

RINGS WITH ANNIHILATOR CHAIN CONDITIONS AND RIGHT DISTRIBUTIVE RINGS

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ABSTRACT. We prove that if a right distributive ring R , which has at least one completely prime ideal contained in the Jacobson radical, satisfies either a.c.c or d.c.c. on principal right annihilators, then the prime radical of R is the right singular ideal of R and is completely prime and nilpotent. These results generalize a theorem by Posner for right chain rings.

INTRODUCTION

The following question occurred in a paper by Posner [9]: Do there exist prime ideals in a right chain ring which are not completely prime? Several authors have approached this problem independently from various points of view (see [1]); however, the question remains open (see [3, 10]), and it is natural to ask for additional conditions which imply that a prime ideal in a right chain ring is completely prime.

In the first part of [9, Theorem 2] it is claimed that if a right chain ring R has either a.c.c. or d.c.c. on right annihilators, then the prime radical $P(R)$ of R is the set of nilpotent elements. This fact implies that the prime radical of R is completely prime. There is a gap in the proof and the chain conditions are needed for right ideals rather than right annihilator ideals, when R is not prime. In fact, it is not proved that the annihilator chain conditions are inherited by $R/P(R)$; however, the result holds and the original motivation of this paper was to find a proof for it.

We say that a ring R is a right distributive ring, or right D -ring for short, if its lattice of right ideals is distributive. It is well known that the class of commutative D -domains coincides with the class of Prüfer domains. The study of noncommutative right D -rings was mainly promoted by a paper of Stephenson [11]. The class of right chain rings (see [1] and the literature quoted therein) is an interesting class of examples. Brungs [2] proved that right D -domains are locally right chain rings. Recently two papers by Mazurek and Puczyłowski [8]

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and Mazurek [7] showed that some features of right chain rings can be carried over to right D -rings.

The purpose of this note is to prove the following:

Theorem 8. *Let R be a right D -ring, which has at least one completely prime ideal contained in the Jacobson radical and satisfies either a.c.c. or d.c.c. on principal right annihilators. Then the prime radical of R equals the right singular ideal of R and is completely prime and nilpotent.*

We say that the ring R satisfies condition (C) if the following holds:

- (C) There exists a completely prime ideal Q of R contained in the Jacobson radical J of R .

This condition first appeared in [11, Proposition 2.1(ii)]. Later it was used in [8], and in [7], where it was shown that the condition is of great interest. Let us point out that it is automatically satisfied for a right chain ring R ; therefore, our result gives an extension of Posner's assertion.

Throughout this paper, every ring R has a unit element. By $J = J(R)$ we denote the *Jacobson radical*, $P = P(R)$ the *prime radical*, and $A = A(R)$ the *generalized nil radical* of R . Further, we write $r(a) = \{x \mid ax = 0\}$ the principal right annihilator of the element a in R . The notations \subset and \supset will mean strict inclusions. Ideals are assumed to be two-sided unless otherwise stated.

PROOF OF THE THEOREM

Let R be any ring. By $Z = Z_r(R) = \{x \in R \mid r(x) \text{ is an essential right ideal of } R\}$ we denote the right singular ideal of R (see [5, pp. 30–36]).

An ideal I of a ring R is said to be right T -nilpotent if for every sequence $(x_i)_{i \in \mathbb{N}}$ of elements of I there exists an n such that $x_n x_{n-1} \cdots x_2 x_1 = 0$. We begin with the following:

Lemma 1. *Suppose that R satisfies a.c.c. on principal right annihilators. Then Z is right T -nilpotent, in particular, $Z \subseteq P(R)$.*

Proof. Suppose $(x_i)_{i \in \mathbb{N}}$ is a sequence of elements in Z such that $x_n \cdots x_2 x_1 \neq 0$ for all $n \in \mathbb{N}$. Since $r(x_1) \subseteq r(x_2 x_1) \subseteq \cdots$ is an ascending chain of principal right annihilators, there exists m with $r(x_m b) = r(b)$, $b = x_{m-1} \cdots x_1$. Now x_m is in Z and $b \neq 0$, so $r(x_m) \cap bR \neq 0$ and there exists $y \in R$ with $by \neq 0$, $x_m b y = 0$, which is a contradiction. The proof is complete since every ideal which is right T -nilpotent is contained in the prime radical (see [4, Proposition 2.3]). \square

We recall the following results [11, Proposition 2.1(ii); 7, Lemma 3.1(ii), Corollary 3.3, and Theorem 3.4] for a right distributive ring R .

