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On waists of right distributive rings

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Abstract. In this paper we prove some general results on waists of right distributive rings which have at least one completely prime ideal contained in the Jacobson radical. Also we give a complete characterization of right D-domains of this type which satisfy ascending chain condition on waists.

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Introduction

A right distributive ring, right D-ring for short, is a ring whose lattice of right ideals is distributive. It is well-known that the class of commutative D-domains coincides with the class of Prüfer domains. Noncommutative right D-rings were investigated in a paper of Stephenson [9]. Brungs proved that right D-domains are locally right chain rings. Recently several papers showed that some features for right chain rings can be carried over to right D-rings ([4], [5], [7], [8]).

In particular, from these last papers the relevance of the following condition became evident:

(MP) There exists a completely prime ideal contained in the Jacobson radical.

Right D-rings with (MP) were first considered in [7] and then also in [4] and [5] where not only the ideal structure of such rings was investigated but also waists were studied. A *waist* is a right ideal I such that for every right ideal K we have either $K \subseteq I$ or $I \subseteq K$. The Theorems 3.9 and 3.11 in [5] give a complete characterization of waists containing the prime radical.

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Throughout this paper R is a right D-ring having an identity element and satisfying condition (MP). By $J(R)$ (resp. $P(R)$) we denote the Jacobson (resp. prime) radical of R. It is well-known that if a completely prime ideal is contained in $J(R)$, then it is a waist ([9], Proposition 2.1 (ii)). Then there exists a largest completely prime ideal contained in $J(R)$. We denote this ideal by Q. The notations $\supset$ and $\subset$ will mean strict inclusions.

The purpose of this paper is to continue the study on waists of right D-rings satisfying (MP). Section 1 is an introductory section. In Section 2 we prove some general results and in Section 3 we study right D-domains which satisfy ascending chain condition on waists. The last Section 4 contains some examples.

The main result of Section 2 states that if L is a semiprime right multiplicative ideal of R contained in Q, then L is a prime right ideal as well as a waist. This result is an extension of several results ([9], Proposition 2.1 (ii); [7], Corollary 3.3; [5], Lemma 3.10). We also obtain some other results in this section.

In his paper Brungs studied right noetherian right D-domains. The main result in our Section 3 is connected with his Theorem 2. We prove here a theorem which gives several equivalent conditions for a right D-domain to satisfy ascending chain condition on waists. In particular, this is the case if and only if every non-zero waist of R is a unique (in the sense we will define afterwards) product of prime ideals contained in Q.

1 Prerequisities

In this section we recall some definitions and basic facts (see for example [5], Sections 2 and 3).

A subset T of R is said to be a *right multiplicative ideal* if for every $a \in T$ and $x \in R$ we have $ax \in T$. A right multiplicative ideal T is said to be *prime* (resp. *semiprime*) if for $a, b \in R$ we have $aRb \subseteq T$ (resp. $aRa \subseteq T$) implies either $a \in T$ or $b \in T$ (resp. $a \in T$).

If T is a right multiplicative ideal of R, the *right associated completely prime ideal* $P_r(T)$ of T is defined by

$$P_r(T) = \{a \in R : \exists\, b \in R \setminus T \text{ with } ba \in T\} \quad \text{(see [5], p. 2701)}.$$

A right ideal I of R is said to be a *waist* if for every right ideal K of R we have either $I \subseteq K$ or $K \subseteq I$. It is clear that I is a waist if and only if for every $a \in R \setminus I$ we have $I \subset aR$.

In this paper we will use frequently the results of ([5], Sections 2, 3). We just recall Lemma 1.2 of that paper which was actually proved by Stephenson [9] and Mazurek [7].

Lemma 1.1 *Let R be a right D-ring and P a completely prime ideal contained in $J(R)$.*

(i) *For any right ideal I of R we have either $I \subseteq P$ or $P \subseteq I$.*

(ii) *For any $a \in R \setminus P$ we have $aP = P$.*

(iii) *For any $a, b \in R$ one of the following holds: $aR \subseteq bR$, $bR \subseteq aR$ or $aP = bP$.*

(iv) *Let I be any two-sided ideal of R with $I \subseteq P$. Then either I is nilpotent or $\bigcap_{n \geq 1} I^n$ is a completely prime ideal.*

Finally, if R is a right D-ring which satisfies (MP), then the prime radical $P(R)$ is prime ([7], Corollary 3.3).

2 Some general results

By ([5], Lemma 2.2 and Theorem 2.6) every completely prime right multiplicative ideal contained in $J(R)$ is a two-sided completely prime ideal as well as a waist. We also know that the prime radical is prime as well as a waist ([5], Lemma 3.10).

The main purpose of this section is to prove the following.

Theorem 2.1 *Let R be a right D-ring and let Q be the largest completely prime ideal in $J(R)$. Then any semiprime right multiplicative ideal L contained in Q is a prime right ideal and a waist.*

Proof. First we show that L is a right ideal. If $L = Q$ we are done, so we may assume $L \subset Q$. Take a, b in L. Then one of the following holds: $a \in bR$, $b \in aR$ or $aQ = bQ$ (Lemma 1.1 (iii)). If one of the first two possibilities occur we easily obtain $a + b \in L$. In the case $aQ = bQ$, then for every $x \in Q$ there exists $y \in Q$ such that $ax = by$. Thus $(a+b)x = b(y+x) \in L$ and we have $(a+b)Q \subseteq L$. However $a + b \in Q$ and so $(a+b)R(a+b) \subseteq (a+b)Q \subseteq L$. Therefore $a + b \in L$, i.e. L is a right ideal.

