Forum Math. **11** (1999), 1–15

Forum Mathematicum

Some remarks on linear structures in right distributive domains

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(Communicated by Rüdiger Göbel)

Abstract. We examine properties of the right ideal structure of right distributive domains. Right distributive domains R are exactly those rings whose localizations at maximal right ideals M are right chain domains R_M. On the one hand, the paper focuses on the question in which way properties of R are carried over to R_M and vice versa. We examine the problem under which conditions two-sided ideals of R are again two-sided in the extension R_M (Lemma 2.2). Further, we observe the relationship between completely prime resp. semiprime ideals of R and the extended ideals in R_M. On the other hand, we prove in particular that for any maximal right ideal $M = R \backslash S_M$ the right-S_M-saturation $I_{[M]}$ of a completely semiprime ideal $I \subseteq M$ of R is completely prime (Theorem 2.9). A central role is played by waists of right distributive rings which are right ideals comparable to each other ideal, in particular there exists a largest waist W which is completely prime. We present a representation theorem in terms of ideals in R_W. We apply these results to the Jacobson radical $J(R)$ of a right distributive domain R. Illustrative examples are given.

1991 Mathematics Subject Classification: 16L30; 16P50, 16N60.

The starting point of our investigation is the well-known result of Brungs ([3], Theorem 1) that right distributive domains, right D-domains for short, are locally right chain domains (see the literature in [3] resp. [17] for some earlier papers on related topics). Note that the class of commutative D-domains coincides with the class of Prüfer domains. Numerous papers have focused on this class of rings with respect to their module properties as well as to algebras over division rings. A survey of recent results can be found in the paper of Brungs, Gräter ([5]). Obviously, the structure or R heavily depends on that of the localized rings R_M and vice versa, where M is a maximal right ideal of R. It is quite natural that the linear structures which are shown by the R_M's must be traced out in R itself.

Stephenson [17] considered another linear structure in right D-rings, namely the existence of so-called waists, which are right ideals comparable (by inclusion) to every other right ideal of the given ring. Further results on this phenomenon are presented in the work of Ferrero and Törner ([8], [9], [10]).

Thus, we want to bring to light properties of right D-domains using insights on right chain domains and reflecting the above mentioned facts on D-domains.

In Section 1 we summarize some well-known facts on right D-domains as well as on right chain rings. In Section 2 we introduce the notions of left- resp. right-S_M-saturated right ideals and the right-S_M-saturation of an ideal. We show that the right-S_M-saturated right ideals of a right D-domain are linearly ordered by inclusion (Proposition 2.3).

The question whether (two-sided) ideals $I \subseteq M$ of a right D-domain R with a maximal right ideal M are again two-sided in the localization R_M leads to the notion of left-S_M-saturation. We prove, using results on right chain rings, that the saturation of any completely semiprime ideal of a right D-rings is a completely prime ideal (Theorem 2.9 resp. Corollary 2.10).

Then waists are described as right ideals which are globally right-saturated (Section 3). In the final Section 4 we describe the 'neighbourhood' of the Jacobson radical and present some examples.

We like to thank Joachim Gräter for several helpful conversations and in particular for Examples 3.11 and 4.2.

1. Terminology and preliminary results

All rings are assumed to be associative with identity 1. Rings without zero-divisors are called *domains*. The *group of units* of a ring R is denoted by $U(R) = U$, whereas $J(R) = J$ stands for the *Jacobson radical* of R. Right ideals I which are closed by left multiplication, hence $RI \subseteq I$ holds, will be called (*two-sided*) *ideals*.

A right ideal $P \neq R$ is called *prime* if and only if $r_1 R r_2 \subseteq P$ implies $r_1 \in P$ or $r_2 \in P$ for $r_1, r_2 \in R$. In case we have $r_1 r_2 \in P$ implying $r_1 \in P$ or $r_2 \in P$, we call P a *completely prime* right ideal. Generalizing this notion we speak of a *semiprime* resp. *completely semiprime* right ideal P provided $rRr \subseteq P$ resp. $r^2 \in P$ leads to $r \in P$. It is a well-known result of Levitzki/Nagata (see [12]) that semiprime ideals are exactly those ideals which are intersections of prime ideals. Not as well-known[1] is the analogous result if one deals with completely prime resp. completely semiprime ideals. Using a theorem of Andrunakievic; Rjabukhin (see [1]) one obtains:

Theorem 1.1. *Let R be a ring. Then each completely semiprime ideal is an intersection of completely prime ideals.*

However, this result can independently be derived for right distributive rings using Lemma 2.4 and Theorem 2.9. Also of interest is the following observation (Théoremè 3, [18]):

[1] The result was pointed out to the authors by Prof. Wisbauer.

