies the following

COROLLARY 2.7 *Let I be a right ideal of a right D-ring R with $P_r(I) \subseteq J$. Then $P_r(I)$ is a two-sided completely prime ideal of R. In particular, if $N_r(R) \subseteq J$, then $N_r(R)$ is an ideal of R.*

Let I be a right ideal of R. We can ask of whether $P_r(I)$ is always a right ideal of R. The following extension of Proposition 2.2 in [9] gives an answer to this question. We will omit the proof since it is similar to that in [9].

First, for a right ideal I of R we denote by M_I the right R-module R/I. If $x \in M$, then $r(x)$ stands for the right annihilator of $x \in M$ in R.

PROPOSITION 2.8 *Let R be a right D-ring and let I be a right ideal of R. Then the following conditions are equivalent:*

(a) $\dim_R M_I = 1$, *where* $\dim_R$ *means Goldie dimension.*

(b) *The sets $r(x)$, $x \in M$, are linearly ordered by inclusion.*

(c) *$P_r(I)$ is a right ideal of R.*

From Corollary 2.7 we have the following

COROLLARY 2.9 *Suppose that I is a right ideal and $P_r(I) \subseteq J$. Then* $\dim_R M_I = 1$.

LEMMA 2.10 *Suppose that P is a completely prime ideal contained in J and $a \in R$. If $aP \neq (0)$, then $P_r(aP) = P$.*

PROOF: We will use freely Lemma 1.2 (i), (ii) and (iii). It is clear that $a \notin aP$ and for any $b \in P$, $ab \in aP$. Then $P \subseteq P_r(aP)$. Assume that there exists $x \in P_r(aP) \backslash P$ and we will get $aP = 0$, a contradiction.

By assumption there exists $c \notin aP$ with $cx \in aP$. So $cx = ap$ for some $p \in P$. We compare c and a. If $a = cr$ for some $r \in R$, then $aP = crP \subseteq cP$. If $c = at$ for some $t \in R$ we have $t \notin P$ since $c \notin aP$. Hence $cP = atP = aP$. So we have $aP \subseteq cP$ in any case. Thus there exists $q \in P$ with $cx = ap = cq$. Therefore $c(q - x) = 0$ and since $x \notin P$ there exists $y \in P$ with $q = xy$. Consequently $cx(y - 1) = 0$ and it follows that $cx = 0$. Then $aP \subseteq cP = cxP = 0$. The proof is complete. ∎

From Corollary 2.9 and Lemma 2.10 we easily deduce

COROLLARY 2.11 *Suppose that P is a completely prime ideal contained in J. Then* $\dim_R (R/aP) = 1$, *for every $a \in R$ such that $aP \neq (0)$.*

Next we will generalize Corollary 2.11. First we get the following

LEMMA 2.12 *Suppose that P is a completely prime ideal contained in J. Then aP is a waist for every $a \in R$.*

PROOF: Suppose that $b \in R \backslash aP$. By Lemma 1.2 (iii) one of the following alternatives is valid: (i) $b = ar$ for some $r \in R$ which leads to $r \notin P$ and so $bR \supset bP = arP = aP$. Secondly (ii) $a = br$ for some $r \in R$ implying $aP = brP \subseteq bP \subset bR$. Finally, the remaining case reads $aP = bP \subset bR$. Hence $aP \subset bR$ in any case. ∎

Now we are ready to prove the second main result of this section:

THEOREM 2.13 *Suppose that R is a right D-ring and I is a nonzero right ideal of R such that for some completely prime ideal $P \subseteq J$ we have $I = \bigcap_{a \notin I} aP$. Then $\dim_R M_I = 1$ and $P_r(I) \subseteq J$ is the smallest completely prime ideal Q of R such that $I = \bigcap_{a \notin I} aQ$.*