To show that L is a waist, take any $b \in R \backslash L$ and assume $L \nsubseteq bR$. Then there exists $a \in L \backslash bR$ and we have neither $aR \subseteq bR$ nor $bR \subseteq aR$. Therefore $bQ = aQ \subseteq L$. Also $b \in Q$ because otherwise $aR \subseteq L \subset Q \subset bR$. It follows that $bRb \subseteq bQ \subseteq L$ and hence $b \in L$, a contradiction. Consequently L is a waist.

It remains to prove that L is prime. For this purpose it is enough to show that L is an intersection of right prime ideals which are contained in Q, since we already know that these prime ideals are waists. Take $a \notin L$. We show that there exists a right prime ideal P containing L such that $a \notin P$. If $a \notin Q$ we may take $P = Q$, so assume that $a \in Q$.

As usually we define a sequence $S = \{a_1, a_2, a_3, \ldots\}$ as follows: put $a_1 = a$; choose an element $r_1 \in R$ such that $a_1 r_1 a_1 \notin L$ and put $a_2 = a_1 r_1 a_1$; choose $r_2 \in R$ with $a_2 r_2 a_2 \notin L$ and put $a_3 = a_2 r_2 a_2$, and so on (see [6], Lemma 10.10, p. 167). Then there exists a right ideal P of R containing L and which is maximal with respect to $P \cap S = \emptyset$. To show that P is prime we first show that P is a waist. By the above it is enough to show that P is semiprime.

In fact, assume that $xRx \subseteq P$ and $x \notin P$. If $P \subset xR$, by the maximality of P there exists an integer i such that $a_i \in xR$. Thus $a_i R a_i \subseteq xRxR \subseteq P$ and we obtain $a_{i+1} \in P$, a contradiction. So we may assume that there exists $b \in P \backslash xR$. Then $bR \nsubseteq xR$ and $xR \nsubseteq bR$; hence $xQ = bQ$. By the maximality of P, the right ideal

$A = xR + P \supset P$ contains an element of S. So there exists an integer j and elements $r \in R$, $p \in P$ such that $a_j = xr + p$. Hence for every $t \in R$ we have $a_j t a_j = xrtxr + xrtp + pta_j \in P + xQ = P + bQ = P$; consequently $a_{j+1} \in P$, a contradiction. Therefore we conclude that P is semiprime and so is a waist.

Now we show that P is prime. Assume that $xRy \subseteq P$, for elements x, y which are not in P. Thus $P \subset xR$ and $P \subset yR$; hence there exist i, j such that $a_i \in xR$ and $a_j \in yR$. We obtain $a_i R a_j \subseteq P$ and it follows as usually that $a_{k+1} \in P$, where $k = max\{i, j\}$ ([6], Lemma 10.10), a contradiction. The proof is complete. □

Remark 2.2 We point out that not necessarily every right multiplicative ideal contained in $J(R)$ is a right ideal. In fact, if a, b are elements of $J(R)$ such that $aR \nsubseteq bR$ and $bR \nsubseteq aR$, then $aR \cup bR$ is a right multiplicative ideal contained in $J(R)$ which is not a right ideal.

Now we prove the following Theorem which is an extension of ([5], Lemmas 2.10 and 2.12).

Theorem 2.3 *Assume that I is a waist with $P(R) \subseteq I \subseteq Q$ and $a \in R$. Then the right ideal aI is a waist. Moreover, if $aI \neq 0$, then $P_r(aI) = P_r(I)$.*

Proof. If $aI = 0$ there is nothing to prove. So assume $aI \neq 0$. Note that in this case we have $ab \notin aI$ if and only if $b \notin I$. In fact, the implication is obvious in one direction and for the other direction assume $b \notin I$ and $ab = ax$, for some $x \in I$. We have $a(b - x) = 0$ where $b - x \notin I$, i.e., $I \subset (b - x)R$. It follows that $aI = 0$, a contradiction.

If $x \in P_r(I)$, then there exists $b \notin I$ with $bx \in I$. So $abx \in aI$ and $ab \notin aI$; hence $x \in P_r(aI)$. Consequently $P_r(I) \subseteq P_r(aI)$.

By ([5], Theorems 3.9 and 3.11) we have that $I = \bigcap_{b \notin I} bP_r(I)$. Now we show that $aI = \bigcap_{b \notin I} abP_r(I)$. From this it follows that aI is a waist since for any $b \in R$, $abP_r(I)$ is a waist.

First, it is clear that $aI \subseteq \bigcap_{b \notin I} abP_r(I)$. Assume that $y \in \bigcap_{b \notin I} abP_r(I)$. Hence $y \in aR$ and so there exists $r \in R$ such that $y = ar$. If $r \in I$ we are done. We assume $r \notin I$ and we will get a contradiction. For any $b \notin I$, $y = abz$, for some $z \in P_r(I)$. Hence $a(r - bz) = 0$ and also $r - bz \in I$, because in the contrary case $I \subset (r - bz)R$ and $aI = (0)$ follows. Thus $r = bz + x$, for some $x \in I$, where $bz \notin I$. In this case $I \subset bzR$ and therefore $x = bzc$, for some $c \in R$. It follows that $r = bz(1 - c) \in bP_r(I)$. Thus $r \in \bigcap_{b \notin I} bP_r(I) = I$, a contradiction.

Now, assume that $c \notin aI$. We compare a and c. If $aR \subseteq cR$ we have $aI \subseteq aP_r(I) \subseteq cP_r(I)$. If $c = ar, r \in R$, we necessarily have $r \notin I$. Thus $cP_r(I) = arP_r(I) \supseteq aI$. The remaining possibility reads $cP_r(I) = aP_r(I) \supseteq aI$. Therefore $aI \subseteq cP_r(I)$ when $c \notin aI$.