Corollary 1.2. *Any completely semiprime ideal P which is prime is also completely prime.*

We call a ring R *right distributive*, a *right D-ring* for short, if for any right ideals $I_1, I_2, I_3 \leq R$ the condition $I_1 \cap (I_2 + I_3) = (I_1 \cap I_2) + (I_1 \cap I_3)$ is satisfied. A ring R is a *right chain ring* if its lattice of right ideals is linearly ordered by inclusion. A *left chain ring* is defined analogously. *Chain rings* are left and right chain rings. Note that in case of right chain rings any intersection of prime resp. completely prime ideals is again a prime resp. completely prime ideal. So by the Levitzki-Nagata result as well by Theorem 1.1 we obtain the following observation which was already mentioned in (Lemma 1.8 (ii), [4]) and to which we will refer later on:

Lemma 1.3. *Let R be a right chain ring. Then any semiprime resp. completely semiprime ideal is prime resp. completely prime.*

As mentioned above we quote the following result from [3]:

Theorem. *Let R be a domain or a right noetherian ring. Then R is a D-ring if and only if $S_M = R\backslash M$ is a right Ore set and $R_M = RS_M^{-1}$ is a right chain ring for every maximal right ideal M of R.*

The next corollary follows directly from the Brungs-Theorem and will be used below:

Corollary 1.4. *Let R be a domain. Then the following assertions are equivalent:*

(a) *R is a right D-ring.*

(b) *Each maximal right ideal M is an ideal where $S_M = R\backslash M$ a right Ore set and R_M a right chain ring.*

Now let P be a completely prime ideal contained in some maximal right ideal M with R a right D-domain. Let $s \notin P$, $a \in P$. Since R_M is a right chain ring, we have $s \cdot rt^{-1} = a$ or $s = rt^{-1}$, whereby the second case leads to a contradiction, since P is completely prime. So $sr = at$ with $t \notin M$ and thus $t \notin P$ which proves that the complement of P is a right Ore set. The localization R_P exists and, as it can be easily checked, R_P is a right chain ring,

Remark 1.5. Let R be a right D-domain. The complement of a completely prime ideal P is a right Ore set. Any localization at a completely prime ideal P of a right D-domain leads to a right chain ring.

2. Saturation of right ideals

We discuss the saturation of a right ideal in R and prove that the set of right-S_P-saturated right ideals contained in P a completely prime ideal is linearly ordered.

We will use the notation introduced in [14] by Gräter, repeat some of his results, and complete them. Next we prove that completely prime ideals are right-S_P-saturated and right-S_P-saturated completely semiprime ideals are completely prime. Further we ask for conditions under which extension ideals of R are two-sided in R_P. This property can be checked within the ring R itself leading to the notion of left-S_P-saturation. We then discuss left saturation for completely prime resp. semiprime ideals and obtain the interesting result that a completely semiprime right-S_P-saturated is already completely prime (Theorem 2.9). This shows also that the right-S_P-saturation of a completely semiprime ideal is completely prime.

2.1. Saturation of right ideals. Let R a be a right D-domain, $P < R$ a completely prime ideal, and $S_P = R \backslash P$. In order to examine the extension $IR_P = I^e \leq R_P$ of a right ideal $I \leq R$ we will introduce the S_P-closure of I in R. *We remark that the following observations in this chapter hold in particular, when P equals some maximal right ideal which is completely prime and two-sided in case of right D-domains.*

Definition 2.1. Let R a be a right D-domain, $P = R \backslash S_P$ a completely prime ideal and $I \subseteq P$ a right ideal.

(i) We call I *right-S_P-saturated* if and only if for any $r \in R$, $rs \in I$ for some $s \in S_P$ implies $r \in I$.

(ii) We call the set $I_{[P]} = \{r \in R \mid \exists s \in S_P : rs \in I\}$ the *right saturation* of I with regard to S_P or simply the *right-S_P-closure* or *right-S_P-saturation* of I.

(iii) An ideal $I \leq R$ is called *left-S_P-saturated* if and only if for any $r \in R$, $s \in S_P$ the containment $sr \in I$ implies $r \in I$.

In the next lemma we will show that $I_{[P]}$ is a right ideal.

Lemma 2.2. *Let R be a right D-domain, $P = R \backslash S_P$ a completely prime ideal, and $I \subseteq P$ a right ideal of R. Then we have:*

(i) *$I_{[P]}$ is a right ideal of R satisfying $I \subseteq I_{[P]} \subseteq P$. Further we have: $I_{[P]} = I^e \cap R = I^{ec}$.*

(ii) *$I_{[P]}$ is right-S_P-saturated.*

(iii) $I = I_{[P]} \Leftrightarrow I = I^{ec}$.

(iv) *If I is an ideal, then $I_{[P]}$ is also an ideal.*

(v) *If I is left-S_P-saturated, then $I_{[P]}$ is also left-S_P-saturated.*

Proof. (i) First we prove $I_{[P]} = I^e \cap R$. That $I_{[P]}$ is a right ideal of R is then an immediate consequence of this observation.

Let $r \in R$ and $x \in I_{[P]}$ with $xs \in I$, $s \in S_P$. Since S_P is a right Ore set, again we find elements r', s' such that $rs' = sr'$ which finally implies $xr \in I_{[P]}$. Further, $x \in P$ for any $x \in I_{[P]}$ since $xs \in I \subseteq P$ and P is completely prime. The rest of the statement is obvious.