PROOF: Assume $x \in P_r(I)$ and take $b \notin I$ with $bx \in I$. Since $b \notin I$ there exists $a \notin I$ with $b \notin aP$ and $bx \in aP$. Therefore $x \in P_r(aP) = P$. Thus $P_r(I) \subseteq P \subseteq J$ and $\dim_R M_I = 1$ by Corollary 2.9. Now, I is a waist by Lemma 2.12 and we obtain $I = \bigcap_{a \notin I} aP_r(I)$ from Corollary 2.4 which finishes the proof. ∎

3 Characterization of waists

The aim of this section is to prove a theorem giving several equivalent characterizations for a right ideal being a waist. By this we also improve some results of Section 2 and throw some new light on these results. First we recall the following (see [5], Proof of Theorem 1).

LEMMA 3.1 *Let P be a completely prime ideal of R and $S = R \backslash P$. Then for every $a \in R$ and $s \in S$ there exist $b \in R$ and $t \in S$ such that $at = sb$ (i.e. S is a right Ore system).*

Let P be a completely prime ideal of R contained in J and $S = R \backslash P$. For any $a \in R$ we set

$$(aR)S^{-1} = \{x \in R \mid \exists s \in S : xs \in aR\}.$$

We have

LEMMA 3.2 *Under the assumptions from above $(aR)S^{-1}$ is a right ideal of R.*

PROOF: Suppose $x, y \in (aR)S^{-1}$, say $xs \in aR$ resp. $yt \in aR$ for some $s, t \in S$. We use Lemma 3.1 to find $u, v \in S$ such that $su = tv$. Then $xsu \in aR$ and

$ytv \in aR$, so $(x-y)su = xsu - ytv \in aR$. Hence $x - y \in (aR)S^{-1}$. Similarly, we get $xr \in (aR)S^{-1}$ for any $r \in R$. ■

We point out that the right ideals $(aR)S^{-1}$ have similar properties as aP. Therefore the next result is analogous to ([9], Lemma 3.1(ii)).

LEMMA 3.3 *Let P be a completely prime ideal of R contained in J and $a, b \in R$. Then either $aR \subseteq bR$ or $(bR)S^{-1} \subseteq (aR)S^{-1}$.*

PROOF: Suppose that aR is not contained in bR. By ([9], Lemma 2.1(ii)) there exist $x, y \in R$ with $x + y = 1$, $ax \in bR$ and $by \in aR$. We have $y \notin J$ since $x = 1 - y$ is not invertible. Hence $y \notin P$ and so $b \in (aR)S^{-1}$. Thus $(bR)S^{-1} \subseteq (aR)S^{-1}$. ■

As a consequence we have

LEMMA 3.4 *Let P be a completely prime ideal of R contained in J, $S = R \backslash P$ and $a, b \in R$. Then $aP = bP$ if and only if $(aR)S^{-1} = (bR)S^{-1}$.*

PROOF: By Lemma 1.2 (iii) and Lemma 3.3, if a and b are not comparable, we have $aP = bP$ as well as $(aR)S^{-1} = (bR)S^{-1}$. So we may assume $aR \subseteq bR$ which implies $(aR)S^{-1} \subseteq (bR)S^{-1}$ and $aP \subseteq bP$.

In case $aP = bP$ and $(aR)S^{-1} \subset (bR)S^{-1}$ we conclude $a = br$ for some $r \in P$ since $b \notin (aR)S^{-1}$. Hence $aP = brP \subset bP$, a contradiction (note that $brP = bP$ will lead to a contradiction $br = brp$ for $p \in P \subseteq J$).

Conversely, assume $(aR)S^{-1} = (bR)S^{-1}$ and $aP \subset bP$. Since $b \in (aR)S^{-1}$ there exist $s \in S$, $r \in R$ with $bs = ar$. Hence $bP = bsP = arP \subseteq aP$, again a contradiction. ■

Let P be a completely prime ideal contained in J. We define an equivalence relation in R. For elements $a, b \in R$ we put

$$a \sim b \Longleftrightarrow aP = bP.$$

For $a \in R$ let $[a] = \sum_{b \sim a} bR$. It is clear that if $a \notin P$ then $[a] = R$.