Combining the former result $aI = \bigcap_{b \notin I} abP_r(I)$ with the last one we obtain $aI = \bigcap_{c \notin aI} cP_r(I)$. Now the proof can easily be completed using ([5], Theorem

2.13). In fact, since $P_r(aI)$ is the smallest completely prime ideal L of R such that $aI = \bigcap_{c \notin aI} cL$ we obtain $P_r(aI) \subseteq P_r(I)$. □

Theorem 2.3 has the following interesting corollary which is an extension of ([4], Lemma 6).

Corollary 2.4 *Assume that I is a waist containing $P(R)$ and K is any right ideal of R. Then $KI = \{ab\colon a \in K, b \in I\}$. In particular, KI is a waist.*

Proof. Set $x = a_1 b_1 + \cdots + a_n b_n \in KI$, where $a_i \in K$, $b_i \in I$, $i = 1, \ldots, n$. Since $a_i I$ is a waist, for $i = 1, \ldots, n$, it follows that there exists some i, say $i = 1$, with $a_j I \subseteq a_1 I$ for $j = 1, \ldots, n$. Thus $a_j b_j = a_1 b'_j$ for some $b'_j \in I$ and so $x = a_1(b_1 + b'_2 + \cdots + b'_n)$ where $b_1 + b'_2 + \cdots + b'_n \in I$. In particular, $KI = \bigcup_{a \in K} aI$ is a waist. □

Remark 2.5 It follows from Corollary 2.4 that if R is a prime ring the product of any finite number of waists is again a waist.

To prove the following proposition we need a lemma.

Recall that if P is a completely prime ideal contained in Q and $T = R \setminus P$, for any $a \in R$, $(aR)T^{-1}$ is defined by

$$(aR)T^{-1} = \{b \in R\colon \exists t \in T \text{ with } bt \in aR\}.$$

Also $(aR)T^{-1}$ is a waist ([5], Proposition 3.6).

We denote by S the complement $R \setminus Q$ of Q, where Q is the maximal completely prime ideal contained in $J(R)$.

Lemma 2.6 *Assume that $a \in Q \setminus P(R)$. Then there does not exist a waist L with $aQ \subset L \subset (aR)S^{-1}$.*

Proof. If L is a waist such that $aQ \subseteq L \subset (aR)S^{-1}$, take any $b \in (aR)S^{-1} \setminus L$. Then $bs \in aR$, for some $s \notin Q$. Also $L \subseteq bP_r(L) \subseteq bQ = bsQ \subseteq aQ$. □

Proposition 2.7 *Let I be a waist which is finitely generated as a right R-module with $I \supset P(R)$. Then there exists $a \in I$ such that $I = (aR)T^{-1}$, where $T = R \setminus P_r(I)$.*

Proof. We may assume that $I = a_1 R + \cdots + a_n R$, where $P(R) \subset a_i R$, $i = 1, \ldots, n$, and $a_i R \nsubseteq a_j R$ for $i \neq j$. Therefore $a_i Q = aQ$ and $(a_i R)S^{-1} = (aR)S^{-1}$, for $i = 1, \ldots, n$, where $a = a_1$ ([5], Lemma 3.3). Take any $b \in I$. We show that $(bR)S^{-1} \subseteq (aR)S^{-1}$. If $b \in P(R)$, then $bR \subset aR$ and we are done. Assume $b \notin P(R)$ and $b = a_1 r_1 + \cdots + a_n r_n$, $r_i \in R$. For any $q \in Q$ we have $bq = a_1 r_1 q + \cdots + a_n r_n q \in aQ$. Hence $bQ \subseteq aQ$ and using Lemma 2.6 we obtain $(bR)S^{-1} \subseteq (aR)S^{-1}$. Take any $x \in (bR)T^{-1}$. Then $xt \in bR \subseteq (aR)S^{-1}$, for some $t \notin P_r(I)$. Thus $xts \in aR$, for some $s \notin Q$, and since $ts \notin P_r(I)$ we obtain $x \in (aR)T^{-1}$. Consequently $I = \bigcup_{b \in I} (bR)T^{-1} = (aR)T^{-1}$. □

3 Right D-rings with a.c.c. on waists

In this section we consider prime right D-rings with (MP) which satisfies *ascending chain condition on waists* (a.c.c.w., for short). The results are then related with the results obtained by Brungs ([2], Sect. 3). In this paper the author studies right noetherian D-domains and his Theorem 2 gives several equivalent conditions for a right noetherian integral domain to be a D-ring. Our main result in this section is Theorem 3.1 which gives several equivalent conditions for a right D-domain to satisfy a.c.c.w. and proves that all prime ideals in Q are completely prime.

We say that R has a.c.c.w. if every family of waists has a maximal member. Also, we say that R satisfies restricted ascending chain condition on waists (r.a.c.c.w., for short) if every family of waists containing the prime radical has a maximal member. Finally, R is said to have ascending chain condition on prime ideals contained in Q (a.c.c.p., for short) if every family of prime ideals of R which are contained in Q has a maximal member. By the definition of a waist (resp. Theorem 2.1) it is clear that if R has a.c.c.w. (resp. a.c.c.p.) every family of waists (resp. prime ideals contained in Q) contains a largest member.

The rings under discussion in this section have the property, as it will be proved later (see Corollary 3.3), that every prime ideal P of R contained in Q is completely prime. Then if $P \supset P'$ and P' is a prime ideal contained in Q, we have $PP' = P'$ (Lemma 1.1 (ii)). Consequently, when we consider products of prime ideals $P_1 P_2 \cdots P_n$ with $P(R) \subseteq P_i \subseteq Q$, $1 \leq i \leq n$, we can cancel a factor P_{i-1} if $P_{i-1} \supset P_i$. Thus the product can be written as a product of the above type, where $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n$. We will call this expression an *standard expression* of the product. Also, we say that *unique representation for products of prime ideals* holds if for any two standard products we have $P_1 P_2 \cdots P_n = P_1' P_2' \cdots P_t'$ implies $n = t$ and $P_i = P_i'$ for all $1 \leq i \leq n$.