Lemma 2. *Let R be a right D -ring and Q a completely prime ideal contained in J .*

- (i) *For every right ideal I of R we have $I \subseteq Q$ or $Q \subseteq I$.*
- (ii) *For any $a, b \in R$ we have: The elements a, b are comparable, that is, $aR \subseteq bR$ or $bR \subset aR$ or otherwise $aQ = bQ$ holds.*
- (iii) *The prime radical P of R is a prime ideal.*
- (iv) *There is no two-sided ideal I of R with $P \subset I \subset A$.*

Note that from Lemma 2(i) condition (C) is satisfied in a right D -ring if and only if the generalized nil radical A of R is completely prime. This was already remarked in [8, p. 469]. Obviously this is automatically true provided R is a right chain ring (see [1]).

Now we can prove the following:

Proposition 3. *Let R be a right D -ring which satisfies condition (C). Then R is right nonsingular if and only if it is a domain.*

Proof. Assume that $Z = 0$ and let Q be a completely prime ideal of R contained in J . If Q equals zero we are done. So we may assume $Q \neq 0$. Take any nonzero elements $a, b \in R$. By Lemma 2(ii) we have the alternatives $aR \subseteq bR$, $bR \subseteq aR$, or $aQ = bQ$. If $aQ = 0$ holds, then $r(a) \supseteq Q$ and, by Lemma 2(i), $r(a)$ is an essential right ideal of R . Hence $a \in Z = 0$. Therefore we have $aR \cap bR \neq 0$ and so the right Goldie dimension of R is one. Thus $Z = 0$ is a completely prime ideal of R by [7, Proposition 1.2(i)]. Consequently, R is a domain. The converse is obvious. \square

Mazurek pointed out to us the following lemma, which was proved by Tuganbaev in a more general setting [12, Lemma 8]. For the sake of completeness, we include an adaption of Tuganbaev's proof to our case.

Lemma 4. *Let R be a right D -ring. Then R/Z has no nonzero nilpotent elements.*

Proof. Assume that $a \notin Z$ and $a^2 \in Z$. Then $H = r(a^2)$ is an essential right ideal of R and $L = r(a)$ is not essential. So there exists a nonzero right ideal of B of R with $L \cap B = 0$ and so $L \cap (H \cap B) = 0$. By [11, Corollary 1(i)'] of Proposition 1.1] we have $\text{Hom}_R(H \cap B, L) = 0$, and since $a(H \cap B) \subseteq L$, we get $a(H \cap B) = 0$. Therefore $H \cap B \subseteq r(a) = L$, hence $H \cap B = 0$, a contradiction. \square

Now we prove some lemmas, which are necessary for the d.c.c. case.

Lemma 5. *Let R be a right D -ring which satisfies condition (C). If I is an ideal of R with $A \not\subseteq I$, then we have $I \subseteq P(R)$.*

Proof. By Lemma 2(i), $I \subset A$. Hence, if $A = P$, we are done; therefore, we may assume $P \subset A$. Suppose there exists $a \in I$ with $a \notin P$, and take any element $b \in P$. Then one of the following three contradictions will follow: (i) $a \in bR \subseteq P$, or (ii) $aA = bA \subseteq P$, which contradicts the primeness of P , or (iii) $b \in aR \subseteq I$. The last possibility would imply $P \subseteq I \subset A$ and so $I = P$. Thus $I \subseteq P$. \square

Lemma 6. *Let R be a right D -ring and Q a completely prime ideal contained in J . Then $Q^2 = \{ab \mid a, b \in Q\}$.*

Proof. By induction, it is enough to prove for $x = a_1b_1 + a_2b_2 \in Q^2$ with $a_i, b_i \in Q$ for $i = 1, 2$ that there exist $a, b \in Q$ with $x = ab$. By Lemma 2(ii) we have either $a_1 = a_2y$ resp. $a_2 = a_1y$, for some $y \in R$ or $a_1Q = a_2Q$. In the second case $a_1b_1 = a_2b'$ follows for some $b' \in Q$. The rest is obvious. \square

Lemma 7. *Let R be a right D -ring and Q a completely prime ideal of R contained in J . Further assume R satisfies d.c.c. on principal right annihilators. Then we have*

- (i) $Z \subseteq Q$;
- (ii) If $Q = Q^2 \neq 0$, then $Z \subset Q$.

Proof. (i) If $Q = 0$, then R is a domain and $Z = 0$. So we may assume $Q \neq 0$. Suppose $Q \subset Z$ and take any $a \in Z$, $a \notin Q$, and $0 \neq b \in Q$. By [7, Lemma 3.1(i)] we have $Q = aQ$. Hence there exists $c \in Q$ such that $b = ac$. So $r(c) \subseteq r(b)$ and $r(a) \cap cR \neq 0$. Thus we find $x \in R$ with $cx \neq 0$ and $acx = 0$, which implies $r(c) \subset r(b)$. Continuing in this way and starting with c instead of b we will reach a contradiction to the d.c.c. Therefore $Z \subseteq Q$, by Lemma 2(i).