Now take $x \in I_{[P]}$, hence $xs = i \in I$ for some $s \in S_P$ and $x = is^{-1} \in I^e \cap R$ follows. On the other hand, let $irs^{-1} \in R$, so $irs^{-1} = a$ leading to $ir = as$ which implies $a \in I_{[P]}$.

(ii) follows since S_P is multiplicatively closed.
(iii) use (i).
(iv) obvious.
(v) follows directly. □

Note that in the commutative case S_P-saturated ideals, with P a maximal right ideal, are called *valuation ideals* (see Gilmer [11], p. 300).

By Lemma 2.2 (iii) we know that right-S_P-saturated right ideals satisfy $I^{ec} = I$. Hence, take any right-S_P-saturated right ideals $A, B \leq R$, so $A^{ec} = A$ resp. $B^{ec} = B$. Since R_P is a right chain domain by Remark 1.5, we have $A^e \subseteq B^e$ or $B^e \subset A^e$, say $A^e \subseteq B^e$. It follows $A^{ec} \subseteq B^{ec}$. Since we assumed A, B to be right-S_P-saturated, it follows $A \subseteq B$. So we have proved that the lattice of right-S_P-saturated right ideals of R is linearly ordered:

Proposition 2.3. *Let R be a right D-domain and $P = R \backslash S_P$ a completely prime ideal. Then the lattice of right-S_P-saturated right ideals of R is linearly ordered.*

2.2. Saturation of semiprime and prime ideals. In this section we apply the notion from above to completely semiprime resp. completely prime ideals.

Lemma 2.4. *Let R be a right D-domain, $P = R \backslash S_P$ a completely prime ideal. Then the following hold:*

(i) *Completely prime right ideals $P' \subseteq P$ are right- and left-S_P-saturated.*

(ii) *Completely prime ideals $P' \subseteq P$ are right- and left-S_P-saturated.*

(iii) *Let I be any completely semiprime ideal $\subseteq P$. The right-S_P-saturation $I_{[P]}$ is again completely semiprime.*

(iv) *Let I be a right-S_P-saturated completely semiprime ideal. Then I is also left-S_P-saturated.*

Proof. (i) Use the fact that $xs, sx \in P'$ with $s \notin P'$ implies $x \in P'$.

(ii) follows from (i).

(iii) Assume $x^2 \in I_{[P]}$, so $x^2 s \in I$ for some $s \in S_P$. Hence $(xsx)(xsx) \in I$ which shows $xsx \in I$. However, $xsxs$ is also an element of I, so $xs \in I$ follows. Hence, $x \in I_{[P]}$ and we are done.

(iv) Let $s \in S_P$ with $sx \in I_{[P]}$, say $sxt \in I$ for some $t \in S_P$. Since $sxt \in I$ and I is assumed to be two-sided, it holds $xt(sxt)s = (xts)^2 \in I$. However, I is completely semiprime, and therefore, $xts \in I$, showing $x \in I_{[P]}$. □

It will be shown in Theorem 2.9 that the right-S_P-saturations of completely semiprime ideal are even completely prime, however, we need some further results.

We remark an obvious consequence:

Corollary 2.5. *Let R be a right D-domain, $P = R \backslash S_P$ a completely prime ideal and $I \subseteq P$ a right ideal. Then there exists no completely prime right ideal P' contained in the interval $I \subset P' \subset I_{[P]}$.*

However, we were not successful to obtain some information on the right-S_P-closure of a prime ideal resp. semiprime ideal which is not completely prime resp. completely semiprime.

Problem 2.6. Describe the right-S_P-saturation of semiprime resp. prime ideals.

2.3. Two-sidedness of extended right ideals. Let again R be a right D-domain, $P = R \backslash S_P$ a completely prime ideal and I a right ideal. It is quite natural to ask whether the extension I^e is an ideal in a localization R_P. Since I^e equals $I_{[P]} R_P$, necessarily the saturated right ideal $I_{[P]}$ has to be an ideal. But this is not sufficient as the following observation shows: Let $a \in I_{[P]}$, so $av \in I$ for some $v \in S_P$. In particular we have to check whether $s^{-1}a$ lies in $I^e = IR_P = I_{[P]} R_P$ for any $s \in S_P$. Since S_P is right Ore, we have $sb = at$ for some $t \in S_P$ and $b \in R$ suitable. It remains to ask for conditions under which $b \in I_{[P]}$ holds. Hence, the notion of left-S_P-saturation (see Definition 2.1 (iii)) is of importance.

Lemma 2.7. *Let R be a right D-domain and $I \subseteq P = R \backslash S_P$ a right ideal with P a completely prime ideal. Further assume that $I_{[P]}$ is an ideal of R. Then the following assertions are equivalent:*

(a) *I^e is an ideal in R_P.*

(b) *$I_{[P]}$ is left-S_P-saturated.*

Proof. (a) $\Rightarrow$ (b) Let be $s \in S_P$ and $sx = a \in I_{[P]}$, so by assertion $x = s^{-1} a \in I^e$, thus $x \in I^{ec} = I_{[P]}$, thus $I_{[P]}$ is left-S_P-saturated.