In this section we will consider the following conditions (w_i) on waists and their corresponding restrictions (rw_i):

(w_1) (resp. (rw_1)) R satisfies a.c.c.w. (resp. r.a.c.c.w.).

(w_2) (resp. (rw_2)) For every waist I of R (resp. with $I \supset P(R)$), the family $\{bQ : b \in I\}$ has a maximal member.

(w_3) (resp. (rw_3)) For every waist I of R (resp. with $I \supset P(R)$) and prime ideal P contained in Q, the family $\{bP : b \in I\}$ has a maximal member.

(w_4) (resp. (rw_4)) Every non-zero waist (resp. waist properly containing $P(R)$) of R can be written as a unique standard product of prime ideals.

(w_5) (resp. (rw_5)) For waists $0 \neq H_1 \subseteq H_2$ (resp. with $P(R) \subset H_1 \subseteq H_2$) there exists a unique waist H such that $H_2 H = H_1$.

(w_6) (resp. (rw_6)) For every non-zero waist I (resp. waist properly containing $P(R)$), we have $IQ \neq I$.

(w_7) (resp. (rw_7)) Every waist (resp. waist properly containing $P(R)$) of R is of the type aP for some $a \in R$ (resp. $a \in R \setminus P(R)$) and P a completely prime ideal in Q (resp. with $P(R) \subset P \subseteq Q$) and for every non-zero prime ideal $L \subseteq Q$ (resp. with $P(R) \subset L \subseteq Q$) we have $LQ \neq L$.

We point out that if $Q \neq P(R)$, then R satisfies the condition (rw_i) if and only if $R/P(R)$ satisfies the condition (w_i), for $i = 1, 2, \ldots, 7$. Note that the case $Q = P(R)$ is a trivial one since Q is the largest waist of R ([5], Corollary 3.13).

The purpose of the section is to prove the following

Theorem 3.1 *Let R be a right D-ring which satisfies* (MP). *Then the conditions* (rw_1), (rw_2), ..., (rw_7) *are equivalent. Furthermore, if these equivalent conditions are fulfilled, then every prime ideal contained in Q is completely prime.*

In particular, if R is a domain, then conditions (w_1), (w_2). ..., (w_7) *are equivalent.*

We prove the theorem in several steps. We begin with the following

Lemma 3.2 *Assume that I is a waist and the family $\{bQ : b \in I\}$ has a maximal member aQ. Then for every completely prime ideal $P \subseteq Q$, aP is a maximal member of $\{bP : b \in I\}$. In particular, $IP = aP$.*

Proof. If $bQ \subseteq aQ$, using Lemma 1.1 (ii), we obtain $bP = bQP \subseteq aQP = aP$. The equality $IP = aP$ follows by Corollary 2.4. □

Corollary 3.3 *Assume that R satisfies one of the conditions* (rw_1), ..., (rw_7) *above. Then every prime ideal P of R with $P(R) \subset P \subseteq Q$ is not idempotent. In particular, every prime ideal contained in Q is completely prime.*

Proof. Let P be any completely prime ideal with $P(R) \subset P \subseteq Q$. If ($rw_2$) holds, by Lemma 3.2 we have $P^2 = aP$, for some $a \in P$. Therefore $P^2 = P$ gives a contradiction. Under the assumptions (rw_4) or (rw_5), $P^2 = P$ is impossible from the unique representation. Finally, if (rw_7) holds we have $P^2 \subseteq PQ \neq P$. The same result follows from (rw_6). Then P is not idempotent. The rest is clear by ([5], Corollary 1.4) and Lemma 1.1 (iv). □

Combining Lemma 3.2 and Corollary 3.3 we immediately have

Corollary 3.4 *The conditions* (w_2) *and* (w_3) *(resp.* (rw_2) *and* (rw_3)) *are equivalent.*

Remark 3.5 Assume that R satisfies one of the conditions (w_1), ..., (w_7) above. Then the same argument used in the proof of Corollary 3.3 shows that if the prime radical is non-zero, then it is not idempotent. So in this case the prime radical is nilpotent.

Now we will construct a transfinite sequence containing all the prime ideals of R contained in Q. Next we will construct another sequence containing all the waists which contain $P(R)$.

For the first we assume that R has no idempotent prime ideal properly containing $P(R)$ and contained in Q. Put $P_1 = Q$ and $P_2 = \bigcap_{n \geq 1} Q^n$. Note that P_2 is also a completely prime ideal different of P_1 (Lemma 1.1 (iv)) and there is no prime ideal P with $P_2 \subset P \subset P_1$. Assume that for some ordinal number λ, a prime ideal P_α has been defined for every $\alpha < \lambda$, such that for $\beta < \alpha < \lambda$ we have $P(R) \subset P_\alpha \subset P_\beta$. Then we define P_λ by:

(a) If there exists some ordinal α with $\lambda = \alpha + 1$ we put $P_\lambda = \bigcap_{n \geq 1} P_\alpha^n$.

(b) If λ is a limit ordinal we put $P_\lambda = \bigcap_{\alpha < \lambda} P_\alpha$.

It is clear that this sequence terminates when $P_{\lambda_0} = P(R)$, for some ordinal λ_0. Also, for $\alpha < \lambda$ we have $P_\lambda \subset P_\alpha$ and there is no further prime ideal between P_α and $P_{\alpha+1}$.