(ii) Assume $Q = Z$ and take any element $0 \neq a \in Q$. By assumption $a = bc$ for some $b, c \in Q$ (use Lemma 6). Hence $r(c) \subseteq r(a)$, and with the same arguments as in (i) we get $r(c) \subset r(a)$. This leads again to a contradiction as in (i). Therefore $Z \subset Q$. \square

Now we are able to prove Theorem 8.

Proof. Case 1. Assume that R satisfies a.c.c. on principal right annihilators. By the symmetric version of Theorem 2.2 and the final remark in [6], $R/P(R)$ is a right nonsingular right D -ring which satisfies (C). Then $P(R)$ is completely prime by Proposition 3. Also, $Z = P(R)$ by Lemmas 1 and 4. Finally by [7, Theorem 3.2], we have that P is either nilpotent or $P = P^2 \neq 0$. Assume $P = P^2 \neq 0$ and take any $0 \neq a \in P$. Then there exists $a_1, b_1 \in P$ with $a = a_1b_1$. Repeating the argument, starting with a_1 instead of a , we have $a = a_2b_2b_1$ for some $a_2, b_2 \in P$. By induction, we get a sequence $\{b_1, b_2, \dots\}$ of elements of P such that for every $m \geq 1$ there exists $a_m \in P$ with $a = a_mb_m \cdots b_1$. On the other hand $P = Z$ is right T -nilpotent, and so we get $a = 0$, a contradiction.

Case 2. Assume that R satisfies d.c.c. on principal right annihilators.

Since the generalized nil radical A is completely prime, $Z \subseteq A$ by Lemma 7. So $Z \subseteq P$ if $A = P$. If $A \neq P$ holds, we have $A = A^2 \neq 0$ by Lemma 2(iv) which implies $Z \subset A$ by Lemma 7 again. Thus, by Lemma 5, $Z \subseteq P$ follows in any case. Therefore by Lemma 4 $P = Z$ and R/P is a prime ring which has no nonzero nilpotent elements. Consequently P is completely prime. Finally, if P is not nilpotent, as in Case 1 we have $P = P^2 \neq 0$. So we get $Z \subset P$, a contradiction. \square

Corollary 9. *Let R be a right D -ring which satisfies condition (C) and a.c.c. on principal right annihilators. Then $P = N_l(R)$, where $N_l(R)$ is the set of left zero-divisors of R .*

Proof. Obviously we have $P \subseteq N_l(R)$. Assume there exists $a \notin P$ with $r(a) \neq 0$. Hence $r(a) \subseteq r(a^2) \subseteq \dots$ and $a^n \notin P$ for every integer n , since P is completely prime. By assumption, there exists m with $r(a^m) = r(a^{m+1})$. Take any $0 \neq b \in r(a)$. Thus $ab = 0$, and it follows that $b \in P \subset a^mR$; therefore, $b = a^mx$ for some $x \in R$, so $a^{m+1}x = 0$, which leads to $b = a^mx = 0$, a contradiction. \square

We were unable to answer the following

Question. Is $P = N_l(R)$ also under d.c.c. for principal right annihilators?

Obviously, it has an affirmative answer if R is prime.

For the sake of completeness, we include a rather obvious example showing the relevance of assumption (C).

Example 10. There exist right D -rings in which the prime radical is not a completely prime ideal, even under strong conditions of finiteness.

Let K_i , $i = 1, 2, \dots, n$, be fields and set $R = K_1 \oplus K_2 \oplus \dots \oplus K_n$. We denote by e_i for $i = 1, \dots, n$ the canonical idempotent $(0, \dots, 1, \dots, 0)$. It is easy to check that every ideal of R is of the type Re with $e = e_{i_1} + \dots + e_{i_j}$, for some idempotents e_{i_k} . So the lattice of ideals of R is finite. By Theorem 1.6 in [11] R is right distributive if and only if for every $a, b \in R$ there exist $x, y \in R$ with $bx \in aR$, $ay \in bR$, and $x + y = 1$. Applying this result it can easily be deduced that the ring R constructed above is right (and left) distributive. We have $J(R) = (0)$ and so R does not satisfy the assumption (C), since (0) is neither completely prime nor prime. Thus the prime radical $P(R) = 0$ is not prime. Moreover, we remark that any nonzero ideal of R is idempotent and so there exist nonzero idempotent ideals, which are not prime provided $n \geq 3$. (We recall from [1] that in right chain rings idempotent ideals are always completely prime.)

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