(b) $\Rightarrow$ (a) We have to show that $I^e = IR_P = I_{[P]} R_P$ is also a left ideal of R_P. Trivially, $RI^e \subseteq I^e$ holds, since $I_{[P]}$ is an ideal of R. Thus it remains to show that $s^{-1} x \in I^e$ for any $x \in I_{[P]}$, $s \in S_P$. Note that $s^{-1} x = at^{-1} \in R_P$ with $a \in R$, $t \in S$. It follows $xt = sa \in I_{[P]}$. By assumption we obtain $a \in I_{[P]}$ and $s^{-1} x = at^{-1} \in I^e$ follows. □

If I is two-sided, then, by Lemma 2.2 (iv), $I_{[P]}$ is an ideal too. Thus Lemma 2.7 holds in particular, if I is an ideal. Note further that by 2.2 (v) I left-S_P-saturated implies the left-saturation of $I_{[P]}$ too.

Problem 2.8. Under which conditions is the Jacobson radical left-S_P-saturated?

2.4. Saturation of completely semiprime ideals. Next we are ready to prove an interesting observation:

Theorem 2.9. *Let R be a right D-domain and $P = R\backslash S_P$ a completely prime ideal. Let P' be a completely semiprime right-S_P-saturated ideal of R. Then P' is also completely prime.*

Proof. Since P' is assumed to be right-S_M-saturated, we have $P' = P'^{ec}$ using Lemma 2.2 (iii). As a completely semiprime ideal P' is obviously left-S_P-saturated (by Lemma 2.4 (iv)). Thus by Lemma 2.7 P'^e is an ideal in the localized ring R_P. Since completely semiprime ideals in a right chain ring are already completely prime (see [4]) and the retraction of a completely prime ideal of R_P is again completely prime in R, it remains to prove that P'^e is completely semiprime.

Let $(xs^{-1})(xs^{-1}) = (xx')(ss')^{-1} \in P'^e$ where $sx' = xs'$. If $x \in P'$, we are done. Otherwise assume $x \notin P'$ and since P' is completely semiprime and saturated neither $x, x^2 \in P'$ nor $xt, x^2t \in P'$ holds for some $t \in S_P$. Take $xs^{-1} \notin P'^e$, hence $x \notin P'$ and even $xt \notin P'$ for any $t \in S_P$ since P' is right-S_P-saturated. Since $sx' = xs' \in P'$, we get $x' \notin P'$. However with $xx' \in P'$ we conclude $x'(xx')x = (x'x)^2 \in P'$, thus $x'x \in P'$ follows.

Next we compare the elements x, x' of the right chain ring R_P. As the situation is symmetric, it suffices to discuss only the case $xR_P \subseteq x'R_P$. Then there exist elements $r \in R$, $t \in S_P$ such that $x = x'(rt^{-1}) = (x'r)t^{-1}$, hence $xt = x'r$. By assumption we have as well $x^2 \notin P$ as $xt = x'r \notin P'$. Multiplying the last equation with x, namely $xxt = xx'r$ we obtain a contradiction, since the left side is not contained in P', however the right side is. Thus, P'^e is completely semiprime in R_P and we are done. □

The next consequence is straightforward:

Corollary 2.10. *Let R be a right D-domain and $P = R\backslash S_P$ a completely prime ideal. Furthermore, let $I \subseteq P$ be a completely semiprime ideal of R. Then the right-S_P-closure $I_{[P]}$ is a completely prime ideal of R.*

Proof. We remind that by Lemma 2.4 (iii) the right-S_P-closure of a completely semiprime ideal is completely semiprime again. The rest follows by Theorem 2.9. □

3. Waists

Refering in particular to the previous papers [9] resp. [10] we extend some of the results in the case of right D-domains. In particular, it is proved that waists are exactly those right ideals with $P_r(I) \subseteq J(R)$. A representation theorem 3.9 is given extending some earlier results. An additional example fitting to the general frame of the theorem is given.

Properties of waists were studied in details in [9] resp. [10]. Here we relate this approach with the concept of saturated right ideals described above.

Definition 3.1. Let R be a ring. A right ideal $I \neq R$ of R is said to be a *waist* if for every right ideal K of R we have either $I \subseteq K$ or $K \subseteq I$.

It is clear that I is a waist if and only if for every $a \in R \backslash I$ we have $I \subset aR$. Since a waist I must be contained in all maximal right ideals I is obviously contained in the Jacobson radical. The zero ideal is a trivial waist. It was already observed by Stephenson [17], the completely prime ideals contained in the Jacobson radical are waists.