LEMMA 3.5 *Let P be a completely prime ideal contained in J, $S = R \backslash P$ and $a \in R$. Then $[a]$ is a waist.*

PROOF: Suppose that $c \notin [a]$ and take any $b \in R$ with $bP = aP$. By assumption $cP \neq bP$ and so b and c are comparable. Then we have $b \in cR$ and so $[a] \subset cR$, since the other possibility will lead to a contradiction: $c \in bR \subseteq [a]$. ■

Now we are ready to prove the following

PROPOSITION 3.6 *Let P be a completely prime ideal contained in J and $a \in R$. Then $(aR)S^{-1} = [a]$ is a waist. Moreover, $(bR)S^{-1} = [a]$ for every $b \sim a$.*

PROOF: If $a \notin P$ we have both $(aR)S^{-1} = R$ and $[a] = R$. So we may assume $a \in P$.

If $x \in (aR)S^{-1}$ there exists $s \in S$ with $xs \in aR$. Hence $xP = xsP \subseteq aP$. If $xP = aP$, then $x \in [a]$. In the other case $xP \subset aP$ we necessarily have $xR \subseteq aR \subseteq [a]$. Therefore $(aR)S^{-1} \subseteq [a]$.

Conversely, suppose $b \sim a$. Then $bP = aP$ and we obtain $bR \subseteq (bR)S^{-1} = (aR)S^{-1}$, by Lemma 3.4. So $[a] \subseteq (aR)S^{-1}$ and Lemma 3.5 completes the proof. ■

LEMMA 3.7 *Let T be a right multiplicative ideal with $P_r(T)$ contained in J. Then $T = \bigcup_{a \in T} (aR)S^{-1}$, where $S = R \backslash P_r(T)$.*

PROOF: It suffices to show that for any $a \in T, (aR)S^{-1} \subseteq T$. In fact, if $xs \in aR$, for some $s \in S$ we have $xs \in T$ and $s \notin P_r(T)$. This implies $x \in T$ by definition of $P_r(T)$. ■

The following corollary improves Corollary 2.4:

COROLLARY 3.8 *Let T be a right multiplicative ideal with $P = P_r(T) \subseteq J$. Then T is a right ideal which is a waist and $T = \bigcap_{a \notin T} aP = \bigcap_{a \notin T} aJ$.*

PROOF: By Theorem 2.6, P is a completely prime ideal of R. Hence T is a right ideal which is a waist, by Proposition 3.6 and Lemma 3.7. Corollary 2.4 completes the proof. ■

Now we are able to prove the main results of this section.

THEOREM 3.9 *Suppose that R is a right D-ring and I a nonzero right ideal of R. Then the following conditions are equivalent:*

(a) $P_r(I) \subseteq J$

(b) *There exist a completely prime ideal P contained in J and a subset $V \subset R$ such that $I = \bigcap_{a \in V} aP$.*

(c) *There exist a completely prime ideal P contained in J and a subset $V' \subset R$ such that $I = \bigcup_{a \in V'} (aR)S^{-1}$, where $S = R \backslash P$.*

Furthermore, under the equivalent conditions above , I is a waist, $\dim_R(R/I) = 1$, *the set $V = R \backslash I$ satisfies* (b) *and the set $V' = I$ satisfies* (c).

PROOF: Using Lemma 2.10 and Corollary 3.8 we easily get (a) is equivalent to (b). The implication (a) $\Rightarrow$ (c) follows from Lemma 3.7.

Assume (c) and let $x \in P_r(I)$. Then there exists $t \notin I$ with $tx \in I$. So for some $a \in V'$ we have $tx \in (aR)S^{-1}$ and there exists $s \in S$ with $txs \in aR$. If $x \notin P$ we get $t \in (aR)S^{-1} \subseteq I$, a contradiction. Thus $x \in P$ and $P_r(I) \subseteq P \subseteq J$ follows.