Proposition 3.6 *Assume that R has no idempotent prime ideal properly containing $P(R)$ and contained in Q. Then for every prime ideal $P \subseteq Q$ there exists an ordinal α_0 with $P = P_{\alpha_0}$. In particular, R satisfies a.c.c.p.*

Proof. The set $\Omega = \{\alpha : \alpha \leq \lambda_0,\ P_\alpha \subseteq P\}$ is a non-empty set of ordinal numbers, where $P_{\lambda_0} = P(R)$. Denote by α_0 the first element of Ω. Then $P_{\alpha_0} \subseteq P$ and $P_\beta \supset P$ for $\beta < \alpha_0$. We easily obtain $P = P_{\alpha_0}$.

In particular, if $\mathscr{P}$ is any family of prime ideals contained in Q we may assume that $\mathscr{P} = \{P_\alpha : \alpha \in \Lambda\}$, where Λ is some set of ordinal numbers. If α_0 is the first element of Λ, then P_{α_0} is the largest member of $\mathscr{P}$. □

Now we start the second construction. We assume here that for every waist I with $I \supset P(R)$ we have $IQ \neq I$.

Put $I_1 = Q$ and $I_2 = Q^2$. Assume that for some ordinal λ, a waist I_α has been defined for every $\alpha < \lambda$ such that for $\beta < \alpha < \lambda$ we have $P(R) \subset I_\alpha \subset I_\beta$. Then we define I_λ by:

(a) If there exists some ordinal α with $\lambda = \alpha + 1$ we put $I_\lambda = I_\alpha Q$.

(b) If λ is a limit ordinal we put $I_\lambda = \bigcap_{\alpha < \lambda} I_\alpha$.

It follows easily that this sequence is a well defined sequence of different waists and there exists an ordinal λ_0 with $I_{\lambda_0} = P(R)$. Also, if $\alpha < \lambda$ we have $I_\lambda \subset I_\alpha$.

Corresponding to Proposition 3.6 we obtain the following result for waists.

Proposition 3.7 *Assume that the condition (rw_6) is satisfied. Then for every waist I with $I \supset P(R)$ there is no further waist L such that $IQ \subset L \subset I$. In particular, for every waist $L \supseteq P(R)$ there exists an ordinal α_0 with $L = I_{\alpha_0}$ and R satisfies r.a.c.c.w.*

Proof. Assume that L is a waist such that $IQ \subseteq L \subset I$ and choose any $x \in I \setminus L$. By ([5], Theorems 3.14 and 3.17) we have $IQ \subseteq L \subseteq xP_r(L) \subseteq IQ$, since $P_r(L) \subseteq Q$. The proof of the remaining part is similar to the proof of Proposition 3.6. □

Remark 3.8 The proof of Proposition 3.7 uses heavily that $I \supset P(R)$ ([5], Theorem 3.11). So we cannot prove that $(w_6) \Rightarrow (w_1)$ using similar argument. Now, if we assume that for every waist I, IQ and I are neighbours in the lattice of waists, then we can continue the transfinite sequence to reach (0) and so obtain (w_1) as a consequence. However, we could not prove the converse of this implication.

We summarize the results that we obtained so far.

Corollary 3.9 *The conditions (rw_1), (rw_2), (rw_3) and (rw_6) of Theorem 3.1 are equivalent.*

Proof. Corollary 3.4 gives the equivalence between (rw_2) and (rw_3) and Proposition 3.7 gives $(rw_6) \Rightarrow (rw_1)$. Also $(rw_1) \Rightarrow (rw_2)$ is obvious. Finally, assume (rw_2) and suppose that I is a waist with $I \supset P(R)$. Then $IQ = aQ$, for some $a \in I$ and $IQ \neq I$ follows. □

We can also obtain the following

Lemma 3.10 *The implications $(w_7) \Rightarrow (w_6)$ and $(rw_7) \Rightarrow (rw_6)$ hold.*

Proof. Assume (w_7) and let I be a non-zero waist. Then $I = aP$, for some $a \in R$ and P a completely prime ideal contained in Q. Then $IQ = aPQ \subset aP = I$, since $PQ \subset P$. The other part is similar. □

Now we will study products of prime ideals contained in Q. We already know that any such product can be written as an standard product $I = P_1 P_2 \cdots P_t$, where $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_t$. By Remark 2.5 any such product is a waist. Also $I \supset P(R)$ if and only if $P_1 \neq P(R)$.

Assume that $I = P_1 P_2 \cdots P_t$ and $K = P_1' P_2' \cdots P_l'$ are two standard products. By induction we easily see that if $P_i' \supseteq P_i$ for $1 \leq i \leq l$ and $t \geq l$ we have $K \supseteq I$. Hence the following is clear.

Lemma 3.11 *Let $I = P_1 P_2 \cdots P_t$ and $K = P_1' P_2' \cdots P_l'$ two standard products of prime ideals. If $K \subset I$, then one of the following conditions holds*

(i) *There exists $i \leq min\{t, l\}$ such that $P_i \supset P_i'$, or*

(ii) $t < l$

Corollary 3.12 *The implications $(w_4) \Rightarrow (w_1)$ and $(rw_4) \Rightarrow (rw_1)$ hold.*

Proof. Assume (w_4) and that $\mathscr{F} = \{I_j : j \in \Gamma\}$ is a family of waists, where Γ is a set of indices. We may assume that $I_j \neq 0$ and hence we have a unique standard decomposition $I_j = P_{j_1} P_{j_2} \cdots P_{j_{t_j}}$, where $0 \neq P_{j_1} \subseteq P_{j_2} \subseteq \cdots \subseteq P_{j_{t_j}} \subseteq Q$. Consider a maximal member in $\{P_{j_1} : j \in \Gamma\}$ (Proposition 3.6), say P_1, and an element $I_{j_1} \in \mathscr{F}$ with $P_{j_1} = P_1$. If I_{j_1} is a maximal member of $\mathscr{F}$ we are done. In the contrary case there exists $I_k \in F$ with $I_k \supset I_{j_1}$. Using Lemma 3.11 and a.c.c.p. it is easy to complete the proof.