3.1. A characterization of waists. First we remind the following notation. Let I be a right ideal, then

$$P_r(I) = \{x \in R \mid \exists t \in R \backslash I : tx \in I\}$$

denotes the *right associated right multiplicative ideal* of I. Obviously, the complement $R \backslash P_r(I)$ is multiplicatively closed. Thus $P_r(P) = P$ in case P is a completely prime right ideal. Using this terminology we obtain the following characterization of waists which was given in (Lemma 2.3, [9]), however we are possible to exclude a special case in the former version. The proof follows arguments given in the cited lemma.

Lemma 3.2. *Let R be a right D-domain, I a right ideal of R and $P = P_r(I)$. Then the following conditions are equivalent:*

(a) *I is a waist.*

(b) $I = \bigcap_{a \notin I} aP$.

Further we make use of some arguments which are part of the proof of (Theorem 3.11, [9]):

Lemma 3.3. *Let R be a right D-domain and I a right ideal which is a waist. Then $P_r(I) \subseteq J(R)$ follows.*

Proof. Assume otherwise $P_r(I) \nsubseteq J(R)$. Hence there exists $x \in P_r(I)$, $x \notin J(R)$, $t \notin I$ such that $tx \in I$. We set $H = \{y \in R \mid ty \in I\}$. Then H is a right ideal which is obviously not contained in $J(R)$, so there exists a maximal right ideal M of R with $H \nsubseteq M$. It follows $I \subseteq tM$, since I is a waist. On the other hand we have $tH \subseteq I \subset tM$, which implies $H \subseteq M$, a contradiction. □

However, the opposite is also true:

Proposition 3.4. *Let R be a right D-domain and I a right ideal. Then the following conditions are equivalent:*

(a) *I is a waist.*

(b) $P_r(I) \subseteq J(R)$.

In addition, $P_r(I)$ is a completely prime ideal if one of the assertions hold.

Proof. (a) ⇒ (b) see Lemma 3.3.

(b) ⇒ (a) Since $P_r(I)$ is a completely prime right multiplicative ideal contained in $J(R)$, we know by (Theorem 2.6, [9]) that $P_r(I)$ is a completely prime ideal. We set $S = R \setminus P_r(I)$. Then by (Proposition 3.6, [9]) we have the right ideal $(aR)S^{-1} = \{x \in R \mid \exists_{s \in S} xs \in aR\}$ as a waist for any $a \in R$. By definition, for any $a \in I$ the right ideal $(aR)S^{-1}$ is contained in I, so $I = \bigcup_{a \in I}(aR)S^{-1}$. However, the union of waists is again a waist.

$P_r(I)$ is a completely prime ideal applying Theorem 2.6 in [9]. □

Corollary 3.5. *Let R be a right D-domain and I a right ideal. Then the following are equivalent:*

(a) *I is a waist.*

(b) *I is right-S_M-saturated for all maximal right ideals $M = R \setminus S_M$.*

Proof. (a) ⇒ (b) By Proposition 3.4 we have $P_r(I) \subseteq J(R)$. So let $s \in S_M$ and $xs \in I$. Since $s \notin J(R)$ and so $s \in R \setminus P_r(I)$ we obtain $x \in I$ proving I to be right-S_M-saturated.

(b) ⇒ (a) By assumption $P_r(I)$ must be contained in every maximal right ideal, so $P_r(I) \subseteq J(R)$ which again proves I to be a waist by Proposition 3.4. □

3.2. The largest waist. The union W of those right ideals which are waists is again a waist. Thus there exists a largest waist. Ferrero/Törner ([9]) already mentioned that W is a completely prime ideal, however under the assertion that condition (MP) holds, that is, there exists a completely prime ideal contained in the Jacobson radical. Note that in D-domains the zero-ideal is a completely prime ideal contained in the Jacobson radical by which (MP) is trivially satisfied. Independently of the mentioned result we deduce this fact directly.

Proposition 3.6. *Let R be a right D-domain. Then the union W of waists of R is a completely prime ideal.*

Proof. If $W = 0$, then W is a completely prime ideal by assertion since R is a domain. So we may assume that there exists a waist $I \neq (0)$. By Proposition 3.4 $P_r(I) \subseteq J(R)$ and since $P_r(I)$ is a completely prime ideal (Proposition 3.4) $P_r(I)$ itself is a waist containing I. Thus any waist leads to a eventually larger waist which is completely prime. So, the union of waists is contained in a union of completely prime ideals which are waists and where the set of completely prime ideals itselves is linearly ordered by inclusion. Hence, W is completely prime. □

Obviously we have:

Corollary 3.7. *Let R be a right D-domain. Then $J(R)$ is a waist if and only if $J(R)$ is completely prime.*

It is natural to ask for more information on the segment $W \subset J(R)$. However, the 'distance' between the largest waist W and the Jacobson radical $J(R)$ may be 'arbitrary' large. Take the intersection of two comaximal (commutative) valuation domains. Then (0) equals W which may differ considerably from $J(R)$; see also Example 3.11.

3.3. A representation theorem for waists.

Theorem 3.8. *Let R be a right D-domain and let W be the largest waist in R. Then $B = R_W$ is a right chain domain and the correspondence*

$$I \mapsto R \cap I$$

induces a bijection between the set of all proper right ideals of B and all waists.