The rest is clear by Theorem 2.13. ∎

If the equivalent conditions of Theorem 3.9 hold, then I is a waist and R satisfies (MP). Under some additional assumption we can also get the converse. But first we prove the following

LEMMA 3.10 *Assume that R satisfies condition (MP). Then the prime radical L is a waist.*

PROOF: If L is completely prime we are done. So assume that the generalized nil radical A of R is different from L. Take $x \notin L$ and let a be any element of L. Since $xR \not\subseteq aR$ and $xA = aA \subseteq L$ contradicts the primeness of L, we have $aR \subseteq xR$. So $L \subset xR$. ∎

THEOREM 3.11 *Suppose that R is a right D-ring and I is a right ideal containing the prime radical of R. Then the conditions* (a), (b) *and* (c) *of Theorem* 3.9 *are also equivalent to the following:*

(d) *R satisfies (MP) and I is a waist.*

(e) *R satisfies (MP) and $I = \bigcap_{a \notin I} aJ$.*

PROOF: First, note that if R satisfies (MP) and $I = L = 0$ we have $P_r(I) = N_r(R)$ which is either zero or A, by Lemma 1.3 (ii). Hence $P_r(I) \subseteq J$ holds in this case. So under the assumption of this theorem the conditions (a), (b) and (c) are always equivalent, even when $I = 0$. Also by Corollary 3.8 condition (a) implies (e). If $I = \bigcap_{a \notin I} aJ$ and $b \notin I$, then $I \subseteq bJ \subseteq bR$. Thus I is a waist and (e) implies (d).

It remains to prove (d) $\Rightarrow$ (a). Thus in the rest of the proof we assume (d).

Case 1: Suppose $I = L$. Then I is prime and $P_r(I)/I = N_r(R/I)$ is either zero or A/I. Hence $P_r(I) \subseteq J$ holds in this case, since J contains a completely prime ideal.

Case 2: Suppose $I \supset L$ and I is a waist. Assume there exists $x \notin J$ such that $x \in P_r(I)$ and let $t \in R$ with $t \notin I$, $tx \in I$. Put $H = \{a \in R \mid ta \in I\}$. Then H is a right ideal with $tH \subseteq I$ and $H \not\subseteq J$. So there exists a maximal right ideal M of R with $H \not\subseteq M$. It follows that $I \subset tM$ because I is a waist.

Hence $tH \subseteq I \subset tM$ and for every $b \in H$ there exists $c_b \in M$ with $tb = tc_b$, i.e., $t(b - c_b) = 0$. We may assume that for some $b \in H$ we have $b - c_b \notin A$, since otherwise we get $b \in A + M \subseteq M$ and so $H \subseteq M$. We may also assume $A \neq L$ since if $A = L$ we get $t \in A \subset I$, a contradiction.

It remains to reach a contradiction when $b - c_b \notin A \neq L$ and $t(b - c_b) = 0$. In this case $A \subset (b - c_b)R$ and so $tA = 0$. Hence $t \in L \subset I$ by the primeness of L, again a contradiction. This completes the proof of the theorem. ∎

The above theorem gives a complete description of a waist containing the prime radical when R satisfies (MP). The next rather trivial example shows the relevance of the assumption (MP).

EXAMPLE 3.12 (see [7], Example 10) Let $R = K_1 \oplus K_2$, where K_i are fields. Then it is easy to see that R is a right D-ring which does not satisfy assumption (MP). The zero ideal is always a waist, but $P_r(0) = N_r(R) = K_1 \cup K_2$, which is neither an ideal nor contained in J. Nevertheless, we can easily see that $P_r(0)$ is the maximum right multiplicative waist.