The other implication can be proved in similar way. □

Now we can give a more precise information provided a.c.c.w. (resp. r.a.c.c.w.) holds. First we note the following

Remark 3.13 Assume that R satisfies (w_2) (resp. (rw_2)), that I is a waist (resp. with $I \supset P(R)$) and H is any product of prime ideals contained in Q. Using the same argument as in Lemma 3.2 we can show that $IH = aH$, for some $a \in I$. Moreover, the element a can be chosen as an element with the property that aQ is a maximal member of $\{bQ : b \in I\}$; so a does not depend on H.

Lemma 3.14 *Assume that R satisfies (w_2) (resp. (rw_2)) and for some products H and L of prime ideals we have $H \subset L$ (we include here the possibility $L = R$ is the empty product). Then for a waist I (resp. with $I \supset P(R)$) such that $IL \neq (0)$ we have $IH \subset IL$.*

Proof. It is clear that $IH \subseteq IL$. Assume (w_2) and that $IH = IL$. By the former remark there exists $a \in I$ such that $IH = aH$ and $IL = aL$. Take any $x \in L \setminus H$. Then there exists $y \in H$ such that $ax = ay$, so $a(x - y) = 0$. Since $x - y \notin H$ and H is a waist we obtain $aH = (0)$. Consequently, $ax = 0$; hence $aL = (0)$, a contradiction.

The other case is similar. □

Corollary 3.15 *Assume that R satisfies (w_2) (resp. (rw_2)) and let $I = P_1 P_2 \cdots P_t$ and $K = P'_1 P'_2 \cdots P'_l$ be two non-zero standard products of prime ideals (resp. with $K \supset P(R)$). Then $K \subset I$ if and only if one of the following possibilities occur.*

(i) *$P_i = P'_i$ for $i = 1, \ldots, t$ and $l > t$, or*

(ii) *There exists $j \leq \min\{t, l\}$ such that $P_i = P'_i$, for $1 \leq i < j$, and $P_j \supset P'_j$.*

In particular, unique representation for non-zero products of prime ideals (resp. prime ideals properly containing $P(R)$) holds.

Proof. Assume (w_2). If (i) holds we may write $K = IP'_{t+1} \cdots P'_l$ and certainly $K \subset I$ since $IQ \subset I$. Under the assumption (ii) we have $P'_j \cdots P'_l \subseteq P'_j \subset P_j \cdots P_t$ and $K \subset I$ follows by Lemma 3.14.

Conversely, assume $K \subset I$. By the first part $P_1' \subseteq P_1$. If $P_1' \subset P_1$ we are done, so assume $P_1 = P_1'$. Therefore $P_2' \cdots P_l' \subset P_2 \cdots P_t$ by the assumption. The proof can easily be completed by induction. The other case is similar. □

Now we are able to complete the proof of the main result.

Proof (of Theorem 3.1) $(rw_1) \Rightarrow (rw_4)$. If there exists a waist containing $P(R)$ which is not a product of prime ideals contained in Q, then we may take the largest one with this property, say I. Then I is not prime. Denote by A the smallest completely prime ideal of R containing I. Clearly A is nilpotent modulo I. So there exists an integer $n \geq 2$ such that $A^n \subseteq I \subset A^{n-1}$. Now, denote by B the largest waist which has the property $A^{n-1}B \subseteq I$. Hence $B \supseteq A \supset I$ and consequently B is a product of prime ideals contained in Q. We show that $A^{n-1}B = I$, which is a contradiciton.

Assume that there exists a waist K such that $B = KQ \subset K$. Then we have $I \subset A^{n-1}K$ and so $I \subseteq A^{n-1}KQ = A^{n-1}B$, by Proposition 3.7. We are done in this case. In the other case $B = \bigcap_\alpha L_\alpha$, where $\{L_\alpha\}$ is a family of waist with $B \subset L_\alpha$ for every α. By Remark 3.13 there exists $a \in A^{n-1}$ such that $A^{n-1}L_\alpha = aL_\alpha$, for every α, and $A^{n-1}B = aB$. Take any $x \in \bigcap_\alpha A^{n-1}L_\alpha$ and write $x = ay_\alpha$, for $y_\alpha \in L_\alpha$. Since R is a domain we obtain $y = y_\alpha = y_\beta \in L_\alpha$, for every α, β, and so $y \in \bigcap_\alpha L_\alpha = B$. Therefore $x \in A^{n-1}B$. It follows that $A^{n-1}B \subseteq I \subseteq \bigcap_\alpha A^{n-1}L_\alpha \subseteq A^{n-1}B$.

The uniqueness of the decomposition was already proved (Corollary 3.15).

$(rw_1) \Rightarrow (rw_5)$ If $H_1 = H_2$ we have $H_1Q \subset H_1 = H_1R$. So the result follows in this case. For the case $H_1 \subset H_2$ the proof can be completed by the same arguments as in $(rw_1) \Rightarrow (rw_4)$.

$(rw_4) \Rightarrow (rw_7)$ It follows easily from Corollary 3.12 and Remark 3.13.

The proof is complete because $(rw_5) \Rightarrow (rw_6)$ is evident. □

Theorem 3.1 gives the equivalence between $(w_1), \ldots, (w_7)$ only when R is domain. Some of the implications have been proved in general. However, we do not know whether the equivalence between all the above conditions remains true for any right D-ring with (MP). In the next proposition we collect the implications which are known to be true. The proof is omitted since the remaining parts are easy to obtain.