Proof. Let I be a proper right ideal of B. Then $(I \cap R)S_W^{-1} = I$, i.e. $(I \cap R)^{ec} = I \cap R$, and $I \cap R = (I \cap R)_{[W]}$ follows. Since $W \subseteq M$ for each maximal right ideal M or R, we conclude $I \cap R = (I \cap R)_{[M]}$ and $I \cap R$ is a waist by Corollary 3.5. Furthermore, if I is a waist, then $P_r(I) \subseteq J(R)$ is a completely prime ideal of R by Proposition 3.4 and therefore $P_r(I)$ is a waist too, i.e., $P_r(I) \subseteq W$. We show $IS_W^{-1} \cap R = I$ where $\supset$ is obvious. Let $r = is^{-1} \in R$, $i \in I$, $s \in R \setminus W$ and let us assume $r \notin I$. Then $i = rs$ implies $s \in P_r(I)$ – a contradiction. This completes the proof. □

By the last result we are able to give for any right ideal I of R a description of the largest waist contained in I and the smallest waist containing I. This was already mentioned in (Proposition 3.15, [9]). Here we apply the theorem of above:

Corollary 3.9. *Let R be a right D-domain, W the largest waist and I a right ideal contained in W. Then $(I) = \bigcap_{a \notin I} aW$ is the largest right ideal contained in I which is a waist. Further $[I] = \bigcup_{a \in I} (aR)_{[W]}$ is the smallest right ideal containing I which is a waist.*

Proof. Obviously aW is a waist in R since $aW = aW \cap R$. With similar arguments $(aR)_{[W]}$ to be a waist follows. □

The noncommutative situation differs considerable from the two-sided situation:

Corollary 3.10. *Let R be a right and left D-domain. Then the maximal waist of R is induced by an overring of R.*

Proof. We apply Theorem 3.8; note that the localizations R_M with M maximal right ideals are chain domains and it exists a chain domain B minimal over the R_M's. The Jacobson radical $J(B)$ of B itself is the largest waist in R. □

The theorem above shows in addition that the general construction of D-domains with waists introduced in [10] basically describes the normal situation. However in the example given there the special property occurs that each waist of R is a right ideal of B and vice versa where Theorem 3.8 only ensures that $\{R \cap I \mid I$ is a right ideal of $B\}$ is the set of all waists of R. The following construction leads to an example which describes the more general situation, in particular that 3.10 is not true in general.

Example 3.11. Let S be a right D-domain with monomorphism $\sigma : S \to S$ and let F be the skew field of fractions of S where σ also denotes the extension of σ to F. In the following σ satisfies the property $\sigma(F) \subseteq S$ which is crucial to our construction. We define the skew power series rings $R = S[\![x, \sigma]\!]$ resp. $B = F[\![x, \sigma]\!]$ over S resp. F in the indeterminate x where B is a right chain ring and we claim that $W := xR = \{xs_1 + x^2 s_2 + \cdots \mid s_1, s_2, \ldots \in S\}$ is a waist in R. Obviously, W is a two-sided ideal of R and we must show $W \subseteq aR$ for each $a \in R \backslash W$. So, let $a = a_0 + xa_1 + x^2 a_2 + \cdots$ where $a_0, a_1, \ldots \in S$ and $a_0 \neq 0$. Then, a is a unit in B, i.e. $a^{-1} = b_0 + xb_1 + \cdots$ where $b_0, b_1, \ldots \in F$ and therefore $a^{-1}x = x\sigma(b_0) + x^2\sigma(b_1) + \cdots$ with $\sigma(b_0), \sigma(b_1), \ldots \in \sigma(F) \subseteq S$ by assumption. This shows $x \in aS$ and $W = xR \subseteq aSR \subseteq aR$. To verify that R is a right D-domain it is enough to prove $aR \cap (bR + cR) = (aR \cap bR) + (aR \cap cR)$ for all $a, b, c \in R$. Without restrictions we can assume that at least one of a, b, and c is not in W since otherwise one can multiply with a suitable power of x^{-1}. We consider the case $a \notin W$ where the other two cases can be treated similarly. If in addition b and c are in W then we see $bR, cR \subseteq W \subseteq aR$ since W is a waist and we are done. If $b \in W$ and $c \notin W$ then $bR \subseteq W \subseteq aR \cap cR$, i.e. $aR \cap (bR + cR) = aR \cap cR = (aR \cap bR) + (aR \cap cR)$, and the same arguments can be used if $c \in W$ and $b \notin W$. But if $b, c \notin W$ then $W \subseteq aR, bR, cR$ and it is sufficient to show that

$$[aR \cap (bR + cR)]/W = [(aR \cap bR) + (aR \cap cR)]/W.$$

But this holds obviously since $R/W \cong S$ is a right D-domain.