Note that any right ideal which is a waist is contained in J. Also there exists a largest waist W. Similarly, if R satisfies (MP) there exists a largest completely prime ideal B of R contained in J, namely the union of all the completely prime ideals of R which are contained in J. From Theorem 3.11 we easily have

COROLLARY 3.13 *If R satisfies condition (MP), then $W = B$.*

COROLLARY 3.14 *Assume that R satisfies (MP) and $A \neq L$. Then $L = \bigcap_{a \notin L} aA = \bigcap_{a \notin L} aJ = \bigcup_{a \in L}(aR)S^{-1}$, where $S = R \backslash A$.*

For a ring R which satisfies (MP) we can give some further information. Suppose I is a right ideal with $I \supseteq L$ and let B be the largest completely prime ideal of R contained in J. If $I \supset B$ there is no waist containing I. So assume $I \subseteq B$ and denote by $m(I)$ (resp. $l(I)$) the minimum (resp. largest) waist of R containing (resp. contained in) I. Note that $m(I)$ and $l(I)$ do exist. We also write

$$(I) = \bigcap_{a \notin I} aB \quad \text{and} \quad [I] = \bigcup_{a \in I}(aR)S^{-1}$$

where $S = R \backslash B$. We have

PROPOSITION 3.15 *Assume R satisfies (MP) and I is a right ideal with $L \subseteq I \subseteq B$. Then $l(I) = (I)$ and $m(I) = [I]$.*

PROOF: Note that $\bigcap_{a\notin I} aB$ is a waist by Lemma 2.12. Also $\bigcap_{a\notin I} aB \subseteq I$. In fact, assume to the contrary that b exists with $b \in \bigcap_{a\notin I} aB \backslash I$. Then $b \in \bigcap_{a\notin I} aB \subseteq bB \subseteq bJ$ and so $b = 0$, a contradiction. Now, suppose H is a waist and $H \subseteq I$. We have either $H \subseteq L$ or $L \subset H$. If $a \notin I$, then $a \notin L$ and so $aB \not\subseteq L$. It follows that $L \subset aB$ and hence $H \subset \bigcap_{a\notin I} aB$ when $H \subseteq L$. In the other case $(L \subset H)$ we have $P_r(H) \subseteq J$ and $H = \bigcap_{a\notin H} aP_r(H)$ by Theorem 3.11. Therefore $H \subseteq \bigcap_{a\notin H} aB \subseteq \bigcap_{a\notin I} aB = (I)$ and consequently $l(I) = (I)$.

Similarly, $[I] = \bigcup_{a\in I}(aR)S^{-1}$ is a waist and $I \subseteq [I]$. If H is a waist with $I \subseteq H$ we have $P_r(H) \subseteq B$. Take any $x \in (aR)S^{-1}$, for some $a \in I$. Then there exists $s \in S$ with $xs \in aR \subseteq H$. Since $s \notin P_r(H)$ we necessarily have $x \in H$. Therefore $[I] \subseteq H$ and it follows that $m(I) = [I]$. ■

4 Right D-rings which are algebraic over the center

As we already said it is natural to ask for additional conditions which imply that every prime ideal in a right D-ring is completely prime. In this section we study the question for a right D-ring which is algebraic over the center.

Hereafter we denote by $C = C(R)$ the center of R. We say that R is algebraic over C if for each $a \in R$ there exists a nonzero polynomial (not necessarily monic) $f(x) \in C[x]$ with $f(a) = 0$.

Some results on right D-domains which are algebraic over the center were obtained in [8]. We begin this section with the following

THEOREM 4.1 *Let R be a right D-ring which is algebraic over the center. Then R is a domain.*

PROOF: By ([9], Corollary 2.4 and Proposition 1.2 (i)) the right Goldie dimension of R is one and the set $N_l(R)$ of left zero divisors of R is a completely prime ideal. Therefore, if $N_l(R) = 0$ we are done.