Proposition 3.16 *Let R be a right D-ring which satisfies* (MP). *Then the following implications hold:* $(w_4) \Rightarrow (w_1) \Rightarrow (w_2) \Leftrightarrow (w_3) \Rightarrow (w_6)$, $(w_4) \Rightarrow (w_5) \Rightarrow (w_6)$ *and* $(w_4) \Rightarrow (w_7) \Rightarrow (w_6)$.

Theorem 3.1 has the following interesting corollaries. The first one is clear.

Corollary 3.17 *Let R be a right D-ring with* (MP) *which satisfies r.a.c.c.w. Then every waist of R which contains $P(R)$ is a two-sided ideal.*

Corollary 3.18 *Let R be a right D-domain which satisfies a.c.c.w. If R is also a left D-ring, then Q is the unique non-zero prime ideal of R contained in $J(R)$ and every non-zero waist of R is a power of Q.*

Proof. Using the symmetric version of Lemma 1.1 (ii), it follows that for prime ideals $P \subset P'$ we have $PP' = P$. Thus the corollary is a consequence of Theorem 3.1 (w_4). □

Corollary 3.19 *Let R be a right D-domain which satisfies a.c.c.w. The set $\mathscr{H}$ of waists $H \subseteq Q$ constitutes a* right invariant right holoid *(see* [3] *for more details).*

Proof. Use Theorem 3.1 (w_5) and Lemma 3.14. □

Remark 3.20 Assume that R is a right D-domain which is right noetherian and let I be any right ideal of R. Then I has a unique representation as a product of prime ideals $I = P_1 P_2 \cdots P_t$, where $P_i \not\supseteq P_j$ if $i<j$, by ([2], Theorem 2). Comparing this representation with ours we conclude that in this case I is a waist if and only if $P_t \subseteq Q$.

4 Examples

The next example was briefly discussed in ([5], Example 1.5). However there are some misprints in [5] and we want to develop it here again, including some additional details.

Example 4.1 Let A be a right D-domain and σ a monomorphism of A. Then the skew field of fractions F of A do exists and the lattice of right A submodules of F is distributive ([9], Proposition 3.3 (ii)). We denote by σ again the extension of σ to a monomorphism of F and by $F[[t;\sigma]]$ the skew power series ring defined by $at = ta^\sigma$ for any $a \in F$.

We put $R = A \oplus tF[[t;\sigma]]$. First we note that if $f = \sum_{i=0}^\infty t^i a_i \in R$, where $0 \neq a_0 \in A$ and $a_i \in F$, there exists $g \in F[[t;\sigma]]$ such that $fg = 1$. Thus $fgth = th$, for every $h \in F[[t;\sigma]]$, where $gth \in R$.

Let I be a right ideal of R and assume that there exists $f = \sum_{i=0}^\infty t^i a_i \in I$ with $a_0 \neq 0$. By the above we obtain $tF[[t;\sigma]] \subseteq I$ and also $a_0 \in I$. Thus $I = (I \cap A) + tF[[t;\sigma]]$, where $I \cap A$ is a right ideal of A. Thus we easily see that I is a waist if and only if $I \cap A$ is a waist of A.

In general, assume that H is a right ideal and that $f = \sum_{i=n}^\infty t^i a_i \in H$ where n is the minimal integer such that $a_n \neq 0$. As above, there exists $g \in F[[t;\sigma]]$ such that for every $h \in F[[t;\sigma]]$ we have $t^{n+1}h = fgth \in fR \subseteq H$. Then $t^{n+1}F[[t;\sigma]] \subseteq H$ and also $t^n a_n \in H$. We write $H_0 = \{a \in F : t^n a \in H\}$ and we easily see that H_0 is a right A-submodule of F such that $H = t^n H_0 + t^{n+1}F[[t;\sigma]]$. Clearly any right ideal L of R with $t^{n+1}F[[t;\sigma]] \subset L \subseteq t^n F[[t;\sigma]]$ is of this type.

Now it is clear that $t^n F[[t,\sigma]]$ is a waist, for every $n \geq 1$. Also the ring R is a right D-domain since the lattice of right A-submodules of F is distributive. The Jacobson radical of R is $J(R) = J(A) \oplus tF[[t;\sigma]]$.

Assume that $J(A) = 0$. Now it is easy to see that $H = t^n H_0 + t^{n+1}F[[t;\sigma]]$ is a waist if and only if H_0 is a waist in the lattice of right A-submodules of F. Thus, if this lattice satisfies ascending chain condition, then R has a.c.c.w. In particular, if F has no non-zero right A-submodule which is a waist, then the waists of R are just the ideals $t^n F[[t;\sigma]]$, $n \geq 1$. The above situation occurs for example for $A = \mathbb{Z}$ (or $A = K[X]$, the polynomial ring in one indeterminate over a field K). This is an example in which we have just one non-zero completely prime ideal contained in $J(R)$.

The general construction can be iterated to construct a right D-ring $T = R \oplus XQ(R)[[X;\sigma]]$, where $Q(R)$ is the skew field of fractions of R. However, this ring T does not satisfies a.c.c.w. anymore (we were unably to clarify this question in general). If, for example, A is a commutative D-domain and $\sigma = \mathrm{id}_A$, then the lattice of R-submodules of $Q(R)$ does not satisfy a.c.c.w. In fact, we put $W_i = \{q \in Q(R) : \exists a \in A \text{ with } qt^i a \in R\}$; it is not hard to show that $W_1 \subset W_2 \subset W_3 \subset \cdots$ is a sequence of right R-submodules of $Q(R)$ which are waists in the lattice of right R-submodules of $Q(R)$.

The next proposition leads to a large class of examples.