If S has only the trivial waist 0 then W is the largest waist in R and by Theorem 3.8 the correspondence $I \mapsto R \cap I$ induces a bijection between the set of all right ideals of R_W and all waists in R. We show that the waists of R need not be right ideals of R_W. But this is easily done since $xF \subseteq W_W$ where $xF \nsubseteq W$ provides $W_W \neq W$. To complete our construction a specific right D-domain S with monomorphism σ is required such that S has only the trivial waist 0 and σ maps the skew field of fractions F of S into S. Let K be a commutative field and let $x_1, x_2, \ldots$ be independent indeterminates over K. We put $S = K(x_2, x_3, \ldots)[x_1]$ which is a principle ideal domain with $J(S) = 0$ and therefore S is a D-domain such that 0 is the only waist in S. Finally, $F = K(x_1, x_2, \ldots)$ is the quotient field of S and if σ denotes the monomorphism of F with $\sigma(k) = k$ for all k in K and $\sigma(x_i) = x_{i+1}$ for al $i = 1, 2, \ldots$ then $\sigma(F) \subseteq S$.

4. The Jacobson radical and further examples

In this section we summarize results on the Jacobson radical and close with an example showing that $J(R)$ has not to be completely prime in general. This construction leads also to a right D-ring with a prime ideal which is not completely prime. The corresponding question for *right chain rings* remained unanswered till 1993 ([6]).

4.1. Some observations. As the intersection of a completely prime ideals the Jacobson radical of a right D-domain is obviously completely semiprime. Thus by Corollary 1.2 we obtain

Corollary 4.1. *$J(R)$ is completely prime provided $J(R)$ is a prime ideal at all.*

Since the Jacobson radical is completely prime if and only if $J(R)$ is a waist one may ask for conditions under which $J(R)$ is completely prime. First we summarize a few observations around the Jacobson radical $J(R)$.

Proposition 4.2. *Let R be a right D-domain. Then the following are equivalent.*

(a) *$J(R)$ is completely prime.*

(b) *$J(R)$ is a waist.*

(c) *The Jacobson radical $J(R)$ is right-S_M-saturated for all maximal ideals M.*

(d) *The Jacobson radical $J(R)$ is right-S_M-saturated for at least some maximal ideal M.*

Proof. The equivalence of (a) and (b) was already stated by Corollary 3.7. Further the statements (b) and (c) are equivalent by Corollary 3.5. Obviously (c) implies (d). Now assert that $J(R)$ is right-S_M-saturated for some maximal right ideal M. Since $J(R)$ is completely semiprime, Theorem 2.9 proves (a). □

The next result is quite obvious:

Lemma 4.3. *Let R be a right D-domain with infinitely many maximal right ideals, however each element $x \notin J(R)$ is contained in only finitely many maximal ideals. Then $J(R)$ is completely prime.*

Proof. Let $xy \in J(R)$ and assume $x \notin J(R)$, then $y \in J(R)$. Otherwise we would find a maximal ideal M not containing both elements and so $xy \notin M$ would follow, a contradiction. □

D-rings where each element R^* is contained in only finitely many maximal ideals were discussed in detail by Gräter [15].

In right noetherian D-domains elements $x \notin J(R)$ which are contained in infinitely many ideals lead to a completely prime ideal which is not maximal.

Lemma 4.4. *Let R be a right noetherian D-domain with infinitely many prime ideals. If there exist elements $x \notin J(R)$ which are lying in infinitely many prime ideals, then there exist a completely prime ideal $\neq (0)$ which is not maximal.*

Proof. Using the factorization of principal right ideals (see [3]) we have $xR = P_1 \cdots P_n$ for finitely many prime ideals $P_1, \ldots, P_n$. If one of the involved prime ideals is not maximal, we are done. Otherwise xR is contained in some maximal ideal M different from all of these P_i's, thus there exists at least some prime ideal P_j strictly contained in a maximal ideal. Contradiction. □

4.2. A further example. In this subsection we describe a method of constructing D-domains which have a prime ideal which is not completely prime. Furthermore, we will be able to present a D-ring with no completely prime ideal inside the Jacobson radical which has also a proper prime ideal which is a waist. Thus condition (MP) used in [10] can not be omitted in general. These examples will be discussed in details later on.

A D-domain with a prime ideal which is not completely prime is given in ([6], [7]) where it is shown that a prime ideal of a chain domain need not be completely prime. Unfortunately, this result is involved in a rather complicated theory which is devoted to the problem of embedding groups into division rings. The construction of D-domains with prime ideals which are not completely prime we use is notably easier than the one given in ([6], [7]) but our domains are intersections of two chain rings instead of chain rings themselves.