Assume, otherwise, $N_l(R) \neq 0$. Using the primeness of R we easily get that if $c \in N_l(R) \cap C$, then $c = 0$. Since the prime radical and the nilradical of R coincide (see the remark following Corollary 2 in [10]) the set of nilpotent elements $N \subset N_l(R)$. Take any $a \in N_l(R)\backslash N$. By assumption there exist $c_0, c_1, \ldots, c_n \neq 0$ in C such that $\sum_{i=0}^{n} c_i a^i = 0$. Hence $c_0 = -\sum_{i=1}^{n} c_i a^i \in N_l(R) \cap C$ and so $c_0 = 0$. Therefore there exist j, $1 \leq j \leq n$, with $c_0 = \ldots = c_{j-1} = 0$ and $c_j \neq 0$. Furthermore, $j < n$ because in the contrary case we have $c_n R a^n = R c_n a^n = 0$, a contradiction since $a^n \neq 0$. Thus $\sum_{i=j}^{n} c_i a^{i-j} \cdot a^j = 0$, so $\sum_{i=j}^{n} c_i a^{i-j} \in N_l(R)$ and consequently $0 \neq c_j = \sum_{i=j}^{n} c_i a^{i-j} - \sum_{i=j+1}^{n} c_i a^{i-j} \in N_l(R) \cap C$, which is a contradiction. ■

We have the following

COROLLARY 4.2 *Let R be a prime right D-ring which is a PI-ring. Then R is a domain.*

PROOF: The assertion follows directly by ([1], Theorem 3). ∎

We say that R is *almost integral* over C if for every $a \in R$ there exists a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in C[x]$ having at least one a_i which is invertible, for $1 \leq i \leq n$, with $f(a) = 0$. It is clear that if R is almost integral over C and I is an ideal of R, then R/I is algebraic over $C(R/I)$. Now we have

THEOREM 4.3 *Let R be a right D-ring which is almost integral over the center. Then every one-sided prime ideal is two-sided and completely prime.*

PROOF: Suppose that P is a right prime ideal of R. Note that $(P : R) = \{a \in R \mid Ra \subseteq P\}$ is the largest two-sided ideal of R which is contained in P. Also $(P : R)$ is prime: $aRb \subseteq (P : R)$ implies $RaRb \subseteq P$, hence either $Ra \subseteq P$ or $Rb \subseteq P$ which shows that either $a \in (P : R)$ or $b \in (P : R)$. So we may assume that $(P : R) = 0$, R is prime and algebraic over C.

By Theorem 4.1 R is a domain. If $P = 0$ we are done. So we assume $P \neq (0)$ and we prove that there exists a two-sided ideal K of R with $0 \neq K \subset P$. This contradiction completes the proof.

Take any $0 \neq a \in P$. Hence there exist $c_0, \ldots, c_{n-1}$, $c_n \neq 0$ in C with $\sum_{i=0}^{n} c_i a^i = 0$. Since $c_n a^n \neq 0$ there exists $r, 0 \leq r < n$ such that $c_j = 0$ for $j < r$ and $c_r \neq 0$. Thus $c_r = -\sum_{i=r+1}^{n} c_i a^i = -\sum_{i=r+1}^{n} a^i c_i \in P \cap C$. Therefore $cP = Pc$ (where $c = c_r$) is a nonzero one-sided ideal of R contained in I. If $I = cI$ we have $c = cb$, for some $b \in P$, so $c(1 - b) = 0$, a contradiction. Consequently $0 \neq cP \subset P$. By the same arguments as above there exists $0 \neq d \in cP \cap C$ and so $K = Rd = dR$ is the required ideal. ∎

5 An additional result on the prime radical

In this section we denote by U the group of units of R. The main purpose here is to prove the following

THEOREM 5.1 *Let R be a right D-ring which satisfies condition (MP). If the index of the subgroup $U \cap C$ in U is finite, then the prime radical of R is completely prime.*

To prove the theorem we need some preparation. We begin with the following Lemma which was proved in [7]. We include the proof here for the sake of completeness.