Proposition 4.2 *Let A be a commutative D-domain with F as its field of fractions and $R_0 = F \oplus J(R_0)$ a right chain domain. Then the subring $R = A \oplus J(R_0)$ of R is a right D-domain with $J(R) = J(A) \oplus J(R_0)$.*

Proof. Obviously R is a domain. Note that every element of the type $1 + j \in R$, $j \in J(R_0)$, is a unit in R. It follows that the Jacobson radical $J(R)$ of R contains $J(R_0)$.

Let M be a maximal right ideal of R and set $M_0 = M \cap R$. Since $J(R_0) \subseteq J(R) \subseteq M$ we easily see that $M = M_0 \oplus J(R_0)$, where M_0 is a maximal ideal of A. Therefore, M is a two-sided ideal. Also, for every maximal ideal N_0 of A, $N_0 + J(R_0)$ is a maximal right ideal of R. It follows that $J(R) = J(A) \oplus J(R_0)$.

Applying a result of [2] it suffices to show that for every maximal right ideal $M = M_0 + J(R_0)$ of R (M_0 a maximal ideal of A) the localization R_M exists and is a right chain domain. First note that $S = R \setminus M = A \setminus M_0$ is a right Ore set in A. So given $x = a + j$, $a \in A$, $j \in J(R_0)$ and $s \in S$, there exists $s' \in S$, $b \in A$ such that $as' = sb$. Thus $(a + j)s' = s(b + s^{-1}js')$, where $b + s^{-1}js' \in R$. Consequently S is a right Ore set in R.

Finally we show that R_M is a right chain domain. Let $x = a + j$, $y = b + k$, arbitrary elements in R, where $a, b \in A$, $j, k \in J(R_0)$. We consider three cases.

Case 1: $a = 0$, $b \neq 0$. We have $y = b(1 + k')$, where $k' = b^{-1}k \in J(R_0)$, so $x = y(1 + k')^{-1}(b^{-1}x) \in yR$.

Case 2: $a \neq 0$, $b \neq 0$. Since $A_{(M_0)}$ is a chain domain we may assume there exists $q \in A_{(M_0)}$ with $a = bq$. We easily check that $(a + j) = b(1 + k')(q + l)$, where $l = (1 + k')^{-1}b^{-1}(j - kab^{-1}) \in J(R_0)$. Thus $x = y(q + l)$, $q + l \in R_M$, i.e., $x \in yR_M$.

Case 3: $a = b = 0$. Since R_0 is a chain domain we may assume there exists $z = cd^{-1} + l \in R_0$, $c, d \in A$, with $j = kz = kcd^{-1}(1 + l')$, $l' \in J(R_0)$. Also, either $cd^{-1} \in A_{(M_0)}$ or $dc^{-1} \in A_{(M_0)}$. Since $A_{(M_0)} \subseteq R_M$ we easily obtain that either $j \in kR_M$ or $k \in jR_M$. The proof is complete. $\square$

Lemma 4.3 *Denote again by $R = A \oplus J(R_0)$ the right D-domain given above and assume that $J(A) = 0$. A subset $H \subset R$ is a right ideal of R which is a waist if and only if H is a right ideal of R_0.*

Proof. If H is a right ideal of R_0, then $H \subseteq J(R_0)$ and H is a right ideal of R. Take $x = a + j \in R \backslash H$, $a \in A$, $j \in J(R_0)$. We show that $H \subseteq xR$. In case $a \neq 0$ this is clear because $H \subseteq J(R_0) \subset xR$ (Proposition 4.2, Case 1). So assume $a = 0$.

For any $h \in H$ there exists $r = cd^{-1} + k \in R_0$, $cd^{-1} \in F$, $k \in J(R_0)$, such that $h = xr$, since R_0 is a right chain domain. If $c \neq 0$, we obtain $x = hr^{-1} \in H$, a contradiction. Thus $c = 0$ and it follows that $h = xk$, $k \in J(R_0) \subseteq R$. Hence, $H \subset xR$ and consequently H is a waist of R.

Now assume that H is a right ideal of R which is a waist. Then $H \subseteq J(R) = J(R_0)$ and HR_0 is a right ideal of R_0. If $HR_0 \subseteq H$ we are done. Assume that there exists $h \in H$, $r_0 \in R_0$ with $hr_0 \in H$. Since H is a waist it follows that $h = hr_0 x$, for some $x \in R$. Thus $x \in P_r(H) \subseteq J(R_0)$ and we get $h = 0$, a contradiction. □

Now we can obtain a right D-domain R satisfying a.c.c.w. which is not noetherian and having a further prime ideal between (0) and $J(R)$.

Example 4.4 Let R_0 be the right noetherian right chain domain of type $\omega^2 + 1$ constructed in ([1], p. 1408). Here we have $F = k(t_1, t_2, \ldots)$ and we take $A = k(t_2, t_3, \ldots)[t_1]$, the polynomial ring over $k(t_2, t_3, \ldots)$ in t_1. Then A is a commutative D-domain with $J(A) = 0$. Consequently, $R = A \oplus J(R_0)$ is a right D-domain in which every waist is a right ideal of R_0, by Lemma 4.3. Since R_0 is noetherian, R satisfies a.c.c.w. We show that R is not right noetherian.

In fact, for any $x \in J(R_0)$ and A-submodule N of F, $x(N + J(R_0))$ is a right ideal of R. Therefore it is enough to show that the lattice of A-submodules of F does not satisfy a.c.c. But this is clear. For any field k and polynomial ring $k[t]$, $W_i = \{f \in k(t) : ft^i \in k[t]\}$ is a $k[t]$-submodule of $k(t)$ and $W_1 \subset W_2 \subset W_3 \subset \ldots$.

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