Example 4.5. We start with a chain domain B which has $M := J(B)$ as its only nontrivial two-sided ideal. An example of such a ring is given in [16]. For our construction we take an arbitrary nonzero a in M and define $R = aBa^{-1} \cap B$. As an intersection of two incomparable chain domains R is a D-domain with exactly two maximal ideals $M \cap R$ and $aMa^{-1} \cap R$. Since aB is a right ideal of B and a left ideal of aBa^{-1} we conclude that $aB \subseteq R$ is a two-sided ideal of R. We show that $P := aB$ is a prime ideal but first we prove that P is not completely prime. Otherwise R_P is a chain domain containing B and $J(R_P)$ is a two-sided ideal of B. This means $J(R_P) = M$ and therefore $aB = M$. But this is a contradiction since then a^2B is a two sided ideal of B by $Ba \subseteq M = aB$ which is strictly contained in M by $a^2B \subset aB$. Let us turn to the proof that P is prime and we describe all two-sided ideals I of R which contain P strictly: $P \subset I$. By [13] we can write I as the intersection of its localizations as right ideal:

$$(*) \qquad I = IB \cap IaBa^{-1}$$

and since IB as well as $IaBa^{-1}$ are (fractional) left ideals of R we also obtain

$$(**) \qquad IB = BIB \cap aBa^{-1}IB, \quad IaBa^{-1} = BIaBa^{-1} \cap aBa^{-1}IaBa^{-1}.$$

Since B has M as its only nontrivial two-sided ideal we see that $BIB = M$ or $BIB = B$. Similarly, $aBa^{-1}IaBa^{-1} = aMa^{-1}$ or $aBa^{-1}IaBa^{-1} = aBa^{-1}$. It remains to discuss $aBa^{-1}IB$ as well as $BIaBa^{-1}$. By $a \in M$ and therefore $aB \subseteq M = BIaB$ the inclusion $aBa^{-1} \subseteq BIaBa^{-1}$ follows. Combining this with (**) and $aBa^{-1}IaBa^{-1} \subseteq aBa^{-1}$ one obtains $IaBa^{-1} = aMa^{-1}$ or $IaBa^{-1} = aBa^{-1}$. To investigate $aBa^{-1}IB$ we first observe that there exists b in I such that $b \notin P = aB$, i.e., $aB \subset bB$ and $b^{-1}a \in M$. Thus $Bb^{-1}a \subset M$ and the inclusion $BaB \subseteq Bb^{-1}a$ cannot hold. There exists $x \in B$ such that $Bb^{-1}a \subset Bax$, i.e. $axa^{-1}b \notin B$. We conclude $BIB \subseteq B \subset axa^{-1}bB \subseteq aBa^{-1}IB$ and $IB = BIB \cap aBa^{-1}IB = BIB$ by (**). This together with (*) shows that I must be one of the following ideals:

$$M \cap aBa^{-1} = M \cap R, \; M \cap aMa^{-1},$$
$$B \cap aBa^{-1} = R, \; B \cap aMa^{-1} = R \cap aMa^{-1}.$$

To prove that P is prime it is enough to verify that $(M \cap aMa^{-1}) \cdot (M \cap aMa^{-1}) \subseteq aB$ does not hold. From the inclusion it would follow that $(M \cap aMa^{-1}) \cdot (M \cap aMa^{-1})B \subseteq aB$. But it will be shown that $(M \cap aMa^{-1}) \cdot (M \cap aMa^{-1})B = M$, so we would get $M \subseteq aB$, a contradiction.

Thus it remains to show that $(M \cap aMa^{-1}) \cdot (M \cap aMa^{-1})B = M$ holds. Since $Ba \subset M$ we see that $BaM \subseteq Ba$ cannot hold and there exists $m \in M$ such that $Ba \subset Bam$. Thus $ama^{-1} \notin B$ as well as $M \subset ama^{-1}B \subseteq aMa^{-1}B$ and $(M \cap aMa^{-1})B = (MB \cap aMa^{-1}B) = M$ follows. This means $(M \cap aMa^{-1})B = M = M^2 = (M \cap aMa^{-1})BM = ((M \cap aMa^{-1}) \cdot (M \cap aMa^{-1}))B$.

We discuss P in terms of localizations and saturations. Let $Q = R \cap M$ and $Q' = R \cap aMa^{-1}$ then P is a prime ideal of R but $P_Q = P$ is only a right ideal of $R_Q = B$. By Lemma 2.2 the right saturation $P_{[Q]} = P$ is a prime ideal. Furthermore, $P_{Q'} = aB \cdot aBa^{-1} = aMa^{-1}$ is a completely prime ideal of $R_{Q'} = aBa^{-1}$ and $P_{[Q']} = R \cap aMa^{-1}$ is a completely prime ideal of R.

We will also use the above example to obtain a D-ring S with no completely prime ideal inside $J(S)$ such that there exists a nontrivial waist which is in addition prime. aM is also a two-sided ideal in $R = B \cap aBa^{-1}$ and we define $S = R/aM$. It remains to show that aB/aM is a waist in S or equivalently that $aB \subseteq aM + bR$ for each b in $R \setminus aB$. We do this by localization where $aB \subseteq aM + bB$ holds obviously. The second inclusion $aBaBa^{-1} \subseteq aMaBa^{-1} + bRaBa^{-1}$ follows by $BaB = MaB$. But this also shows that aB/aM is a minimal right ideal of $S = R/aM$.

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Received March 15, 1997; revised and in final form December 8, 1997

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