LEMMA 5.2 *If Q is a completely prime ideal contained in J, then $Q^2 = \{ab \mid a, b \in Q\}$.*

PROOF: By induction it is enough to prove that for $x = a_1b_1 + a_2b_2 \in Q^2$, $a_i, b_i \in Q$, $i = 1,2$ there exist $a, b \in Q$ with $x = ab$. By Lemma 1.2 (iii) we have either $a_1 = a_2y$ resp. $a_2 = a_1y$, for some $y \in R$ or $a_1Q = a_2Q$. In the second case $a_1b_1 = a_2b'$ follows for some $b' \in Q$. The rest is obvious. ∎

Let R be a right D-ring satisfying (MP). It is known that $A = NR$ ([9], Proposition 3.5). The following improves this result.

PROPOSITION 5.3 *Let R be a right D-ring satisfying (MP). Then $A = NU$. Further, if $A \neq L$ then $A = NA = NJ$.*

PROOF: We know that $L \subseteq N \subseteq A$ and if $N = L$, then L is completely prime, i.e., $L = A$. Hence in this case we clearly have $A = NU$. Thus we may assume $L \subset N$ and so $A \neq L$.

Suppose $A = aR$ for some $a \in R$. Then $A = A^2 = aRA = aA = a^2R$ and so $a = a^2x$, for some $x \in R$. It follows that $a(1 - ax) = 0$ and hence $a = 0$, because $1 - ax \in U$. Thus $A = 0$ which is a contradiction. Therefore A is not right principal.

Take $a \in A \backslash N$. Then $aR \subset A$ and there exists $t \in N$ with $tR \not\subseteq aR$, since $A = \sum_{t \in N} tR$ (Proposition 3.5 in [9]). Thus we have $aA \subseteq tA$ by Lemma 1.2 (iii).

Take any $x \in A$. By Lemma 5.2 there exist $b, c \in A$ with $x = bc$. If b lies in N we have $x \in NA$. If otherwise $b \notin N$ there exists $t \in N$ with $bA \subseteq tA$, by the argument above. Therefore $x = bc = ty$ for some $y \in A$ and we get $A = NA$.

Consequently $A = NR \supseteq NJ \supseteq NA = A$. Take any $x \in A$ again. Then there exists $t \in N$, $y \in A$ with $x = ty$ and so $x = t - t(1 - y) \in NU$. The proof is complete. ∎

In the proof of Theorem 5.1 we need also the following

REMARK 5.4 If R satisfies (MP) and $A \neq L$, then A/L is a simple ring. In fact, if K is a proper ideal of A/L, the Andrunakievich lemma immediately implies that $K = 0$ by Lemma 1.3.

PROOF: (of Theorem 6.1) Assume that the prime radical is not completely prime. Then $L \subset N \subset A$. Since N is the sum of the nilpotent ideals of N ([10], Corollary 2), there exists an ideal I of N which is nilpotent and not contained in L. Also, for every $u \in U$, uIu^{-1} is a nilpotent ideal of N. Then $K = \sum_{u \in U} uIu^{-1}$ is an ideal of N and $UK \subseteq KU$ as is easy to see.

We prove $KU = A$. Note that $KU \not\subseteq L$ and $KU \subseteq A$. Furthermore, since $A = NU = UN$ (Proposition 6.3 and Lemma 3 in [10]) we have $KUA = KUNU = KNU^2 \subseteq KU$ and $AKU = NUKU \subseteq NKU^2 \subseteq KU$. Thus KU is an ideal of A and it follows that $KU = A$, by Remark 5.4.

Define $H = \{v \in V / vIv^{-1} = I\}$. If the index $[U : U \cap C] < \infty$, we have $[U : H] < \infty$ because $U \cap C \subseteq H$. Assume $[U : H] = n < \infty$ and take $u_1, \ldots, u_n \in U$ such that $U = \bigcup_{i=1}^{n} u_i H$. We can easily see that $K = \sum_{i=1}^{n} u_i I u_i^{-1}$ is nilpotent. Therefore A is nilpotent because $A^m = K^m U$, for every $m \geq 1$. This is a contradiction and so the proof is complete. ■

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